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An inversion formula for the attenuated X-ray transformation

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Abstract. The problem of inversion of the attenuated X-ray transformation is solved by an explicit formula. Several subsequent results are also given.

1. Introduction

Consider some real-valued sufficiently regular function a on $\mathbf{R}^d$ sufficiently rapidly vanishing at infinity. We say also that a is an attenuation coefficient. Consider the attenuated X-ray transformation P_a defined by the formula

$$P_a f(x,\theta)=\int_{\mathbf{R}} \exp(-Da(x+s\theta,\theta))f(x+s\theta)\,ds, \quad x\in\mathbf{R}^d,\ \theta\in\mathbf{S}^{d-1}, \tag{1.1}$$

for any real-valued sufficiently regular function f on $\mathbf{R}^d$ sufficiently rapidly vanishing at infinity, where Da is the divergent beam transform of a, i.e.

$$Da(x,\theta)=\int_0^{+\infty} a(x+s\theta)\,ds, \quad x\in\mathbf{R}^d,\ \theta\in\mathbf{S}^{d-1}. \tag{1.2}$$

In addition,

$$P_a f(x,\theta)=P_a f(\pi_\theta x,\theta), \quad x\in\mathbf{R}^d,\ \theta\in\mathbf{S}^{d-1}, \tag{1.3}$$

where

$$\begin{gathered}\pi_\theta \text{ is the orthogonal projector of } \mathbf{R}^d \text{ on the subspace}\\ X_\theta=\{x\in\mathbf{R}^d \mid x\theta=0\}.\end{gathered} \tag{1.4}$$

Due to (1.3), $P_a f$ on $\mathbf{R}^d\times\mathbf{S}^{d-1}$ is uniquely determined by $P_a f$ on $T\mathbf{S}^{d-1}$, where

$$T\mathbf{S}^{d-1}=\{(x,\theta)\mid x\in\mathbf{R}^d,\ \theta\in\mathbf{S}^{d-1} \text{ and } x\theta=0\}. \tag{1.5}$$

We interpret $T\mathbf{S}^{d-1}$ as the set of all oriented straight lines in $\mathbf{R}^d$. If $\gamma=(x,\theta)\in T\mathbf{S}^{d-1}$, then $\gamma=\{y\in\mathbf{R}^d \,|\, y=x+t\theta \text{ and } t\in\mathbf{R}\}$ (up to orientation) and θ gives the orientation of γ.

As a basic problem, we consider the problem of finding $f|_Y$, where Y is a two-dimensional plane in $\mathbf{R}^d$, $d\geq 2$, from $P_a f|_{T\mathbf{S}^1(Y)}$, where $T\mathbf{S}^1(Y)$ is the set of all oriented straight lines lying in Y, under the condition that $a|_Y$ is known. For this problem the case when $d\geq 3$ is reduced to the case when $d=2$.

This problem comes from the emission tomography (see [Na]). In the emission tomography setting, f is the density of emitters, a is the linear attenuation coefficient of the medium, $P_a f(\gamma)$, $\gamma=(x,\theta)\in T\mathbf{S}^{d-1}$, is the measured emission intensity in the direction θ at a detector at $+\infty$ on γ (at a detector on the connected component of $\gamma\backslash(\operatorname{supp} f\cup\operatorname{supp} a)$ containing $+\infty$ on γ for compactly supported f and a).

We carry out the basic considerations of Section 2 assuming that

$$a, f\in C^{\alpha,1+\varepsilon}(\mathbf{R}^d,\mathbf{R}) \quad \text{for some } \alpha\in]0,1[\text{ and } \varepsilon>0, \tag{1.6}$$

where

$$C^{\mu,\sigma}(\mathbf{R}^d,\mathbf{C})=\{u\in C(\mathbf{R}^d,\mathbf{C}) \mid \|u\|_{\mu,\sigma}<+\infty\}, \tag{1.7}$$

$$\|u\|_{\mu,\sigma}=\max(\|u\|_{0,\sigma},\|u\|'_{\mu,\sigma}), \qquad \mu\neq 0, \tag{1.8a}$$

$$\|u\|_{0,\sigma}=\sup_{x\in\mathbf{R}^d}(1+|x|)^\sigma|u(x)|, \tag{1.8b}$$

$$\|u\|'_{\mu,\sigma}=\sup_{\substack{x,y\in\mathbf{R}^d\\ 0<|y|\leq 1}}(1+|x|)^\sigma\frac{|u(x+y)-u(x)|}{|y|^\mu}, \quad \mu\neq 0, \tag{1.8c}$$

where $\mu\in[0,1[$ and $\sigma\geq 0$.

In addition, we extend the final results of Section 2 to the case when

$$a, f\in L^{\infty,1+\varepsilon}(\mathbf{R}^d,\mathbf{R}) \quad \text{for some } \varepsilon>0, \tag{1.9}$$

where

$$\begin{aligned} &L^{\infty,\sigma}(\mathbf{R}^d,\mathbf{C})=\{u\in L^\infty(\mathbf{R}^d,\mathbf{C}) \mid \|u\|_{0,\sigma}<+\infty\},\\ &\|u\|_{0,\sigma}=\operatorname{ess\,sup}_{x\in\mathbf{R}^d}(1+|x|)^\sigma|u(x)|, \qquad \sigma\geq 0. \end{aligned} \tag{1.10}$$

Our main result is Theorem 2.1 stating that, under the assumptions (1.6) for $d=2$, and under the condition that a is known, the transform $P_a f$ on $T\mathbf{S}^1$ uniquely determines f on $\mathbf{R}^2$ by the explicit formulas (2.12). We obtain Theorem 2.1 using techniques of [MZ], [FN] and [N2]. Theorem 2.1 implies Corollary 2.5 stating that, under the assumptions (1.6) for $d \geq 2$, and under the condition that a is known, the transform $P_a f$ on $\mathcal{M}(S)$ uniquely determines f on $\mathbf{R}^d$ by the scheme (2.68) (where $\mathcal{M}(S)$, defined by (2.67) for some great circle S in $\mathbf{S}^{d-1}$, is a d-dimensional submanifold of $T\mathbf{S}^{d-1}$ for $d \geq 2$ and $\mathcal{M}(S) = T\mathbf{S}^1$ for $d=2$). In addition, Theorem 2.1 and Corollary 2.5 remain valid under the assumptions (1.9) in place of (1.6) according to our Remark 2.6. Note also that, under the assumptions (1.9) and $d=2$, the formula (2.12) splits up into (2.20a–c) and (2.19), (2.20b–c). Actually, the formulas (2.20a–c) are our inversion formula for the transformation P_a for $d=2$, the formulas (2.19) and (2.20b–c) are new necessary range conditions for P_a for $d=2$. In Section 3 we give several subsequent results concerning finding f from $P_a f$, under the condition that a is known, for $d \geq 2$, and concerning the range characterization for P_a for $d=2$.

To our knowledge, the aforementioned results of Theorem 2.1 and Corollary 2.5 are completely new even on the level of pure uniqueness (without an inversion method) under the assumptions that a and f are real-valued functions of the Schwartz class on $\mathbf{R}^d$ in place of (1.6). The work [ABK] comes rather close to the proof of the uniqueness results (without the inversion formulas (2.12) and (2.68)) contained in our Theorem 2.1 and Corollary 2.5, at least, under the assumptions that

$$a, f \in C_0^\infty(\mathbf{R}^d, \mathbf{R}) \tag{1.11}$$

(the space of real-valued infinitely smooth compactly supported functions on $\mathbf{R}^d$) in place of (1.6). In addition, the question whether the two-dimensional transformation P_a is injective (even being restricted to $C_0^\infty(\mathbf{R}^2, \mathbf{R})$ and for $a \in C_0^\infty(\mathbf{R}^2, \mathbf{R})$) was known as an open problem for a long time see, e.g., [Na].

Concerning known uniqueness, nonuniqueness and other results for generalized Radon transformations, of which P_a is an example, see [M], [BG], [MQ], [Na], [F], [BQ], [S], [B], [KLM], [ABK] and references given there. To our knowledge, the strongest local uniqueness result given in the literature for the two-dimensional transformation P_a was obtained in [F]. (Local uniqueness means that every point x has a neighbourhood U_x so that no non-trivial f supported in U_x lies in the kernel of P_a.) It is indicated also in [F] that $P_a f \equiv 0$ on $T\mathbf{S}^{d-1}$ implies $f \equiv 0$ on $\mathbf{R}^d$ for $d \geq 3$, under the assumptions (1.11). (Note, however, that the problem of finding f on $\mathbf{R}^d$ from $P_a f$ on $T\mathbf{S}^{d-1}$, under the condition that a is known, is strongly overdetermined for $d \geq 3$ (i.e. $\dim T\mathbf{S}^{d-1} = 2d-2 > d$ for $d \geq 3$).) In [F], the proofs of the aforementioned results of [F] contain no inversion methods for the transformation P_a for known a. In [ABK] an inversion method is developed for the two-dimensional attenuated X-ray transformation P_a for the case when a is a real-valued twice differentiable compactly supported function on $\mathbf{R}^2$ and P_a is restricted to the space of real-valued sufficiently regular compactly supported functions f on $\mathbf{R}^2$. The inversion method of [ABK] is based on a Cauchy type formula for generalized A-analytic l^2-valued functions. Our inversion method based on the explicit formula (2.12) is drastically simpler than the inversion method of [ABK].

In the literature (see [TM], [Na], [AK], [P]) the theory of the transformation P_a defined by (1.1) is well developed for the case when f is supported in Ω and a is constant in Ω, where Ω is an open convex bounded domain in $\mathbf{R}^d$. For this case our Theorem 2.1 complemented by Remark 2.6 gives a new inversion formula.

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2. Inverse scattering for the attenuated X-ray equation

Consider the equation

$$\theta\partial_x\psi(x,\theta)+a(x)\psi(x,\theta)=f(x), \quad x\in\mathbf{R}^d,\ \theta\in\mathbf{S}^{d-1}, \tag{2.1}$$

where θ is a spectral parameter, $\partial_x=(\partial/\partial x_1,\dots,\partial/\partial x_d)$, $\theta\partial_x=\sum_{j=1}^{d}\theta_j\partial/\partial x_j$ and where a and f satisfy (1.6).

For any $\theta\in\mathbf{S}^{d-1}$ consider the real-valued continuous solution $\psi^+(\cdot,\theta)$ of (2.1) specified by

$$\lim_{s\to-\infty}\psi^+(x+s\theta,\theta)=0 \quad \text{for } x\in\mathbf{R}^d. \tag{2.2}$$

For any $\theta\in\mathbf{S}^{d-1}$ such a solution exists and is unique. In the emission tomography setting, $\psi^+(x,\theta)$ is the emission intensity through the point x in the direction θ, where a is the linear attenuation coefficient of the medium and f is the density of emitters. We say that equation (2.1) is the attenuated X-ray equation.

The following formula holds:

$$\psi^+(x,\theta)=\exp(-Da(x,-\theta))\int_{-\infty}^{0}\exp(Da(x+t\theta,-\theta))f(x+t\theta)\,dt, \tag{2.3}$$

$x\in\mathbf{R}^d$, $\theta\in\mathbf{S}^{d-1}$, where Da is defined by (1.2).

Note that for any $\theta\in\mathbf{S}^{d-1}$ the factor

$$\varphi^+(\cdot,\theta)=\exp(-Da(\cdot,-\theta)) \tag{2.4}$$

of the right-hand side of (2.3) is the real-valued continuous solution of the equation

$$\theta\partial_x\varphi(x,\theta)+a(x)\varphi(x,\theta)=0,\quad x\in\mathbf{R}^d,\ \theta\in\mathbf{S}^{d-1}, \tag{2.5}$$

specified by

$$\lim_{s\to-\infty}\varphi^+(x+s\theta,\theta)=1\quad \text{for } x\in\mathbf{R}^d. \tag{2.6}$$

The following formula holds:

$$P_af(x,\theta)=\lim_{s\to+\infty}\psi^+(x+s\theta,\theta),\quad x\in\mathbf{R}^d,\ \theta\in\mathbf{S}^{d-1}, \tag{2.7}$$

where P_af is defined by (1.1).

We say that P_af is the relative scattering matrix for the equation (2.1), where a is considered as a background parameter and f is considered as a perturbation.

Under the assumptions (1.6), the functions φ^+, ψ^+ and P_af have, in particular, the properties

$$\varphi^+,\psi^+,P_af\in C(\mathbf{R}^d\times\mathbf{S}^{d-1},\mathbf{R}), \tag{2.8}$$

$$|\varphi^+(x,\theta)-1|\le\exp(c_1(\varepsilon,\theta,x)\|a\|_{0,1+\varepsilon})-1, \tag{2.9a}$$

$$|\varphi^+(x+y,\theta)-\varphi^+(x,\theta)|\le(\exp(2c_1(\varepsilon,\theta,x)\|a\|_{\alpha,1+\varepsilon})-1)|y|^\alpha, \tag{2.9b}$$

$$\|\varphi^+(\cdot,\theta)\|_{\alpha,0}\le\exp\left(\frac{2^{5/2}\|a\|_{\alpha,1+\varepsilon}}{\varepsilon}\right), \tag{2.9c}$$

$$|\psi^+(x,\theta)|\le\exp\left(\frac{2^{5/2}\|a\|_{0,1+\varepsilon}}{\varepsilon}\right)c_1(\varepsilon,\theta,x)\|f\|_{0,1+\varepsilon}, \tag{2.10a}$$

$$|\psi^+(x+y,\theta)-\psi^+(x,\theta)|\le4\exp\left(\frac{2^{7/2}\|a\|_{\alpha,1+\varepsilon}}{\varepsilon}\right)c_1(\varepsilon,\theta,x)\|f\|_{\alpha,1+\varepsilon}|y|^\alpha, \tag{2.10b}$$

$$|P_af(x,\theta)|\le\exp\left(\frac{2^{5/2}\|a\|_{0,1+\varepsilon}}{\varepsilon}\right)c_3(\varepsilon,|\pi_\theta x|)\|f\|_{0,1+\varepsilon}, \tag{2.11a}$$

$$|P_af(x+y,\theta)-P_af(x,\theta)|\le4\exp\left(\frac{2^{7/2}\|a\|_{\alpha,1+\varepsilon}}{\varepsilon}\right)c_3(\varepsilon,|\pi_\theta x|)\|f\|_{\alpha,1+\varepsilon}|y|^\alpha, \tag{2.11b}$$

where $c_1(\varepsilon,\theta,x)$ is given by (A.8), $c_3(\varepsilon,s)$ is given by (A.13), π_θ is defined by (1.4), $x,y\in\mathbf{R}^d$, $|y|\le1$ and $\theta\in\mathbf{S}^{d-1}$. We obtain (2.8)–(2.11) using Lemmas A.1, A.2, A.4 and the formulas (2.3), (2.4) and (2.7).

For $d \geq 2$ we consider the problem of finding $f|_Y$, where Y is a two-dimensional plane in $\mathbf{R}^d$, from $P_a f|_{T\mathbf{S}^1(Y)}$, where $T\mathbf{S}^1(Y)$ is the set of all oriented straight lines lying in Y, under the condition that $a|_Y$ is known. We say that this problem is an inverse scattering problem for the equation (2.1). For this problem the case when $d \geq 3$ is reduced to the case when $d=2$. Further, for the latter case we obtain the following result.

Theorem 2.1. *Suppose that a and f satisfy* (1.6) *for $d=2$. Then, under the condition that a is known, $P_a f$ on $T\mathbf{S}^1$ uniquely determines f on $\mathbf{R}^2$ by the formulas*

$$f(x) = -\frac{1}{4\pi}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)\int_{\mathbf{S}^1} \varphi(x,\theta)(\theta_1 + i\theta_2)\, d\theta, \tag{2.12a}$$

$$\varphi(x,\theta) = \exp(-D_{-\theta}a(x))m(x\theta^\perp,\theta), \tag{2.12b}$$

$$\begin{aligned} m(s,\theta) = {} & \frac{1}{2i}\exp\left(-\frac{H_+ P_\theta^\perp a(s)}{2i}\right) H_+ \exp\left(\frac{H_- P_\theta^\perp a}{2i}\right) P_{a,\theta}^\perp f(s) \\ & + \frac{1}{2i}\exp\left(\frac{H_- P_\theta^\perp a(s)}{2i}\right) H_- \exp\left(-\frac{H_+ P_\theta^\perp a}{2i}\right) P_{a,\theta}^\perp f(s), \end{aligned} \tag{2.12c}$$

where D_θ, $P_\theta^\perp$, $P_{a,\theta}^\perp$, $H_\pm$ and $\exp(\pm(2i)^{-1}H_\mp P_\theta^\perp a)$ are the operators such that

$$D_\theta u(x) = \int_0^{+\infty} u(x+t\theta)\, dt, \tag{2.13}$$

$$P_\theta^\perp u(s) = \int_{\mathbf{R}} u(s\theta^\perp + t\theta)\, dt, \tag{2.14}$$

$$\begin{aligned} P_{a,\theta}^\perp u(s) &= \int_{\mathbf{R}} \exp(-D_\theta a(s\theta^\perp + t\theta))u(s\theta^\perp + t\theta)\, dt \\ &= P_a u(s\theta^\perp,\theta), \end{aligned} \tag{2.15}$$

$$H_\pm v(s) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{v(t)}{s \pm i0 - t}\, dt, \tag{2.16}$$

$$\exp\left(\pm\frac{H_\mp P_\theta^\perp a}{2i}\right) v(s) = \exp\left(\pm\frac{H_\mp P_\theta^\perp a(s)}{2i}\right) v(s), \tag{2.17}$$

where u and v are test functions, $x=(x_1,x_2)\in\mathbf{R}^2$, $\theta=(\theta_1,\theta_2)\in\mathbf{S}^1$, $\theta^\perp=(-\theta_2,\theta_1)$, $s\in\mathbf{R}$ and $d\theta$ is the standard element of arc length on $\mathbf{S}^1$.

Remark 2.2. If $a \equiv 0$ in (2.12), then the formulas (2.12) turn into the formulas (4.6) of [N1] which are similar to the formula (1.12) of [FN].

Remark 2.3. Under the conditions (1.6) and $d=2$, the formulas (2.12) imply that

$$\operatorname{Re}\varphi(x,\theta) = 0, \quad \operatorname{Re} m(s,\theta) = 0, \tag{2.18}$$

$$\frac{\partial}{\partial x_1}\int_{\mathbf{S}^1}\operatorname{Im}\varphi(x,\theta)\theta_1 d\theta+\frac{\partial}{\partial x_2}\int_{\mathbf{S}^1}\operatorname{Im}\varphi(x,\theta)\theta_2\,d\theta=0, \tag{2.19}$$

$$f(x)=-\frac{1}{4\pi}\left(-\frac{\partial}{\partial x_1}\int_{\mathbf{S}^1}\operatorname{Im}\varphi(x,\theta)\theta_2\,d\theta+\frac{\partial}{\partial x_2}\int_{\mathbf{S}^1}\operatorname{Im}\varphi(x,\theta)\theta_1\,d\theta\right), \tag{2.20a}$$

$$\operatorname{Im}\varphi(x,\theta)=\exp(-D_{-\theta}a(x))\operatorname{Im}m(x\theta^\perp,\theta), \tag{2.20b}$$

$$\operatorname{Im}m(s,\theta)=-\operatorname{Re}\left(\exp\left(-\frac{H_+P_\theta^\perp a(s)}{2i}\right)\right. \tag{2.20c}$$
$$\left.\times(c_+H_++c_-H_-)\exp\left(\frac{H_-P_\theta^\perp a}{2i}\right)P_{a,\theta}^\perp f(s)\right)$$

for any real c_+ and c_- such that $c_++c_-=1$, where $x\in\mathbf{R}^2$, $\theta\in\mathbf{S}^1$ and $s\in\mathbf{R}$.

Remark 2.4. Under the conditions (1.6) and $d=2$, the formulas (2.12) and (2.18)–(2.20) are valid pointwise.

Proof of Theorem 2.1. Consider the equation

$$\theta\partial_x\psi(x,\theta)+a(x)\psi(x,\theta)=f(x),\quad x\in\mathbf{R}^2,\ \theta\in\Sigma, \tag{2.21}$$

where

$$\Sigma=\{\theta\in\mathbf{C}^2\mid\theta^2=\theta_1^2+\theta_2^2=1\}. \tag{2.22}$$

The equation (2.21) is the equation (2.1) for $d=2$ with complexified θ.

For any $\theta\in\Sigma\backslash\mathbf{S}^1$ we consider the complex-valued continuous solution $\psi(\cdot,\theta)$ of (2.21) specified by

$$\psi(x,\theta)\to 0,\quad \text{as } |x|\to\infty. \tag{2.23}$$

For any $\theta\in\Sigma\backslash\mathbf{S}^1$ such a solution exists and is unique.

The following formula holds:

$$\psi(x,\theta)=\int_{\mathbf{R}^2}R(x,y,\theta)f(y)\,dy,\quad x\in\mathbf{R}^2,\ \theta\in\Sigma\backslash\mathbf{S}^1, \tag{2.24}$$

where

$$R(x,y,\theta)=\exp(-G_\theta a(x))G(x-y,\theta)\exp(G_\theta a(y)), \tag{2.25}$$

$$G(x,\theta)=\frac{\operatorname{sgn}(\operatorname{Re}\theta_1\operatorname{Im}\theta_2-\operatorname{Re}\theta_2\operatorname{Im}\theta_1)}{-2\pi i(\theta_2x_1-\theta_1x_2)}, \tag{2.26}$$

(2.27a) G_θ is the convolution operator with the function $G(\cdot,\theta)$,

$$G_\theta a(x)=\int_{\mathbf{R}^2}G(x-y,\theta)a(y)\,dy. \tag{2.27b}$$

Note that for any $\theta\in\Sigma\backslash\mathbf{S}^1$ the factor

$$\varphi(\,\cdot\,,\theta)=\exp(-G_\theta a(\,\cdot\,)) \tag{2.28}$$

on the right-hand side of (2.25) is the complex-valued continuous solution of the equation

$$\theta\partial_x\varphi(x,\theta)+a(x)\varphi(x,\theta)=0, \quad x\in\mathbf{R}^2,\ \theta\in\Sigma, \tag{2.29}$$

specified by

$$\varphi(x,\theta)\to 1, \quad \text{as } |x|\to\infty. \tag{2.30}$$

Note also that for any $\theta\in\Sigma\backslash\mathbf{S}^1$ the function $G(\,\cdot\,,\theta)$ is the complex-valued solution continuous outside of zero of the equation

$$\theta\partial_x G(x,\theta)=\delta(x), \quad x\in\mathbf{R}^2,\ \theta\in\Sigma, \tag{2.31}$$

specified by

$$G(x,\theta)\to 0, \quad \text{as } |x|\to\infty. \tag{2.32}$$

The function $\psi(\,\cdot\,,\theta)$ is not defined for $\theta\in\mathbf{S}^1$, in general. For $\theta\in\mathbf{S}^1$ we consider the functions $\psi_\pm(\,\cdot\,,\theta)$ defined by the formula

$$\psi_\pm(x,\theta)=\lim_{\tau\to 0+}\psi(x,\omega(\pm\tau)) \quad \text{for any } x\in\mathbf{R}^2, \tag{2.33}$$

where

(2.34)
$$\omega(\tau)=\theta\sqrt{1+\tau^2}+i\tau\theta^\perp, \quad \tau\in\mathbf{R},\ \theta=(\theta_1,\theta_2)\in\mathbf{S}^1,\ \sqrt{1+\tau^2}>0,\ \theta^\perp=(-\theta_2,\theta_1).$$

Note that

$$\omega(\tau)\in\Sigma\backslash\mathbf{S}^1 \quad \text{for } \tau\in\mathbf{R}\backslash\{0\}. \tag{2.35}$$

The following formula holds:

$$\psi_\pm(x,\theta)=\int_{\mathbf{R}^2}R_\pm(x,y,\theta)f(y)\,dy, \quad x\in\mathbf{R}^2,\ \theta\in\mathbf{S}^1, \tag{2.36}$$

where

$$R_\pm(x,y,\theta)=\exp(-G_{\pm,\theta}a(x))G_\pm(x-y,\theta)\exp(G_{\pm,\theta}a(y)), \tag{2.37}$$

$$G_\pm(x,\theta)=\frac{\pm 1}{2\pi i(\theta^\perp x\mp i0\,\mathrm{sgn}\,(\theta x))}, \tag{2.38}$$

$G_{\pm,\theta}$ are the convolution operators with the generalized functions $G_{\pm}(\,\cdot\,,\theta)$,

$$G_{\pm,\theta}a(x)=\int_{\mathbf{R}^2} G_{\pm}(x-y,\theta)a(y)\,dy. \tag{2.39}$$

The following formulas are valid:

$$G_{\pm}(x,\theta)=\frac{\pm 1}{2\pi i(\theta^{\perp}x\pm i0)}+\delta(\theta^{\perp}x)\chi_{+}(\theta x), \tag{2.40a}$$

$$G_{\pm}(x,\theta)=\frac{\pm 1}{2\pi i(\theta^{\perp}x\mp i0)}-\delta(\theta^{\perp}x)\chi_{+}(-\theta x), \tag{2.40b}$$

$$G_{\pm,\theta}u(x)=D_{-\theta}u(x)\pm\frac{1}{2i}H_{\pm}P_{\theta}^{\perp}u(\theta^{\perp}x), \tag{2.41a}$$

$$G_{\pm,\theta}u(x)=-D_{\theta}u(x)\pm\frac{1}{2i}H_{\mp}P_{\theta}^{\perp}u(\theta^{\perp}x), \tag{2.41b}$$

where

$$\chi_{+}(s)=\begin{cases}1 & \text{for } s>0,\\ 0 & \text{for } s\le 0,\end{cases} \tag{2.42}$$

D_{θ}, $P_{\theta}^{\perp}$, $H_{\pm}$ are defined by (2.13), (2.14) and (2.16), respectively, u is a test function, $x\in\mathbf{R}^2$ and $\theta\in\mathbf{S}^1$.

The fact that the limits in (2.33) exist and that the formulas (2.36) and (2.37) are valid follows from (2.24), (2.25), (A.25), (A.31), (A.32), (2.41), (A.15), (A.16), (A.19), (A.24) and the formula

$$\begin{aligned}G_{\omega(\pm\tau)}(\exp(G_{\omega(\pm\tau)}a)f)=G_{\pm,\theta}(\exp(G_{\pm,\theta}a)f)+(G_{\omega(\pm\tau)}-G_{\pm,\theta})(\exp(G_{\omega(\pm\tau)}a)f)\\ +G_{\pm,\theta}((\exp((G_{\omega(\pm\tau)}-G_{\pm,\theta})a)-1)\exp(G_{\pm,\theta}a)f).\end{aligned}$$

Using (2.41a), (A.31), (A.33), (A.15), (A.16), (A.19), (A.24) and (A.6) we obtain, in particular, that

$$\psi_{\pm}\in C(\mathbf{R}^2\times\mathbf{S}^1,\mathbf{C}), \tag{2.43}$$

$$\psi_{\pm}(x,\theta) \text{ satisfy (2.21) for } x\in\mathbf{R}^2,\ \theta\in\mathbf{S}^1, \tag{2.44}$$

$$\lim_{s\to-\infty}\psi_{\pm}(x+s\theta,\theta)=m_{\pm}(\theta^{\perp}x,\theta), \tag{2.45}$$

where

$$m_{\pm}(\theta^{\perp}x,\theta)=\pm\frac{1}{2i}\exp\left(\mp\frac{H_{\pm}P_{\theta}^{\perp}a(\theta^{\perp}x)}{2i}\right)H_{\pm}P_{\theta}^{\perp}\exp(G_{\pm,\theta}a)f(\theta^{\perp}x) \tag{2.46}$$

for $x\in\mathbf{R}^2$ and $\theta\in\mathbf{S}^1$, $\exp(G_{\pm,\theta}a)$ denotes the operator such that

$$\exp(G_{\pm,\theta}a)u(x)=\exp(G_{\pm,\theta}a(x))u(x), \quad x\in\mathbf{R}^2, \tag{2.47}$$

and u is a test function. In addition,

$$m_\pm(s,\theta)=\pm\frac{1}{2i}\exp\left(\mp\frac{H_\pm P_\theta^\perp a(s)}{2i}\right)H_\pm P_\theta^\perp g(s), \tag{2.48a}$$

where, using (2.41b),

$$\begin{aligned} g(x)&=\exp(G_{\pm,\theta}a(x))f(x)\\ &=\exp\left(\pm\frac{H_\mp P_\theta^\perp a(\theta^\perp x)}{2i}\right)\exp(-D_\theta a(x))f(x), \quad s\in\mathbf{R},\ \theta\in\mathbf{S}^1,\ x\in\mathbf{R}^2. \end{aligned} \tag{2.48b}$$

In addition, using (2.48) and the definitions (2.14), (2.15) and (2.17) we obtain that

$$P_\theta^\perp g(s)=\exp\left(\pm\frac{H_\mp P_\theta^\perp a(s)}{2i}\right)P_{a,\theta}^\perp f(s), \tag{2.49}$$

$$m_\pm(s,\theta)=\pm\frac{1}{2i}\exp\left(\mp\frac{H_\pm P_\theta^\perp a(s)}{2i}\right)H_\pm\exp\left(\pm\frac{H_\mp P_\theta^\perp a}{2i}\right)P_{a,\theta}^\perp f(s), \tag{2.50}$$

$s\in\mathbf{R}$, $\theta\in\mathbf{S}^1$, $x\in\mathbf{R}^2$.

The following formula holds:

$$\psi_+(x,\theta)-\psi_-(x,\theta)=\varphi(x,\theta), \quad x\in\mathbf{R}^2,\ \theta\in\mathbf{S}^1, \tag{2.51}$$

where φ is given by (2.12b–c).

To obtain (2.51) we use that if φ is defined by the left-hand side of (2.51), then, by virtue of (2.43)–(2.45) and (2.50),

(2.52) $\varphi\in C(\mathbf{R}^2\times\mathbf{S}^1,\mathbf{C})$,

(2.53) $\varphi(x,\theta)$ satisfies (2.29) for $x\in\mathbf{R}^2$, $\theta\in\mathbf{S}^1$,

(2.54) $\lim_{s\to-\infty}\varphi(x+s\theta,\theta)=m_+(\theta^\perp x,\theta)-m_-(\theta^\perp x,\theta)=m(\theta^\perp x,\theta)$, $x\in\mathbf{R}^2$, $\theta\in\mathbf{S}^1$,

where m is given by (2.12c), and that the properties (2.52)–(2.54) imply (2.12b).

The Riemann surface Σ defined by (2.22) admits the following parametrization by $\lambda\in\mathbf{C}\setminus\{0\}$:

$$\begin{aligned} &\lambda(\theta)=\theta_1+i\theta_2 && \text{for } \theta=(\theta_1,\theta_2)\in\Sigma,\\ &\theta_1(\lambda)=\frac{1}{2}\left(\lambda+\frac{1}{\lambda}\right),\ \theta_2(\lambda)=\frac{1}{2i}\left(\lambda-\frac{1}{\lambda}\right) && \text{for } \lambda\in\mathbf{C}\setminus\{0\}. \end{aligned} \tag{2.55}$$

In the variable λ the circle $\mathbf{S}^1$ takes the form

$$T=\{\lambda\in\mathbf{C}\mid |\lambda|=1\}. \tag{2.56}$$

Using (2.24), (2.33), (2.55) and Lemma A.7 we obtain that for any $x\in\mathbf{R}^2$ the following formulas hold:

$$\frac{\partial}{\partial\bar\lambda}\psi(x,\theta(\lambda))=0 \quad \text{for } \lambda\in(\mathbf{C}\backslash\{0\})\backslash T \tag{2.57}$$

(i.e. $\psi(x,\theta(\lambda))$ is a holomorphic function in $\lambda\in(\mathbf{C}\backslash\{0\})\backslash T$),

$$\psi(x,\theta(\lambda))=\frac{1}{\lambda}\bar Cf(x)+o\left(\frac{1}{\lambda}\right), \quad \text{as } \lambda\to\infty, \tag{2.58}$$

$$\psi(x,\theta(\lambda))=\lambda Cf(x)+o(\lambda), \quad \text{as } \lambda\to 0, \tag{2.59}$$

$$\psi(x,\theta(\lambda(1\pm 0)))=\psi_{\mp}(x,\theta(\lambda)) \quad \text{for } \lambda\in T, \tag{2.60}$$

where C and $\bar C$ are the operators such that

$$Cu(x)=-\frac{1}{\pi}\int_{\mathbf{R}^2}\frac{u(y)\,dy}{(y_1+iy_2)-(x_1+ix_2)}, \tag{2.61}$$

$$\bar Cu(x)=-\frac{1}{\pi}\int_{\mathbf{R}^2}\frac{u(y)\,dy}{(y_1-iy_2)-(x_1-ix_2)}, \tag{2.62}$$

where $x=(x_1,x_2)\in\mathbf{R}^2$, $y=(y_1,y_2)\in\mathbf{R}^2$ and u is a test function. In addition, note that for G given by (2.26) the following formula holds:

$$G(x,\theta(\lambda))=\frac{\operatorname{sgn}(1-|\lambda|)}{2\pi i\left(\frac{1}{2}i\right)(\lambda\bar z-z/\lambda)}, \quad x\in\mathbf{R}^2,\ \lambda\in(\mathbf{C}\backslash\{0\})\backslash T, \tag{2.63}$$

where

$$z=x_1+ix_2, \quad \bar z=x_1-ix_2. \tag{2.64}$$

The properties (2.57)–(2.60) imply that

$$\begin{aligned}\bar Cf(x)&=-\frac{1}{2\pi i}\int_T(\psi_+(x,\theta(\lambda))-\psi_-(x,\theta(\lambda)))\,d\lambda\\&=-\frac{1}{2\pi}\int_{\mathbf{S}^1}(\psi_+(x,\theta)-\psi_-(x,\theta))(\theta_1+i\theta_2)\,d\theta.\end{aligned} \tag{2.65}$$

The formulas (2.65), (2.51) and the formula

$$\frac{1}{2}\left(\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_2}\right)\bar{C}f(x)=f(x),\quad x\in\mathbf{R}^2, \tag{2.66}$$

imply the formula (2.12a). □

Let S be a great circle in $\mathbf{S}^{d-1}$, $X(S)$ be the linear span of S in $\mathbf{R}^d$, $X^{\perp}(S)$ be the orthogonal complement of $X(S)$ in $\mathbf{R}^d$, and

$$\mathcal{M}(S)=\{\gamma=(x,\theta)\in T\mathbf{S}^{d-1}\,|\,\theta\in S\}. \tag{2.67}$$

Note that $\dim\mathcal{M}(S)=d$ for $d\geq 2$ and $\mathcal{M}(S)=T\mathbf{S}^1$ for $d=2$.

Theorem 2.1 implies the following corollary.

Corollary 2.5. *Suppose that a and f satisfy (1.6), where $d\geq 2$. Then, under the condition that a is known, $P_a f$ on $\mathcal{M}(S)$ uniquely determines f on $\mathbf{R}^d$ by the following scheme*

$$P_a f|_{T\mathbf{S}^1(Y)} \overset{(2.12)}{\longrightarrow} f|_Y \tag{2.68a}$$

for each two-dimensional plane Y of the form

$$Y=X(S)+y,\quad y\in X^{\perp}(S), \tag{2.68b}$$

where $T\mathbf{S}^1(Y)$ is the set of all oriented straight lines lying in Y.

Note that

$$\mathcal{M}(S)=\bigcup_{y\in X^{\perp}(S)} T\mathbf{S}^1(X(S)+y). \tag{2.69}$$

Remark 2.6. Theorem 2.1, Remark 2.3 and Corollary 2.5 remain valid under the assumptions (1.9) (in place of (1.6)). In this case the formulas (2.12c) and (2.20c) are valid in $L^p(\mathbf{R},\mathbf{C})$ for any $\theta\in\mathbf{S}^1$, where $\max(1,\varepsilon^{-1})<p<+\infty$; the formulas (2.12a), (2.19) and (2.20a) are valid in the sense of distribution theory; and the scheme (2.68) is valid for almost any Y of the form (2.68b). Note also that Theorem 2.1 and Corollary 2.5 remain valid under the assumption that $a,f\in C^{\alpha,1+\varepsilon}(\mathbf{R}^d,\mathbf{C})$ for some $\alpha\in]0,1[$ and $\varepsilon>0$ (in place of (1.6)). In this case the formulas (2.12) are valid pointwise.

3. Subsequent results (added in October 2000)

The main results of this section are Propositions 3.1, 3.4, 3.5, 3.9, Corollary 3.3 and Remark 3.8. For $d=2$, Proposition 3.1 relates holomorphic moments of a function f and the attenuated X-ray transform $P_a f$ by formula (3.2). For $d\geq 2$, Corollary 3.3 relates, in particular, the classical Radon transform (along two-dimensional planes) $R_{2,d}f$ and the attenuated X-ray transform $P_a f$ by means of the formula (3.2) with $n=0$. For $d=2$, in Proposition 3.4 we deal with the reconstruction of f from $P_a f$ on a subset of $T\mathbf{S}^1$, under the condition that a is (completely) known. For $d=2$, in Proposition 3.5 we give new necessary conditions on $P_a f$; these conditions can be considered as a generalization of the well-known symmetry (3.16) for the classical X-ray transform Pf. In Remark 3.8 we relax the assumptions of Propositions 3.1, 3.4, 3.5 and Corollary 3.3. For $d=2$, in Proposition 3.9 we give, in particular, sufficient conditions for a function g on $T\mathbf{S}^1$ in order to be in the range of P_a defined on $C^{\infty,1+\varepsilon}(\mathbf{R}^2,\mathbf{R})$, $\varepsilon\in]0,1[$, for $a\in C_0^\infty(\mathbf{R}^2,\mathbf{R})$. We plan to develop Proposition 3.9 in a subsequent paper.

Proposition 3.1. *Let*

$$a\in C^{\alpha,1+\varepsilon_a}(\mathbf{R}^2,\mathbf{R}),\quad f\in C^{0,1+\varepsilon_f}(\mathbf{R}^2,\mathbf{R}) \tag{3.1}$$

for some $\alpha\in]0,1[$, $\varepsilon_a>0$ *and* $\varepsilon_f>1$. *Then*

$$\int_{\mathbf{R}^2}(x_1\pm ix_2)^n f(x)\,dx=\frac{\pm 2i}{2\pi}\int_{\mathbf{S}^1}(\theta_1\pm i\theta_2)^n\int_{\mathbf{R}}s^n\exp\Big(\pm\frac{H_\mp P_\theta^\perp a(s)}{2i}\Big)P_{a,\theta}^\perp f(s)\,ds\,d\theta \tag{3.2}$$

for any $n\in\mathbf{N}\cup\{0\}$ *such that* $n<\varepsilon_f-1$.

Proof. Consider the functions

$$I_n(\theta)=\int_{\mathbf{R}^2}(-\theta_2x_1+\theta_1x_2)^n\exp(G_\theta a(x))f(x)\,dx\quad\text{for }\theta\in\Sigma\backslash\mathbf{S}^1, \tag{3.3}$$

$$I_{\pm,n}(\theta)=\lim_{\tau\to 0+}I_n(\omega(\pm\tau))\qquad\text{for }\theta\in\mathbf{S}^1, \tag{3.4}$$

where $\omega(\tau)$ is given by (2.34) and where $n\in\mathbf{N}\cup\{0\}$, $n<\varepsilon_f-1$.

The following formula holds:

$$\begin{aligned}I_{\pm,n}(\theta)&=\int_{\mathbf{R}^2}(\theta^\perp x)^n\exp(G_{\pm,\theta}a(x))f(x)\,dx\\&=\int_{\mathbf{R}^2}(\theta^\perp x)^n\exp\Big(-D_\theta a(x)\pm\frac{H_\mp P_\theta^\perp a(\theta^\perp x)}{2i}\Big)f(x)\,dx\\&=\int_{\mathbf{R}}s^n\exp\Big(\pm\frac{H_\mp P_\theta^\perp a(s)}{2i}\Big)P_{a,\theta}^\perp f(s)\,ds,\end{aligned} \tag{3.5}$$

where $\theta=(\theta_1,\theta_2)\in\mathbf{S}^1$ and $\theta^\perp=(-\theta_2,\theta_1)$.

The fact that (under the conditions of Proposition 3.1) the limits in (3.4) exist and the formula (3.5) is valid follows from (A.25), (A.31), (A.32), (2.41b) and (2.15).

Using (3.3), (3.4), (2.55) and Lemma A.7 we obtain that

$$\frac{\partial}{\partial\bar\lambda} I_n(\theta(\lambda)) = 0 \quad \text{for } \lambda \in (\mathbf{C}\backslash\{0\})\backslash T, \tag{3.6}$$

$$I_n(\theta(\lambda)) = \lambda^n \left(\frac{i}{2}\right)^n \int_{\mathbf{R}^2} (x_1 - ix_2)^n f(x)\, dx + o(\lambda^n), \qquad \text{as } \lambda \to \infty, \tag{3.7}$$

$$I_n(\theta(\lambda)) = \frac{1}{\lambda^n}\left(-\frac{i}{2}\right)^n \int_{\mathbf{R}^2} (x_1 + ix_2)^n f(x)\, dx + o(\lambda^{-n}), \quad \text{as } \lambda \to 0, \tag{3.8}$$

$$I_n(\theta(\lambda(1\pm 0))) = I_{\pm,n}(\theta(\lambda)) \qquad \text{for } \lambda \in T. \tag{3.9}$$

The properties (3.6)–(3.9) imply that

$$\begin{aligned} \left(\mp\frac{i}{2}\right)^n \int_{\mathbf{R}^2} (x_1 \pm ix_2)^n f(x)\, dx &= \frac{1}{2\pi i}\int_T \lambda^{\pm n-1} I_{\pm,n}(\theta(\lambda))\, d\lambda \\ &= \frac{1}{2\pi}\int_{\mathbf{S}^1} (\theta_1 \pm i\theta_2)^n I_{\pm,n}(\theta)\, d\theta. \end{aligned} \tag{3.10}$$

The formula (3.2) follows from (3.10) and (3.5). □

Remark 3.2. If the conditions (3.1) are valid, then the formulas (3.5)–(3.9) imply also that

$$\int_{\mathbf{S}^1} (\theta_1 \pm i\theta_2)^m \int_{\mathbf{R}} s^n \exp\left(\pm\frac{H_{\mp} P_\theta^\perp a(s)}{2i}\right) P_{a,\theta}^\perp f(s)\, ds\, d\theta = 0, \tag{3.11}$$

for any $n \in \mathbf{N} \cup \{0\}$ and $m \in \mathbf{N}$ such that $n < m$ and $n < \varepsilon_f - 1$. For the case when a and f are real-valued functions of the Schwartz class on $\mathbf{R}^2$, the formula (3.11) (in slightly different form) was given previously in Theorem 6.2 of [Na] as necessary conditions on the range of the attenuated Radon transformation.

Let $\mathcal{H}_{p,d}$ denote the set of all p-dimensional planes in $\mathbf{R}^d$, $d \ge p$. Consider the Radon transformation $R_{2,d}$ defined by the formula

$$R_{2,d} f(Y) = \int_Y f(x)\, dx, \quad Y \in \mathcal{H}_{2,d},\ d \ge 2, \tag{3.12}$$

for any real-valued sufficiently regular function f on $\mathbf{R}^d$ sufficiently rapidly vanishing at infinity, where dx is the standard element of area on Y.

Proposition 3.1 implies the following corollary.

Corollary 3.3. *Suppose that*

$$a\in C^{\alpha,1+\varepsilon_a}(\mathbf{R}^d,\mathbf{R}),\quad f\in C^{0,1+\varepsilon_f}(\mathbf{R}^d,\mathbf{R}),\quad d\geq 2, \tag{3.13}$$

for some $\alpha\in]0,1[$, $\varepsilon_a>0$ *and* $\varepsilon_f>1$. *Then for any* $Y\in\mathcal{H}_{2,d}$, *under the condition that* $a|_Y$ *is known, one can find* $R_{2,d}f(Y)$ *from* $P_af|_{T\mathbf{S}^1(Y)}$ *by means of* (3.2) *with* $n=0$. *In addition, for* $d\geq 3$ *under the condition that* $a|_X$ *is known, one can find* $f|_X$ *from* $P_af|_{T\mathbf{S}^2(X)}$ *for any* $X\in\mathcal{H}_{3,d}$, *using* (3.2) *with* $n=0$ *and the classical Radon inversion formula for the transformation* $R_{2,3}$. *(Here* $T\mathbf{S}^{p-1}(h)$ *is the set of all oriented straight lines lying in* $h\in\mathcal{H}_{p,d}$.*)*

Proposition 3.4. *Let*

$$\begin{aligned} &a\in C^{\alpha,1+\varepsilon}(\mathbf{R}^2,\mathbf{R})\quad \textit{for some } \alpha\in]0,1[\textit{ and } \varepsilon>0,\\ &f\in C_0(\mathbf{R}^2,\mathbf{R}) \end{aligned} \tag{3.14}$$

($C_0(\mathbf{R}^2,\mathbf{R})$ *being the space of real-valued continuous compactly supported functions on* $\mathbf{R}^2$*). Let* Θ *be a subset of* $\mathbf{S}^1$ *of positive length. Then, under the condition that a is known,* $P^{\perp}_{a,\theta}f(s)$ *given for all* $(\theta,s)\in\Theta\times\mathbf{R}$ *uniquely determines* f.

Proof. Consider the functions I_n, $I_{\pm,n}$ defined by (3.3) and (3.4), where $n\in\mathbf{N}\cup\{0\}$. Under the conditions (3.14), the functions I_n and $I_{\pm,n}$ are well-defined by (3.3) and (3.4), and the formulas (3.5)–(3.9) hold for any $n\in\mathbf{N}\cup\{0\}$.

The proposition follows from the following statements (being valid under the assumptions (3.14)):

(1) Under the condition that a is known, the transform $P^{\perp}_{a,\theta}f(s)$ given for all $(\theta,s)\in\Theta\times\mathbf{R}$ uniquely determines $I_{\pm,n}(\theta)$ for all $\theta\in\Theta$ and $n\in\mathbf{N}\cup\{0\}$ by (3.5).

(2) The function $I_{j,n}(\cdot)$ on Θ uniquely determines this function on $\mathbf{S}^1$ via analytic continuation according to (3.6)–(3.9) for any $j\in\{+,-\}$ and $n\in\mathbf{N}\cup\{0\}$.

(3) Under the condition that a is known, the sequence $I_{j,0}$, $I_{j,1}$, $I_{j,2}$, ... , uniquely determines $P^{\perp}_{a,\theta}f(\cdot)$ on $\mathbf{R}$ by means of (3.5) and the inverse moment problem for any $j\in\{+,-\}$ and $\theta\in\mathbf{S}^1$.

(4) Under the condition that a is known, the transform $P^{\perp}_{a,\theta}f(s)$ given for all $(\theta,s)\in\mathbf{S}^1\times\mathbf{R}$ uniquely determines f by (2.12). □

Proposition 3.5. *Under the assumptions* (1.6) *and* $d=2$, *the following formula holds*:

$$\int_{\mathbf{S}^1}\operatorname{Im}\varphi(x,\theta)\,d\theta=0\quad \textit{for any } x\in\mathbf{R}^2, \tag{3.15}$$

where $\operatorname{Im}\varphi(x,\theta)$ *is given by* (2.20b–c).

Remark 3.6. We consider the formulas (3.15), (2.20b–c) as necessary conditions on the attenuated X-ray transform $P^{\perp}_{a,\theta}f(s)$, $\theta\in\mathbf{S}^1$, $s\in\mathbf{R}$. The necessary conditions

(3.15), (2.20b–c) differ from Natterer's necessary conditions (3.11): under the assumptions (1.6) and $d=2$, the integral over $\mathbf{S}^1\times\mathbf{R}$ in (3.11) can diverge for $n\geq\varepsilon-1$ (i.e. even for $n=0$ if $\varepsilon\leq 1$), whereas the formulas (3.15), (2.20b–c) are completely well-defined. For the case when $a\equiv 0$, the necessary conditions (3.15), (2.20b–c) are a corollary of the following well-known property

$$P_\theta^\perp f(s)=P_{-\theta}^\perp f(-s),\quad \theta\in\mathbf{S}^1,\ s\in\mathbf{R}. \tag{3.16}$$

For this case the property (3.16) implies the formula $\operatorname{Im}\varphi(x,\theta)=-\operatorname{Im}\varphi(x,-\theta)$, $x\in\mathbf{R}^2$, $\theta\in\mathbf{S}^1$, implying (3.15).

Remark 3.7. Using that φ of Theorem 2.1 and Remark 2.3 satisfies (2.29), $\theta\in\mathbf{S}^1$, the formula (2.19) can be rewritten as

$$a(x)\int_{\mathbf{S}^1}\operatorname{Im}\varphi(x,\theta)\,d\theta=0,\quad x\in\mathbf{R}^2. \tag{3.17}$$

The formula (3.15) strengthens (3.17) for those x where $a(x)=0$.

Proof of Proposition 3.5. The properties (2.57)–(2.60) imply that

$$\begin{aligned}0=\lim_{\lambda\to 0}\psi(x,\theta(\lambda))&=\frac{1}{2\pi i}\int_T(\psi_+(x,\theta(\zeta))-\psi_-(x,\theta(\zeta)))\,\frac{d\zeta}{\zeta}\\&=\frac{1}{2\pi}\int_{\mathbf{S}^1}(\psi_+(x,\theta)-\psi_-(x,\theta))\,d\theta.\end{aligned} \tag{3.18}$$

The formula (3.15) follows from (3.18), (2.51) and (2.20b–c). □

Remark 3.8. Proposition 3.1, the formula (3.11) and the formulas (3.5)–(3.9) for the functions I_n and $I_{\pm,n}$ defined by (3.3) and (3.4) remain valid under the assumptions

$$a\in L^{\infty,1+\varepsilon_a}(\mathbf{R}^d,\mathbf{R}),\ f\in L^{\infty,1+\varepsilon_f}(\mathbf{R}^d,\mathbf{R})\quad\text{for some } \varepsilon_a>0 \text{ and } \varepsilon_f>1, \tag{3.19}$$

$d=2$, in place of (3.1). Corollary 3.3 remains valid under the assumptions (3.19), $d\geq 2$, in place of (3.13) for the only $Y\in\mathcal{H}_{2,d}$ for $d=2$, for almost any $Y\in\mathcal{H}_{2,d}$ for $d\geq 3$, for the only $X\in\mathcal{H}_{3,d}$ for $d=3$, and for almost any $X\in\mathcal{H}_{3,d}$ for $d\geq 4$. Proposition 3.4 remains valid under the assumptions

$$\begin{aligned}&a\in L^{\infty,1+\varepsilon_a}(\mathbf{R}^2,\mathbf{R})\quad\text{for some } \varepsilon_a>0,\\&f_\delta\in L^\infty(\mathbf{R}^2,\mathbf{R})\qquad\ \text{for some } \delta>0,\end{aligned} \tag{3.20}$$

where $f_\delta(x)=e^{\delta|x|}f(x)$, $x\in\mathbf{R}^2$, in place of (3.14). Proposition 3.5 remains valid under the assumptions (1.9) in place of (1.6) for (3.15) being valid, say, in $L^1(\mathbf{R}^2,\mathbf{R})$ and (2.20c) being valid as in Remark 2.6. Note also that, under the assumption that $a,f\in C^{\alpha,1+\varepsilon}(\mathbf{R}^2,\mathbf{C})$ for some $\alpha\in]0,1[$ and $\varepsilon>0$, the following formula holds

$$\int_{\mathbf{S}^1}\varphi(x,\theta)\,d\theta\quad\text{for any } x\in\mathbf{R}^2,$$

where φ is given by (2.12b–c).

Proposition 3.9. (I) *Suppose that a and f satisfy* (1.11) *for d=2. Then*

$$g\in C_0^\infty(\mathbf{R}\times\mathbf{S}^1,\mathbf{R}),\quad \textit{where } g(s,\theta)=P_{a,\theta}^\perp f(s),\ s\in\mathbf{R},\ \theta\in\mathbf{S}^1, \tag{3.21}$$

and the conditions (3.15), (2.20b–c) *are valid.*

(II) *Suppose that*

$$a\in C_0^\infty(\mathbf{R}^2,\mathbf{R}), \tag{3.22a}$$

$$g\in C_0^\infty(\mathbf{R}\times\mathbf{S}^1,\mathbf{R}) \tag{3.22b}$$

and that the conditions (3.15), (2.20b–c) *for g are valid, i.e.*

$$\int_{\mathbf{S}^1}\exp(-D_{-\theta}a(x))\,\mathrm{Re}\bigg(\exp\Big(-\frac{H_+P_\theta^\perp a(\theta^\perp x)}{2i}\Big) \times(H_++H_-)\exp\Big(\frac{H_-P_\theta^\perp a}{2i}\Big)g_\theta(\theta^\perp x)\bigg)\,d\theta=0,\quad x\in\mathbf{R}^2, \tag{3.23}$$

where g_θ denotes the function on $\mathbf{R}$ *such that $g_\theta(s)=g(s,\theta)$, $s\in\mathbf{R}$. Then there is a function f such that*

$$g(s,\theta)=P_{a,\theta}^\perp f(s),\qquad s\in\mathbf{R},\ \theta\in\mathbf{S}^1, \tag{3.24}$$

$$f\in C^{\infty,1+\varepsilon}(\mathbf{R}^2,\mathbf{R})\quad \textit{for any } \varepsilon\in]0,1[, \tag{3.25}$$

where

$$C^{\infty,\sigma}(\mathbf{R}^2,\mathbf{R})=\{f\mid \partial^j f\in C^{0,\sigma}(\mathbf{R}^2,\mathbf{R})\ \textit{for } j=(j_1,j_2)\in(\mathbf{N}\cup\{0\})^2\},\ \sigma\ge 0, \tag{3.26}$$
$$\partial^j f(x)=\frac{\partial^{j_1+j_2}f(x)}{\partial x_1^{j_1}\partial x_2^{j_2}},\quad x\in\mathbf{R}^2.$$

Remark 3.10. Proposition 3.9 is a result on the range characterization for the two-dimensional attenuated X-ray transformation P_a. In part I we deal with necessary conditions, and in part II we deal with sufficient conditions.

Remark 3.11. For the case when $a\equiv 0$, part II of Proposition 3.9 is also valid with

$$g(s,\theta)=g(-s,-\theta),\quad s\in\mathbf{R},\ \theta\in\mathbf{S}^1, \tag{3.27}$$

in place of (3.23). To our knowledge, even the latter result was not given explicitly in the literature.

Remark 3.12. As a corollary of Proposition 3.9 and Theorem 2.1, under the assumption that $a\in C_0^\infty(\mathbf{R}^2,\mathbf{R})$, one can give a range characterization for P_a defined on $C_{\mathcal{D}}^\infty(\mathbf{R}^2,\mathbf{R})=\{f\in C_0^\infty(\mathbf{R}^2,\mathbf{R})\,|\,\operatorname{supp} f\subseteq\mathcal{D}\}$, where $\mathcal{D}$ is a compact in $\mathbf{R}^2$, just complementing the necessary conditions for g of part I of Proposition 3.9 or, which is the same, the sufficient conditions for g of part II of Proposition 3.9 by the condition that f constructed from g by means of (2.20) is identically zero on $\mathbf{R}^2\backslash\mathcal{D}$.

Remark 3.13. Part I of Proposition 3.9 follows from Proposition 3.5 and well-known facts. We plan to give the proof of part II of Proposition 3.9 in a subsequent paper.

Appendix: Estimates for operators

We present, first, some estimates for the operators D_θ, P_θ and $P_\theta^\perp$. Here D_θ and P_θ are defined by

$$D_\theta u(x)=\int_0^{+\infty}u(x+t\theta)\,dt, \tag{A.1}$$

$$P_\theta u(x)=\int_{\mathbf{R}}u(x+t\theta)\,dt, \tag{A.2}$$

where $x\in\mathbf{R}^d$, $\theta\in\mathbf{S}^{d-1}$ and u is a test function; $P_\theta^\perp$ is defined by (2.14). We use π_θ defined by (1.4) and χ_+ defined by (2.42).

Lemma A.1. *Let*

$$u\in C^{\alpha,1+\varepsilon}(\mathbf{R}^d,\mathbf{C}), \tag{A.3}$$

$$\|u\|_{0,1+\varepsilon}\le U_1, \tag{A.4}$$

$$\|u\|'_{\alpha,1+\varepsilon}\le U_2, \tag{A.5}$$

where $0<\alpha<1$ *and* $\varepsilon>0$. *Then*

$$|D_{-\theta}u(x)|\le c_1(\varepsilon,\theta,x)U_1, \tag{A.6}$$

$$|D_{-\theta}u(x+y)-D_{-\theta}u(x)|\le c_1(\varepsilon,\theta,x)U_2|y|^\alpha, \tag{A.7}$$

$$c_1(\varepsilon,\theta,x)\overset{\text{def}}{=}\frac{2^{(1+\varepsilon)/2}(1+\chi_+(\theta x))}{\varepsilon\big(\sqrt{2}+|\pi_\theta x|-\theta x\chi_+(-\theta x)\big)^\varepsilon}, \tag{A.8}$$

$$|D_{-\theta}u(x)-D_{-\theta'}u(x)|\le c_2(\varepsilon,\beta,|x|)U|\theta-\theta'|^\beta, \tag{A.9}$$

$$c_2(\varepsilon,\beta,r)\overset{\text{def}}{=}\frac{2^{(3+\varepsilon)/2}}{\varepsilon}\left(\frac{3\varepsilon-2\beta}{\varepsilon-\beta}+3\cdot 2^{\varepsilon/2}r^\varepsilon\right) \tag{A.10}$$

for $x,y\in\mathbf{R}^d$, $|y|\le1$, $\theta,\theta'\in\mathbf{S}^{d-1}$, $|\theta-\theta'|\le1$, $0<\beta<\varepsilon$, $\beta\le\alpha$ *and* $U=\max(U_1,U_2)$.

Lemma A.2. *Under the assumptions of Lemma* A.1, *the following estimates hold:*

$$|P_\theta u(x)| \le c_3(\varepsilon, \theta, |\pi_\theta x|)U_1, \tag{A.11}$$

$$|P_\theta u(x+y) - P_\theta u(x)| \le c_3(\varepsilon, \theta, |\pi_\theta x|)U_2|y|^\alpha, \tag{A.12}$$

$$c_3(\varepsilon, s) \stackrel{\text{def}}{=} \frac{2^{(3+\varepsilon)/2}}{\varepsilon\big(\sqrt{2}+|s|\big)^\varepsilon}, \tag{A.13}$$

$$|P_\theta u(x) - P_{\theta'} u(x)| \le 2c_2(\varepsilon, \beta, |x|)U|\theta-\theta'|^\beta, \tag{A.14}$$

in addition, for $d=2$,

$$|P_\theta^\perp u(s)| \le c_3(\varepsilon, s)U_1, \tag{A.15}$$

$$|P_\theta^\perp u(s+\delta) - P_\theta^\perp u(s)| \le c_3(\varepsilon, s)U_2|\delta|^\alpha, \tag{A.16}$$

$$|P_\theta^\perp u(s) - P_{\theta'}^\perp u(s)| \le c_4(\varepsilon, \beta)U|\theta-\theta'|^\beta, \tag{A.17}$$

$$c_4(\varepsilon, \beta) \stackrel{\text{def}}{=} \frac{2^{(5+\varepsilon)/2}}{\varepsilon}\left(\frac{3\varepsilon-2\beta}{\varepsilon-\beta} + 3\left(\frac{2}{\sqrt{3}}\right)^\varepsilon\right), \tag{A.18}$$

where $x, y \in \mathbf{R}^d$, $|y| \le 1$, $\theta, \theta' \in \mathbf{S}^{d-1}$, $|\theta-\theta'| \le 1$, $0<\beta<\varepsilon$, $\beta \le \alpha$, $U = \max(U_1, U_2)$, $s, \delta \in \mathbf{R}$ *and* $|\delta| \le 1$.

Lemmas A.1 and A.2 follow from Lemmas $A.1_a$ and $A.2_a$ of [N2], respectively.

Remark A.3. Let $u \in C^{0,1+\varepsilon}(\mathbf{R}^2, \mathbf{C})$, where $\varepsilon>0$. Then

$$\begin{aligned} \|D_{-\theta} u\|_{0,0} &\le 2R \sup_{|x|\le R} u(x) + c_3(\varepsilon, R)\|u\|_{0,\varepsilon}, \\ \|P_\theta^\perp u\|_{0,0} &\le 2R \sup_{|x|\le R} u(x) + c_3(\varepsilon, R)\|u\|_{0,\varepsilon}, \end{aligned} \tag{A.19}$$

where $\theta \in \mathbf{S}^1$ and $R \ge 0$.

Lemma A.4. *Let* $u \in C^{\alpha,\sigma}(\mathbf{R}^d, \mathbf{C})$ *and* $v \in C^{\alpha,\tau}(\mathbf{R}^d, \mathbf{C})$, *where* $0<\alpha<1$, $\sigma \ge 0$ *and* $\tau \ge 0$. *Then* $uv \in C^{\alpha,\sigma+\tau}(\mathbf{R}^d, \mathbf{C})$ *and*

$$\|uv\|_{0,\sigma+\tau} \le \|u\|_{0,\sigma}\|v\|_{0,\tau}, \tag{A.20}$$

$$\|uv\|_{\alpha,\sigma+\tau} \le 2^{1+\min(\sigma,\tau)}\|u\|_{\alpha,\sigma}\|v\|_{\alpha,\tau}. \tag{A.21}$$

This lemma is elementary.

We present now some estimates for the operators $H_\pm$ defined by (2.16).

Lemma A.5. *Let $v\in C^{\alpha,\varepsilon}(\mathbf{R},\mathbf{C})$, where $0<\alpha<1$ and $\varepsilon>0$. Let $H_\pm v(s)$ be defined by* (2.16) *for $s\in\mathbf{R}\cup\mathbf{C}_\pm$, where $\mathbf{C}_\pm=\{s\in\mathbf{C}\,|\,\pm\operatorname{Im} s>0\}$. Then*

$$\frac{\partial}{\partial\bar s}H_\pm v(s)=0 \quad \text{for } s\in\mathbf{C}_\pm \tag{A.22}$$

and the following estimates hold:

$$\begin{aligned}|H_\pm v(s)|&\le c_5(\alpha,\varepsilon,\varepsilon')\frac{\|v\|_{\alpha,\varepsilon}}{(1+|s|)^{\varepsilon'}},\\ |H_\pm v(s+\delta)-H_\pm v(s)|&\le c_5(\alpha,\varepsilon,\varepsilon')\frac{\|v\|_{\alpha,\varepsilon}|\delta|^\alpha}{(1+|s|)^{\varepsilon'}},\end{aligned} \tag{A.23}$$

where $c_5(\alpha,\varepsilon,\varepsilon')$ is a positive constant, for $s,s+\delta\in\mathbf{R}\cup\mathbf{C}_\pm$, $|\delta|\le1$ and $0\le\varepsilon'<\min(1,\varepsilon)$;

$$\|H_\pm v\|_{0,0}\le\frac{2}{\pi\alpha}\|v\|'_{\alpha,0}h^\alpha+\left(1+\frac{2}{\pi}\ln\left(\frac{r}{h}\right)\right)\|v\|_{0,0}+c_6(\varepsilon,\beta)\frac{\|v\|_{0,\varepsilon}}{r^\beta}, \tag{A.24}$$

where $c_6(\varepsilon,\beta)$ is a positive constant, for $0<h\le1$, $r\ge1$ and $0\le\beta<\min(1,\varepsilon)$.

The formulas (A.22) and (A.23) are given in Lemma I.3 of [Fa]. The estimates (A.24) follow from the estimate (A.80) of [N2].

We now present estimates for the operators G_θ and $G_{\pm,\theta}$ defined by (2.27a) and (2.39).

Lemma A.6. *Let $u\in C^{\alpha,1+\varepsilon}(\mathbf{R}^2,\mathbf{C})$, where $0<\alpha<1$ and $\varepsilon>0$. Then*

$$G_{\omega(\pm\tau)}u(x)\to G_{\pm,\theta}u(x), \quad \text{as } \tau\to0+, \tag{A.25}$$

uniformly in u and x for $\|u\|_{\alpha,1+\varepsilon}\le A$ and $|x|\le R$, where A, R, α and ε are fixed and $\omega(\tau)$ is defined by (2.34).

(To prove (A.25), we decompose u into the sum $u(\cdot)=u_{1,\theta^\perp x}(\cdot)+u_{2,\theta^\perp x}(\cdot)$, where

$$\begin{aligned}u_{1,\theta^\perp x}(y)&=\chi(\theta^\perp y-\theta^\perp x)u(y),\\ u_{2,\theta^\perp x}(y)&=(1-\chi(\theta^\perp y-\theta^\perp x))u(y),\\ \chi(s)&=\begin{cases}1, & |s|\le1,\\ 2-|s|, & 1\le|s|\le2,\\ 0, & |s|\ge2.\end{cases}\end{aligned}$$

The proof of (A.25) with $u_{2,\theta^\perp x}$ in place of u is rather elementary. To prove (A.25) with $u_{1,\theta^\perp x}$ in place of u, we take $p{=}\theta y$, $q{=}\theta^\perp y$ as the variables of integration instead of $y{=}(y_1,y_2)$, we integrate, first, with respect to q and use the estimates (A.23).)

In the next lemma we use $\theta(\lambda){=}(\theta_1(\lambda),\theta_2(\lambda))$ of (2.55).

Lemma A.7. (1) *Let* $u{\in}C^{0,1+\varepsilon}(\mathbf{R}^2,\mathbf{C})$, *where* $\varepsilon{>}0$. *Then*

$$G_{\theta(\lambda)}u\in C^{\beta,\varepsilon'}(\mathbf{R}^2,\mathbf{C}),\quad 0\le\beta<1,\ 0\le\varepsilon'<\min(\varepsilon,1),\ |\lambda|\ne 1, \tag{A.26}$$

and, in particular,

$$\|G_{\theta(\lambda)}u\|_{0,0}\le\frac{4}{\varepsilon}\frac{|\lambda|}{1-|\lambda|^2}\|u\|_{0,1+\varepsilon},\qquad |\lambda|<1, \tag{A.27}$$

$$\|G_{\theta(\lambda)}u-\lambda Cu\|_{0,0}\le\frac{4}{\varepsilon}(1-|\lambda|^2)|\lambda|^3\|u\|_{0,1+\varepsilon},\qquad |\lambda|<1, \tag{A.28}$$

$$\|G_{\theta(\lambda)}u\|_{0,0}\le\frac{4}{\varepsilon}\frac{1}{(1-|\lambda|^{-2})|\lambda|}\|u\|_{0,1+\varepsilon},\qquad |\lambda|>1, \tag{A.29}$$

$$\left\|G_{\theta(\lambda)}u-\frac{1}{\lambda}\bar{C}u\right\|_{0,0}\le\frac{4}{\varepsilon}\frac{1}{(1-|\lambda|^{-2})|\lambda|^3}\|u\|_{0,1+\varepsilon},\qquad |\lambda|>1, \tag{A.30}$$

where $\lambda{\in}(\mathbf{C}\backslash\{0\})\backslash T$ *and the operators* C *and* $\bar{C}$ *are defined by* (2.61) *and* (2.62).

(2) *Let* $u_\lambda{\in}C^{0,1+\varepsilon}(\mathbf{R}^2,\mathbf{C})$ *for some* $\varepsilon{>}0$ *and any* $\lambda{\in}(\mathbf{C}\backslash\{0\})\backslash T$. *Let* $\|u_\lambda\|_{0,1+\varepsilon}$ *be bounded in* λ *on each compact of* $(\mathbf{C}\backslash\{0\})\backslash T$. *Let* $u_\lambda(x)$ *be a holomorphic function in* $\lambda{\in}(\mathbf{C}\backslash\{0\})\backslash T$ *for any* $x{\in}\mathbf{R}^2$. *Then* $G_{\theta(\lambda)}u_\lambda(x)$ *is a holomorphic function in* $\lambda{\in}(\mathbf{C}\backslash\{0\})\backslash T$ *for any* $x{\in}\mathbf{R}^2$.

Lemma A.8. *Let* $u{\in}C^{\alpha,1+\varepsilon}(\mathbf{R}^2,\mathbf{C})$, *where* $0{<}\alpha{<}1$ *and* $\varepsilon{>}0$. *Then*

$$\|G_{\pm,\theta}u\|_{\alpha,0}\le c_7(\alpha,\varepsilon)\|u\|_{\alpha,1+\varepsilon},\quad \theta\in\mathbf{S}^1, \tag{A.31}$$

$$\|G_{\theta}u\|_{\alpha,0}\le c_7(\alpha,\varepsilon)\|u\|_{\alpha,1+\varepsilon},\quad \theta\in\Sigma\backslash\mathbf{S}^1, \tag{A.32}$$

(A.33)
$$|G_{\pm,\theta}u(x)-G_{\pm,\theta'}u(x)|\le c_8(\alpha,\varepsilon,\beta)(1+|x|)^\varepsilon\|u\|_{\alpha,1+\varepsilon}\big(1+\big|\log|\theta-\theta'|\big|\big)|\theta-\theta'|^\beta,$$

$\theta,\theta'{\in}\mathbf{S}^1$, $x{\in}\mathbf{R}^2$, $0{<}\beta{<}\varepsilon$, $\beta{\le}\alpha$, *where* $c_7(\alpha,\varepsilon)$ *and* $c_8(\alpha,\varepsilon,\beta)$ *are positive constants.*

Lemmas A.6–A.8 are obtained in [N2].

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