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Hartogs–Bochner type theorem in projective space

Frédéric Sarkis[1]

Abstract. We prove the following Hartogs–Bochner type theorem: *Let M be a connected C^2 hypersurface of $P_n(\mathbf{C})$ $(n\geq 2)$ which divides $P_n(\mathbf{C})$ in two connected open sets Ω_1 and Ω_2. Suppose that M has at most one open CR orbit. Then there exists $i\in\{1,2\}$ such that C^1 CR functions defined on M extends holomorphically to Ω_i.*

1. Introduction

The classical Hartogs–Bochner theorem states that if $\Omega\Subset\mathbf{C}^n$ $(n\geq 2)$ is a domain the boundary $\partial\Omega$ of which is smooth and connected, then every continuous CR function defined on $\partial\Omega$ extends holomorphically to Ω. A natural question is to ask if such an extension phenomenon is valid for domains included in a complex manifold X. Of course, in the case when X is compact, there is no hope to expect such a result. Indeed, if the Hartogs–Bochner phenomenon is valid in X, then CR functions on $\partial\Omega$ would extend to Ω but also to $X\backslash\Omega$ and thus are constant which is impossible in general. Nevertheless, the following Hartogs–Bochner type phenomenon has been conjectured in $P_2(\mathbf{C})$: *Let M be a connected C^2 hypersurface of $P_2(\mathbf{C})$ which divides $P_2(\mathbf{C})$ into two connected open sets Ω_1 and Ω_2. Then CR functions on M extend holomorphically to one of these sets.*

This conjecture has interested many authors at least since 1996 when E. Porten communicated to me the question with reference to R. Dwilewicz. In [17], we proved that holomorphic (resp. meromorphic) functions defined in a connected neighborhood of M extend holomorphically (resp. meromorphically) to one of the two sides of M and repeated the question about the extension of CR functions. Recently, Dwilewicz and Merker [4] gave a simplification of this proof in the holomorphic case and raised the question again. In [10], Henkin and Iordan gave a proof of the

([1]) Supported by the TMR network.

conjecture for M of Lipschitz class but only under the hypothesis that one of the two sides of M contains a weakly concave domain with smooth boundary.

In this paper, we prove the following Hartogs–Bochner type theorem.

Theorem 1. *Let M be a connected C^2 hypersurface of $P_n(\mathbf{C})$ $(n\geq 2)$ which divides $P_n(\mathbf{C})$ in two connected open sets Ω_1 and Ω_2. Suppose that M has at most one open CR orbit. Then there exists $i\in\{1,2\}$ such that for any C^k $(k\geq 1)$ CR function $f\colon M\to\mathbf{C}$, there exists a holomorphic function $F\in\mathcal{O}(\Omega_i)\cap C^k(\overline{\Omega}_i)$ such that $F|_M=f$.*

The above mentioned results of [17], [4] and of [10] in the smooth case can be obtained as a consequence of our theorem.

Moreover, if M admits a nonconstant CR function f, then the hypothesis that M has only one open CR orbit is a necessary condition for the Hartogs–Bochner phenomenon to be valid (see Corollary 1).

As the Hartogs–Bochner theorem is already known for restrictions of holomorphic functions defined in a neighborhood of M, one natural idea is to apply the analytic disc techniques in order to extend continuous CR functions on M to a one-sided neighborhood of M. Then by deforming Ω_i (resp. M) in this one-sided neighborhood, we are reduced to the case of holomorphic functions in the neighborhood of $\overline{\Omega}_i$ (resp. M). This idea has already been applied by Jöricke, Merker and Porten in order to obtain many results about extension and removability of singularities of CR functions. In the case of the study of the Hartogs–Bochner phenomenon, Jöricke [11] proved that compact hypersurfaces of $\mathbf{C}^n$ are globally minimal (i.e. consist of a single CR orbit). Thus, using the propagation results of Trépreau [21] of analytic extension along CR orbits, one obtains that CR functions defined on M extend holomorphically to a one-sided neighborhood of M. Thus, in the case of $\mathbf{C}^n$, the Hartogs–Bochner extension theorem can be reduced to the classical Hartogs extension theorem (this has been used for example in [16] and [17] in order to prove CR meromorphic extension results). In the case of a compact hypersurface of $P_n(\mathbf{C})$, it is conjectured in [17] and also in [4] that compact hypersurfaces are also globally minimal, but unfortunately this is not known and is related to the following question of E. Ghys (see [7]): *Does there exist a nontrivial compact set laminated by Riemann surfaces in $P_2(\mathbf{C})$?*

Indeed, in the case of a connected compact hypersurface M of $P_n(\mathbf{C})$, CR orbits are either open subsets of M or injectively immersed complex hypersurfaces whose closure is a compact subset of M laminated by complex manifolds of dimension $n-1$. Of course, if there exists no such laminated compact set in $P_n(\mathbf{C})$, then M has to be globally minimal (i.e. has only one CR orbit which is open). Let K be the union of all nonopen CR orbits of M. Then K is a laminated compact subset of $P_n(\mathbf{C})$. The hypothesis that M has only one CR orbit which is open is then equivalent to the fact that $M\setminus K$ is connected. If $K=\emptyset$, M is globally minimal and we are reduced to the result of [17]. If $K\neq\emptyset$, it is known that $P_n(\mathbf{C})\setminus K$ is Stein. Then, we apply the boundary problem result of Chirka [1] to the graph of CR functions over $M\setminus K$ in order to obtain the needed holomorphic extension.

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2. Preliminaries

2.1. Decomposition into CR orbits

Let M be an oriented and compact real hypersurface of class C^2 of a complex manifold of dimension n. For any point $p\in M$, we call $H_p(M)=T_p(M)\cap iT_p(M)$ the *holomorphic tangent space to* M at the point p (where $T_p(M)$ is the tangent space to M at p and i is the imaginary unit). As M is of class C^2, the set of holomorphic tangent spaces to M is a vector bundle of complex rank $n-1$. A C^1 curve $\gamma\colon[0,1]\to M$ is called a *CR curve* if for any point $t\in[0,1]$, $\gamma'(t)\in H_{\gamma(t)}(M)$. Let $x\in M$, the set of points $y\in M$ which can be joined to x by a piecewise CR curve is called the *CR orbit* $\mathcal{O}_{CR}(x)$ of x in M. It is well known that CR orbits are CR submanifolds injectively immersed in M and of the same CR dimension. Thus, for any point $x\in M$, $\mathcal{O}_{CR}(x)$ is either an open set (and we will say that M is globally minimal at the point x) or a complex manifold η_x of dimension $n-1$ injectively immersed in M. In this last case, the CR orbits are tangent to the bundle $H(M)$ of complex tangent vectors to M. As M is of class C^2, $H(M)$ is of class C^1, thus any point $p\in M$ has a neighborhood U_p such that η_x is a product of the unit ball of $\mathbf{C}^{n-1}$ by a topological set $T\subset\mathbf{C}$. More precisely, $\bar\eta_x$ is a compact set laminated by complex manifolds of dimension $n-1$ (see [7]).

Definition 1. Let N be a compact topological space and $\{U_i\}_{i\in I}$ be an open covering of N such that for any $i\in I$, there exists a homeomorphism h_i of U_i on $\mathbf{B}\times T_i$, where $\mathbf{B}$ is the unit ball of $\mathbf{C}^n$ and T_i is a topological space. We say that these open sets define an atlas of a structure of lamination by complex manifolds of dimension n if the change of charts $h_{ij}=h_j\circ h_i^{-1}$ on their domain of definition are of the form

$$h_{ij}(z,t)=(f_{ij}(z,t),\gamma_{ij}(t)),$$

where f_{ij} depends holomorphically on the variable z and continuously on the variable t. Two atlases on N are equivalent if there union is an atlas. A topological space is called laminated if an equivalence class of atlases is given.

Moreover, let

$$K=\{x\in M\,;M \text{ is not globally minimal at the point } x\}$$

then K is also a compact set of M (as its complement is open by definition) and is laminated by complex manifolds of dimension $n-1$ (see [19] and [12] for a precise study of the structure of CR orbits).

As remarked in [11], in order to prove the global minimality of compact hypersurfaces of $\mathbf{C}^m$, one has to show that there exists no such laminated compact sets in Stein manifolds.

Proposition 1. *Let X be a Stein manifold. Then there is no laminated compact set $Y\subset X$.*

Proof. Let us suppose that there exists such a compact set in X. By embedding X in $\mathbf{C}^n$, we obtain a laminated compact set $Y\subset\mathbf{C}^n$. Let $r>0$ be the infimum of the real numbers $s>0$ such that $Y\subset\overline{B(0,s)}$, where $B(0,s)$ is the ball of center 0 and radius s. Let $z\in Y\cap\partial B(0,r)$, let $\mathbf{C}_z$ be the complex line containing the segment $[0,z]$ and let $\pi\colon\mathbf{C}^n\to\mathbf{C}_z$ be the projection on $\mathbf{C}_z$. Let D_z be a complex manifold contained in Y and passing through the point z. Then the restriction of π_z on D_z is a nonconstant holomorphic function whose modulus has a maximum at the point z, which contradicts the maximum principle. □

2.2. Laminated compact sets of $P_n(\mathbf{C})$

Let Y be a compact subset of $P_n(\mathbf{C})$ laminated by complex manifolds of dimension $n-1$. Then by the definition of Y, $P_n(\mathbf{C})\setminus Y$ is pseudoconvex (as at any point of its boundary there is a piece of complex hypersurface included in the boundary). So, according to [20], [5], [6] and [13], $P_n(\mathbf{C})\setminus Y$ is Stein. As a direct consequence, we obtain the following result.

Proposition 2. *Let Y be a compact set of $P_n(\mathbf{C})$ $(n\geq 2)$ laminated by complex manifolds of dimension $n-1$, and $f\colon Y\to\mathbf{C}$ be a continuous function on Y whose restriction on any complex manifold contained in Y is holomorphic. Then f is constant on Y.*

Proof. In order to prove that $f(Y)$ contains only one point, it is sufficient to prove that its topological boundary $\partial f(Y)$ contains only one point. First, let us remark that for any point $x\in\partial f(Y)$, $f^{-1}(x)$ is a laminated compact set of $P_n(\mathbf{C})$. Indeed, let $y\in Y$ be a point such that $f(y)=x$. From the open mapping theorem, f is constant on the maximal complex manifold passing through the point y. Thus, it is constant on its closure and we obtain that $f^{-1}(x)$ is a laminated compact subset of Y that we will denote by Y_x. Now, let us suppose that $\partial f(Y)$ contains two different points x_1 and x_2. Then the sets Y_{x_1} and Y_{x_2} are two compact sets of $P_n(\mathbf{C})$ laminated by complex manifolds of dimension $n-1$ which do not intersect, as f takes a different value on each one. But, $P_n(\mathbf{C})\setminus Y_{x_1}$ is a pseudoconvex open set of $P_n(\mathbf{C})$, following [20], [5], [6] and [13], $P_n(\mathbf{C})\setminus Y_{x_1}$ is Stein and $Y_{x_2}\subset P_n(\mathbf{C})\setminus Y_{x_1}$ which contradicts Proposition 2. □

Let M be a real hypersurface of a complex manifold X, then for any point $x\in M$, there exists an open connected neighborhood V_x of x such that $V_x\setminus M$ is a disjoint union of two connected open sets (which are called *one-sided neighborhoods of M at the point x*). We will say that $\mathcal{W}$ is a *one-sided neighborhood of M* if for any point $x\in M$, $\mathcal{W}$ contains a one-sided neighborhood of M at x (the side can change). Applying Proposition 2 and the results on propagation of CR extension of [21], [15] and [11], we obtain the following proposition.

Proposition 3. *Let M be a connected C^2 real hypersurface of $P_n(\mathbf{C})$ which divides $P_n(\mathbf{C})$ into two connected open sets. Let K be the set of points $x\in M$ such that M is not globally minimal at the point x. Then continuous CR functions on M are constant on K. Moreover, any continuous CR function defined on $M\setminus K$ extends holomorphically to a one-sided neighborhood of any point of $M\setminus K$.*

Corollary 1. *Let M be a connected C^2 real hypersurface of $P_n(\mathbf{C})$ which divides $P_n(\mathbf{C})$ into two connected open sets Ω_1 and Ω_2. Suppose that there exists a nonconstant continuous CR function f on M and that the Hartogs–Bochner phenomenon is valid for M. Then M admits at most one open CR orbit.*

Proof. Let us suppose the contrary. Let M_1 and M_2 be two open CR orbits of M and let us suppose that $f(K)=0$ and that f is not constant on M_1. Indeed, according to Proposition 3, f is constant on K, so it has to be nonconstant on one of the open CR orbits of M. Let g be the function defined by $g=f$ on M_1 and $g=0$ on $M\setminus M_1$. Then, of course, g is a continuous CR function defined on M. As we have supposed that the Hartogs–Bochner phenomenon is valid for M, let G be the holomorphic extension of g on the side Ω_i of M. Then $G=0$ on all of Ω_i as its boundary value vanishes on an open subset of M. So $g=0$ on all of M which contradicts that $g|_{M_1}=f|_{M_1}$ is not constant. □

2.3. Holomorphic decomposition of CR functions

In this section, we give a proof (communicated to us by C. Laurent-Thiébaut) of the classical decomposition theorem for CR functions as differences of boundary values of holomorphic functions.

Proposition 4. *Let X be a complex manifold and $M \subset X$ be a C^k compact real hypersurface which divides X in two open sets Ω_1 and Ω_2. Let $f: M \to \mathbf{C}$ be a CR function of class $C^{l+\alpha}$ ($0 \leq l < k$, $0 < \alpha < 1$) on M. Let us suppose that $H^{0,1}(X) = 0$. Then there exist two holomorphic functions f_1 and f_2 defined, respectively, on Ω_1 and Ω_2 such that*

(1) $f_i \in C^{l+\alpha}(\bar{\Omega}_i)$, $i=1,2$;

(2) $f = f_1|_M - f_2|_M$.

Proof. Let $T = f[M]^{0,1}$ where $[M]^{0,1}$ is the part of bidegree $(0,1)$ of the integration current over M. As f is a CR function, the current T is $\bar{\partial}$-closed. The hypothesis $H^{0,1}(X) = 0$ implies that there exists a distribution S such that

$$\bar{\partial} S = T \text{ in } X.$$

The support of the current T being included in M, $\bar{\partial} S = 0$ on $X \setminus M$. Thus S defines two holomorphic functions f_1 and f_2 defined, respectively, in Ω_1 and Ω_2. If $z_0 \in M$, let us consider a neighborhood V of z_0 biholomorphic to a ball in $\mathbf{C}^n$. One can then solve the $\bar{\partial}$ problem on V and obtain a distribution S_0 on V such that

$$\bar{\partial} S_0 = T \quad \text{in } V.$$

Thus, on V we have that

$$\bar{\partial}(S - S_0) = 0$$

which implies that $S - S_0$ is a holomorphic function on V and so is C^∞. Thus, the regularity of S in a neighborhood of z_0 is the same as the regularity of S_0 itself whose jump is $C^{l+\alpha}$ as can be checked using the kernels of Henkin in the ball. □

2.4. Complex boundary problem

Let X be a complex Riemannian manifold of dimension n and M be a closed and oriented C^1 submanifold of X of dimension $2p-1$ with $p \geq 1$ (we let $[M]$ be the integration current associated to M). We will call any locally finite linear combination of analytic subsets of $X \setminus M$ with integer coefficients a *holomorphic p-chain.* Of course, holomorphic p-chains define closed currents of bidimension (p,p) of $X \setminus M$. The volume of a holomorphic p-chain $[T] = \sum_i n_i [T_i]$, is the expression

$$\mathrm{Vol}[T] = \sum_j |n_j| \, \mathrm{Vol}\, V_j.$$

where $\mathrm{Vol}\, V_j$ (or $\mathcal{H}_{2p}(V_j)$) is the $2p$-dimensional Hausdorff measure of the analytic set V_j. The volume $\mathrm{Vol}[T]$ is also equal to the mass of the associated current $[T]$. If a holomorphic p-chain is of locally finite mass in X, the associated current in $X\setminus M$ will have an extension to a current of X which is not closed in general in X. The question of finding necessary and sufficient conditions for $[M]$ to be the boundary (in the current sense) of a holomorphic p-chain of $X\setminus M$ of locally finite volume in X is called the *complex boundary problem.*

Two necessary conditions for $[M]$ to have a solution to the boundary problem are that $[M]$ can be decomposed into the sum of two currents of bidimension $(p,p-1)$ and $(p-1,p)$ (such a current will be said *maximally complex*) and that $[M]$ is a closed current. Indeed, suppose that there exists a holomorphic p-chain $[T]$ of $X\setminus M$ such that $[T]$ is of locally finite volume in X and satisfy $[M]=d[T]$. Then necessarily we have

$$\mathrm{d}[M]=\mathrm{d}(\mathrm{d}[T])=0$$

and as $[T]$ is a current of bidimension (p,p),

$$[M]=\mathrm{d}[T]=(\partial+\bar\partial)[T]=\partial[T]+\bar\partial[T]$$

is maximally complex. Of course, the property for the current $[M]$ to be maximally complex is equivalent to the fact that for any point $p\in M$, $\dim_{\mathbf{C}} H_p(M)=p-1$, where $H_p(M)$ is the holomorphic tangent bundle to M at the point p.

In the case of $X=\mathbf{C}^n$, $p\geq 2$ and M is compact, Harvey and Lawson [9] proved that these two conditions are in fact sufficient for the boundary problem for M to have a solution. Then, many authors studied the boundary problem for more general manifolds (see for example [1], [3], [2], [18] and [14]). In this section we would like to mention the following result that will be used in the present article.

Proposition 5. (Chirka [1]) *Let Y be a polynomially convex compact set of $\mathbf{C}^n$ and Γ a closed, oriented and maximally complex C^1 submanifold of $\mathbf{C}^n\setminus Y$ of dimension $2p-1$ $(p\geq 2)$ such that $\Gamma\cup Y$ is a compact set of $\mathbf{C}^n$. Then there exists a unique holomorphic p-chain $[T]$ of $\mathbf{C}^n\setminus(Y\cup M)$, of locally finite volume in $\mathbf{C}^n\setminus Y$, which is the solution to the boundary problem for $[\Gamma]$ in $\mathbf{C}^n\setminus Y$ (i.e. $d[T]=[\Gamma]$).*

3. Proof of the main theorem

Let

$$K=\{x\in M\,;\mathcal{O}_{CR}(x)\text{ is not an open subset of }M\}.$$

3.1. The case when M is globally minimal (i.e. $K=\emptyset$)

According to Proposition 3, if the compact set K is empty, continuous CR maps on M extend holomorphically to a one-sided neighborhood of M. Thus, by deforming M into this one-sided neighborhood and by remarking that, if $x \in M$ is such that CR maps on M extend to its two sides then they are restrictions of holomorphic map in the neighborhood of x, we are reduced to the following result.

Proposition 6. ([17]) *Let M be a connected C^1 hypersurface of $P_n(\mathbf{C})$ ($n \geq 2$) which divides $P_n(\mathbf{C})$ in two connected open sets Ω_1 and Ω_2. Suppose that there exists a nonconstant holomorphic function defined in a connected neighborhood of M. Then holomorphic functions defined in a connected open neighborhood of M extend holomorphically to Ω_1 or to Ω_2.*

We will give a proof of the proposition which is slightly different from the one in [17]. Let V be the connected open neighborhood of M on which there exists a nonconstant holomorphic function $g: V \to \mathbf{C}$.

Lemma 1. *There exists a connected neighborhood $\widetilde{V}$ of M, which is relatively compact in V and two holomorphic functions $f_1 \in \mathcal{O}(\Omega_1 \cup \widetilde{V})$ and $f_2 \in \mathcal{O}(\Omega_2 \cup \widetilde{V})$ such that $g|_{\widetilde{V}} = f_1|_{\widetilde{V}} - f_2|_{\widetilde{V}}$.*

Proof. Let $\widetilde{V}$ be a connected open neighborhood of M which is relatively compact in V. Let ϕ be a smooth function defined on $P_n(\mathbf{C})$ such that $\operatorname{supp} \phi \subset V$ and $\phi|_{\widetilde{V}} \equiv 1$. For any $i \in \{1, 2\}$, let us consider the smooth forms ω_i defined by

$$\omega_i = \begin{cases} \bar{\partial}(g\phi) & \text{on } \Omega_i, \\ 0 & \text{on } P_n(\mathbf{C}) \setminus \Omega_i. \end{cases}$$

Then, for each $i \in \{1, 2\}$, ω_i is a $\bar{\partial}$-closed $(0, 1)$ smooth form whose support is included in $\Omega_i \setminus \widetilde{V}$. As $H^{0,1}(P_n(\mathbf{C})) = 0$, there exists a smooth function u_i defined on $P_n(\mathbf{C})$ such that $\omega_i = \bar{\partial} u_i$. Of course, we have

$$\bar{\partial}(g\phi) = \omega_1 + \omega_2 = \bar{\partial} u_1 + \bar{\partial} u_2 = \bar{\partial}(u_1 + u_2).$$

So the smooth function $u_1 + u_2 - g\phi$ is holomorphic on $P_n(\mathbf{C})$ and thus is constant. Let us denote by c the constant that satisfy

$$c = u_1 + u_2 - g\phi.$$

Then the functions f_1 defined on $\Omega_1 \cup \widetilde{V}$ by $f_1 = g\phi - u_1 + \frac{1}{2}c$ and f_2 defined on $\Omega_2 \cup \widetilde{V}$ by $f_2 = u_2 - g\phi - \frac{1}{2}c$ are holomorphic on there respective domains of definition and satisfy that on $\widetilde{V}$ (as $\phi|_{\widetilde{V}} \equiv 1$) we have $f_1 - f_2 = g$. $\square$

Proof of Proposition 6. As g is supposed nonconstant, one of the two holomorphic functions f_1 and f_2 has also to be nonconstant. Let us suppose, for example, that f_1 is nonconstant. Then, according to [20], [5], [6] and [13], the envelope of holomorphy W of $\Omega_1 \cup \widetilde{V}$ is Stein. So the domain $\Omega_1 \cup \widetilde{V}$ embeds in its envelope and in particular Ω_1 can be seen as a bounded domain of a Stein space W and we are reduced to the classical Hartogs extension theorem. $\square$

3.2. The case when M is not globally minimal (i.e. $K \neq \emptyset$)

First, let us remark that Theorem 1 is trivial in the case when there are no nonconstant C^1 CR functions on M. So in all the following we will always assume that $g: M \to \mathbf{C}$ is a nonconstant C^1 CR function.

Lemma 2. *The compact set K must verify the following properties:*
(1) *K is a compact set laminated by complex manifolds of dimension $n-1$;*
(2) *the CR function g is constant on K (we can suppose that $g(K)=0$);*
(3) *K is of null $(2n-1)$-dimensional Hausdorff measure;*
(4) *the open set $U = P_n(\mathbf{C}) \setminus K$ is Stein.*

Proof. The two first points are a consequence of Proposition 3. According to Proposition 4, $g = f_1 - f_2$ where $f_i \in \mathcal{O}(\Omega_i) \cap C^0(\overline{\Omega}_i)$. As g is nonconstant, one of the two functions f_i has also to be nonconstant. But, as they are constant on K, the set K has to be of null measure in M (which proves the third point). Finally, as the compact set K is supposed nonempty,

$$U = P_n(\mathbf{C}) \setminus K$$

is a pseudoconvex open subset of $P_n(\mathbf{C})$. According to [20], [5], [6] and [13], U is Stein. □

3.2.1. *Semi-local solution to the boundary problem.* According to Proposition 4, $g = f_1|_M - f_2|_M$, where $f_i \in \mathcal{O}(\Omega_i) \cap C^0(\Omega_i)$. As g is nonconstant, one of these two functions is also nonconstant (let suppose for example that f_1 is nonconstant, that $f_1(K) = \{0\}$ and that the orientation of M has been chosen such that M is the oriented boundary of Ω_1). Let us prove the following proposition which implies Theorem 1.

Proposition 7. *Under the previous hypothesis, any continuous CR function $f: M \setminus K \to \mathbf{C}$ extends holomorphically to $\Omega_1 \setminus \{z; f_1(z) = 0\}$. Moreover, if f is bounded, then its extension is also bounded and according to the Riemann extension theorem, it extends holomorphically to all of Ω_1.*

According to Proposition 3, up to deforming M, we can always assume that f_1 and f are smooth on $M \setminus K$. We will prove that the graph of the restrictions of f over the level sets $\{z; f_1(z) = c\}$ admits solutions to the boundary problem. More precisely, let us consider the graph of the map (f_1, f) over the set $M \setminus \{z; f_1(z) = 0\} \subset M \setminus K$:

$$\widetilde{\Gamma}_{f_1,f} = \{(w, y, z) \in (\mathbf{C} \setminus \{0\}) \times \mathbf{C} \times P_n(\mathbf{C});\, z \in M,\ f_1(z) \neq 0,\ w = f_1(z),\ y = f(z)\}.$$

Lemma 3. *There exists a holomorphic n-chain $[\widetilde{T}]$ of $((\mathbf{C}\setminus\{0\})\times\mathbf{C}\times P_n(\mathbf{C}))\setminus\widetilde{\Gamma}_{f_1,f}$, of locally finite mass in $(\mathbf{C}\setminus\{0\})\times\mathbf{C}\times P_n(\mathbf{C})$, which is a solution to the boundary problem for $[\widetilde{\Gamma}_{f_1,f}]$. (Moreover, for any compact $R\subset\mathbf{C}\setminus\{0\}$, the set $\widetilde{T}\cap(R\times\mathbf{C}\times P_n(\mathbf{C}))$ is compact in $(\mathbf{C}\setminus\{0\})\times\mathbf{C}\times U$.)*

Proof. We recall that $U=P_n(\mathbf{C})\setminus K$ is Stein. Let $D(0,\varepsilon)$ be the closed disc of center 0 and radius ε, $C(0,\varepsilon)$ its boundary and $\pi:\mathbf{C}\times\mathbf{C}\times U\to\mathbf{C}$ be the projection on the first member. For any $\varepsilon>0$, $\widetilde{\Gamma}_{f_1,f}\cap(C(0,\varepsilon)\times\mathbf{C}\times U)$ is bounded. Thus, there exists a holomorphically convex compact set B_ε in $\mathbf{C}\times U$ such that $[\widetilde{\Gamma}_{f_1,f}\cap((\mathbf{C}\setminus D(0,\varepsilon))\times\mathbf{C}\times U)]\cup(D(0,\varepsilon)\times B_\varepsilon)$ is a compact subset of $\mathbf{C}\times\mathbf{C}\times U$. According to Proposition 5 with $Y=D(0,\varepsilon)\times B_\varepsilon$, the boundary problem for $[\widetilde{\Gamma}_{f_1,f}]$ admits a unique solution in $(\mathbf{C}\times\mathbf{C}\times U)\setminus(Y\cup\widetilde{\Gamma}_f)$. By uniqueness of the solution and by letting ε tend to zero, we obtain that the boundary problem for $[\widetilde{\Gamma}_{f_1,f}]$ has a unique solution $[\widetilde{T}]$ to the boundary problem in $(\mathbf{C}\setminus\{0\})\times\mathbf{C}\times U$. As $[\widetilde{\Gamma}_{f_1,f}]$ has a solution to the boundary problem in $(\mathbf{C}\setminus\{0\})\times\mathbf{C}\times U$ and as $U\subset P_n(\mathbf{C})$ we obtained in fact that $[\widetilde{T}]$ is a solution to the boundary problem for $[\widetilde{\Gamma}_{f_1,f}]$ in $(\mathbf{C}\setminus\{0\})\times\mathbf{C}\times P_n(\mathbf{C})$. □

For any $c\in\mathbf{C}\setminus\{0\}$, let $\gamma_c=(\{c\}\times\mathbf{C}\times P_n(\mathbf{C}))\cap\widetilde{\Gamma}_{f_1,f}$ be the graph of f over the level set $\{z;f_1(z)=c\}$. According to Sard's theorem, for almost all $c\in\mathbf{C}\setminus\{0\}$, γ_c is a smooth real submanifold of dimension $2n-3$ and the intersection current $[\gamma_c]$ obtained by intersecting $[\Gamma_{f_1,f}]$ with the fiber $\{c\}\times\mathbf{C}\times P_n(\mathbf{C})$ is well defined and corresponds to the integration current over γ_c. Moreover, for almost all $c\in\mathbf{C}\setminus\{0\}$, the boundary in the current sense of the intersection current (denoted $[S_c]$) obtained by slicing the current $[\widetilde{T}]$ by the fiber $\{c\}\times\mathbf{C}\times P_n(\mathbf{C})$ is equal to the intersection current of $[\widetilde{\Gamma}_{f_1,f}]$ by this same fiber. So we obtain:

Lemma 4. *For almost all $c\in\mathbf{C}\setminus\{0\}$, there exists a holomorphic $(n-1)$-chain $[S_c]$ of $(\{c\}\times\mathbf{C}\times U)\setminus\gamma_c$, of finite mass in $\{c\}\times\mathbf{C}\times U$, which is a solution to the boundary problem for $[\gamma_c]$ (i.e. $d[S_c]=[\gamma_c]$).*

Proof of Proposition 7. By hypothesis, $M\setminus K$ is connected (and not empty because f_1 is supposed nonconstant on M). So $\widetilde{\Gamma}_{f_1,f}$ is also connected. Thus, the holomorphic n-chain $[\widetilde{T}]$ is irreducible and with multiplicity ±1. For almost all $c\in\mathbf{C}\setminus\{0\}$, the intersection current $[\gamma_c]$ of $[\widetilde{\Gamma}_{f_1,f}]$ by the fiber $\{c\}\times\mathbf{C}\times P_n(\mathbf{C})$ is well defined. According to Lemma 4, for almost all $c\in\mathbf{C}\setminus\{0\}$, we have that $[\gamma_c]=d[S_c]$, where $[S_c]$ is the intersection current of $[\widetilde{T}]$ by the fiber $\{c\}\times\mathbf{C}\times P_n(\mathbf{C})$. Let $\Pi_3:\mathbf{C}\times\mathbf{C}\times P_n(\mathbf{C})\to P_n(\mathbf{C})$ be the projection on the last member. For almost all $c\in\mathbf{C}\setminus\{0\}$ the integration currents $[\gamma_c^1]$ and $[S_c^1]$, respectively, over the level sets $\{z\in M;f_1(z)=c\}$ and $\{z;f_1(z)=c\}$ are well defined and satisfy $[\gamma_c^1]=d[S_c^1]$. Then, we have for almost all $c\in\mathbf{C}\setminus\{0\}$,

$$d\Pi_{3*}([S_c])=\Pi_{3*}(d[S_c])=\Pi_{3*}([\gamma_c])=[\gamma_c^1]=d[S_c^1].$$

By the uniqueness of the solution to the boundary problem in the Stein manifold $\{c\}\times\mathbf{C}\times U$ we have that

$$\Pi_{3*}([S_c])=[S_c^1].$$

As the current $[\widetilde{T}]$ is of multiplicity ± 1, the currents $[S_c]$ have also to be of multiplicity ± 1. But, as by construction, the current $[S_c^1]$ is of multiplicity $+1$ and satisfy $\Pi_{3*}([S_c])=[S_c^1]$, the current $[S_c]$ has also to be of multiplicity $+1$. This proves that f extends holomorphically on almost all the level lines S_c^1 of f_1 (i.e. the Riemann surface S_c is the graph of a holomorphic function defined on S_c^1). Thus, Π_3 defines a proper and one-to-one projection of $\widetilde{T}$ on $\Omega_1\setminus\{z:f_1(z)=0\}$. Let Π_3^{-1} be the inverse map of Π_3 and $\pi_2:\mathbf{C}\times\mathbf{C}\times U\to\mathbf{C}$ the projection defined by $\pi_2(z_1,z_2,z_3)=z_2$. Then the holomorphic function $F=\pi_2\circ\Pi_3^{-1}$ is defined on $\Omega_1\setminus\{z:f_1(z)=0\}$ and is a holomorphic extension of f on $\Omega_1\setminus\{z:f_1(z)=0\}$.

In the case when f is bounded, by construction of the solution to the boundary problem the extension has also to be bounded. □

The only remaining point in the proof of Theorem 1 is the regularity of the extension. According to [9], Theorem 5.2, p. 249, the regularity up to the boundary of the holomorphic extension is the same as the one of the considered CR function on M.

4. Related problems

In the case when M admits a nonconstant C^1 CR function, the hypothesis that M has at most one open CR orbit is necessary. In order to give a complete answer to the conjecture one have to answer the following question.

Problem 1. Let M be a connected C^2 hypersurface of $P_n(\mathbf{C})$ which divides $P_n(\mathbf{C})$ in two connected open sets Ω_1 and Ω_2. Suppose that M admits a nonconstant C^1 CR function. Can M have two distinct open CR orbits?

We do not know if Theorem 1 is still valid if we assume less regularity for M or for the CR functions. For example, in the case when M is Lipschitz and f is in the Sobolev space $W^{-1/2}(M)$, a counterexample is given by Henkin and Iordan in [10]. Nevertheless, by analogy with the extension result they obtain, one might expect the following problem to have a positive answer.

Problem 2. (Henkin) Let $\Omega\subset P_n(\mathbf{C})$ $(n\geq 2)$ be a domain with Lipschitz boundary $\partial\Omega$ which admits a nonconstant holomorphic function. Let f be a CR function which is in the Sobolev space $W^{1/2}(\partial\Omega)$. Does f admit a holomorphic extension in $\mathcal{O}(\Omega)\cap W^1(\Omega)$?

As we have seen, in the case when M is of class C^2, the main difficulty is the possible existence of laminated compact subset K of M. Thus the following problems become natural.

Problem 3. Let M be a connected C^2 hypersurface of $P_n(\mathbf{C})$ $(n \geq 2)$ which divides $P_n(\mathbf{C})$ in two connected open sets Ω_1 and Ω_2. Then

(1) does there exist $i \in \{1, 2\}$ such that smooth CR maps $f: M \to P_1(\mathbf{C})$ extend meromorphically to Ω_i?

(2) is M globally minimal?

(3) does there always exist a nonconstant C^1 CR function on M?

Let $U \subset P_n(\mathbf{C})$ be an open set. If U contains a laminated compact set K then holomorphic functions on U have to be constant and meromorphic functions have to be rational. As we have proved, continuous CR functions on K are also constant. Thus one could expect the following problem to have a positive answer.

Problem 4. Let $K \subset P_2(\mathbf{C})$ be a compact set C^2 laminated by Riemann surfaces. Let f be a C^2 CR map from K to $P_1(\mathbf{C})$ (i.e. for any analytic disc $\Delta \subset K$, $f|_\Delta$ is a holomorphic map). Does there exist a rational map $Q: P_2(\mathbf{C}) \to P_1(\mathbf{C})$ such that $Q|_K = f$?

Of course, in the case when it is known that there exist no non-trivial laminated compact subset of $P_2(\mathbf{C})$, Problems 1, 3 and 4 are obvious.

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Frédéric Sarkis
UFR de Mathématiques
Université Lille I
FR-59655 Villeneuve d'Ascq Cedex
France
email: sarkis@agat.univ-lille1.fr