Critical points of Green's function, harmonic measure, and the corona problem

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1. Introduction

Let $\mathscr{R}$ denote a Riemann surface and let $H^\infty(\mathscr{R})$ denote the collection of bounded analytic functions on $\mathscr{R}$. We assume that the functions in $H^\infty(\mathscr{R})$ separate the points of $\mathscr{R}$. Let $\mathscr{M}=\mathscr{M}(H^\infty(\mathscr{R}))$ denote the collection of all complex homomorphisms on $H^\infty(\mathscr{R})$, i.e. $\mathscr{M}$ is the maximal ideal space of $H^\infty(\mathscr{R})$. Each point $\zeta\in\mathscr{R}$ corresponds in a natural way (point evaluation) to an element of $\mathscr{M}$. This paper is concerned with the corona problem for $H^\infty(\mathscr{R})$: Is $\mathscr{M}$ the closure (in the Gelfand topology) of $\mathscr{R}$? More concretely, the problem is:

Given $f_1, \dots, f_n \in H^\infty(\mathscr{R})$ and $\delta>0$ such that $1 \geqq \max_j |f_j(\zeta)| > \delta$ for all $\zeta\in\mathscr{R}$, is it possible to find $g_1, \dots, g_n\in H^\infty(\mathscr{R})$ with

$$\sum_{j=1}^n f_j g_j = 1?$$

We refer to $f_1, \dots, f_n$ as "corona data", $g_1, \dots, g_n$ as "corona solutions", and $\max \|g_j\|_\infty$ as a "bound on the corona solutions". We reserve n and δ exclusively for the above use. This problem for $\mathscr{U}$, the unit disk in **C**, was conjectured by S. Kakutani in 1941, and solved by L. Carleson [12] in 1962. Not only has the theorem itself been of great interest in classical function theory, but also the proof introduced tools which have been of fundamental importance during the last twenty years. See Garnett's book [25] for an excellent discussion.

After the disk, the next most complicated Riemann surface is an annulus. The simplest proof of the corona theorem in this case is due independently to S. Scheinberg [38] and E. L. Stout [41]. We reproduce it here as motivation for our approach. First one pulls back corona data $\{f_j\}$ on the annulus, A, to functions on a vertical strip S by the map e^z. Since a strip is simply-connected, we may solve

$$1 = \sum f_j(e^z) g_j(z) \quad z\in S,\ g_j\in H^\infty(S) \tag{1.1}$$

by Carleson's theorem. Define operators $T_N(g)=\frac{1}{2N+1}\sum_{-N}^{N} g(z+2\pi ki)$ and apply them to equation (1.1) to obtain

$$1 = \sum f_j(e^z)T_N(g_j)(z).$$

Since $\|T_N(g_j)\|_\infty \leqq \|g_j\|_\infty$ and $\|T_N(g)(z)-T_N(g)(z+2\pi i)\|_\infty \leqq \|g\|_\infty/N$, we may find normal limits G_j of a subsequence of $\{T_N(g_j)\}$ with the properties

$$G_j(z+2\pi i) = G_j(z)$$

and

$$\sum f_j(e^z)G_j(z) = 1, \quad z\in S.$$

Clearly $G_j(z)=h_j(e^z)$ for some $h_j\in H^\infty(A)$ and

$$1 = \sum f_j(z)h_j(z), \quad z\in A.$$

For a more general Riemann surface $\mathcal{R}$, if $H^\infty(\mathcal{R})$ contains non-constant functions, the uniformization theorem tells us that the unit disk $\mathcal{U}$ is the universal covering space of $\mathcal{R}$. Moreover, there is a Fuchsian group Γ of linear fractional transformations of $\mathcal{U}$ onto $\mathcal{U}$ such that

$$H_\Gamma^\infty \equiv \{f\in H^\infty(\mathcal{U})\colon\ f\circ\gamma = f \text{ for all } \gamma\in\Gamma\}$$

is naturally isomorphic to $H^\infty(\mathcal{R})$ under the covering map $\pi\colon \mathcal{U}\to\mathcal{R}$. Solving the corona problem for $H^\infty(\mathcal{R})$ is then equivalent to finding solutions invariant under the group Γ in $H^\infty(\mathcal{U})$ when the corona data are invariant. We can retrieve the surface $\mathcal{R}$ from $\mathcal{U}$ and Γ by defining the normal fundamental domain

$$\mathcal{R}_0 = \{z\in\mathcal{U}\colon\ \varrho(z,0) < \varrho(z,\gamma(0)) \text{ for all } \gamma\in\Gamma\setminus\text{id}\}$$

where ϱ is the pseudo-hyperbolic metric on $\mathcal{U}$:

$$\varrho(z,w) = \left|\frac{z-w}{1-\bar{w}z}\right|.$$

The boundary of $\mathcal{R}_0$, denoted by $\partial\mathcal{R}_0$, consists of arcs of circles orthogonal to $\partial\mathcal{U}$, plus a subset of $\partial\mathcal{U}$. The map π is one-to-one on $\mathcal{R}_0$ and identifies the arcs of circles in pairs. An explicit construction of π and Γ in the case when $\mathcal{R}=\mathbf{C}\setminus E$ for $E\subset\mathbf{R}$ is given, for example, in [37]. Although Γ is countable, the averaging process in the above proof works only for the annulus. The contribution of this paper is to present a "weighted" averaging process to demonstrate the corona theorem on some surfaces. Our averaging process is closely related to a method introduced by Forelli [20].

A number of authors have proved the corona theorem for finite bordered Riemann surfaces, e.g. [5], [6], [18], [20], [40], [41], [42]. A general Riemann surface can always be exhausted by a sequence of finite bordered surfaces. However, to solve the corona problem on the larger surface, one needs control on the norms of the corona solutions, $\|g_j\|_\infty$, in the approximating surfaces in order to take normal limits. Unfortunately, the above cited methods do not produce such solutions. M. Behrens [7], [8] was the first to discover a class of infinitely connected planar domains for which he corona theorem is true. These are so-called "roadrunner sets" $\mathscr{R}=\mathscr{U}\setminus\bigcup_{j=1}^{\infty}\Delta_j$ where Δ_j is a disk centered at c_j and radius r_j such that

$$\sum_1^\infty r_j/|c_j| < \infty \quad \text{and} \quad |c_{j+1}/c_j| < \lambda < 1 \quad \text{for all } j.$$

The summability restriction has been improved somewhat in [16] and [17]. Behrens [8] also proved that if the corona theorem fails for a plane domain, then it fails for a domain of the form $\mathscr{R}=\mathscr{U}\setminus\bigcup_{j=1}^{\infty}\Delta_j$ where $\{\Delta_j\}$ is a sequence of disks clustering only at the origin. Gamelin [21] showed that the corona problem is "local" in the sense that it depends only on the behavior of $\mathscr{R}$ near each boundary point. Around 1970, B. Cole (see [22]) constructed a Riemann surface for which the corona theorem fails. He constructed a sequence of finite bordered Riemann surfaces $\mathscr{R}^{(k)}$ and functions $f_1^{(k)}, f_2^{(k)}\in H^\infty(\mathscr{R}_k)$ with $\max_j |f_j^{(k)}(t)|\geqq\delta$ on $\mathscr{R}^{(k)}$, but where any solutions $f_1^{(k)}g_1^{(k)}+f_2^{(k)}g_2^{(k)}=1$ must satisfy $\sup_k (\|g_1^{(k)}\|_\infty+\|g_2^{(k)}\|_\infty)=\infty$. Constructing such a sequence of *planar* domains is equivalent to the failure of the corona problem for a planar domain, as shown in Gamelin [21].

In 1980, Carleson [15] considered planar domains whose complements were at the opposite extreme from those considered by Behrens. A measurable set $E\subset\mathbf{R}$ is *homogeneous* if there is an $\varepsilon>0$ such that

$$|(x-r, x+r)\cap E| \geqq \varepsilon r \tag{1.2}$$

for all $r>0$ and all $x\in E$, where $|F|$ denotes the Lebesgue measure of a subset F of $\mathbf{R}$. Carleson proved the corona theorem for the surfaces $\mathscr{R}=\mathbf{C}\setminus E$. His proof is based on the recent solution by Jones [28] of the $\bar\partial$ problem, various difficult estimates on the harmonic measure of subsets of E in $\mathbf{C}\setminus E$, and a detailed examination of the corresponding Fuchsian groups. The motivation for our work was to understand this result. At the heart of our paper is another proof of Carleson's theorem. The general problem for planar domains is still open, as is the problem in several variables for the ball and polydisk.

It has long been known that the corona problem and interpolation problems are intimately related. A sequence $\{\beta_j\}\subset\mathscr{R}$ is an *interpolating* sequence for $H^\infty(\mathscr{R})$ provided the restriction of $H^\infty(\mathscr{R})$ to $\{\beta_j\}$ is ℓ^∞. In other words, for every bounded sequence $\{w_j\}$, there is an $f\in H^\infty(\mathscr{R})$ such that $f(\beta_j)=w_j$ for all j. When $\mathscr{R}=\mathscr{U}$, Carleson [11] proved that $\{\beta_j\}\subset\mathscr{U}$ is an interpolating sequence for $H^\infty(\mathscr{U})$ if and only of $\inf_j \prod_{i=1, i\neq j}^{\infty} \varrho(\beta_i, \beta_j)>0$. The problem of characterizing interpolating sequences for finite bordered Riemann surfaces is solved in [13], [40], [41], [43], and for certain infinitely connected planar domains in [9].

In Section 2, we show that it is enough to solve the corona problem at the critical points of Green's function, i.e. at $\{\zeta\colon \nabla G(\zeta,\zeta')=0\}$. For example, the corona theorem is true if the critical points lift, under the map π^{-1}, to an interpolating sequence for $H^\infty(\mathscr{U})$, or if the critical points form an interpolating sequence for $H^\infty(\mathscr{R})$. We express each of these conditions in terms of Green's function on $\mathscr{R}$. We also show, by example, these conditions are not the same. By comparing Green's function and harmonic measure we verify the interpolating condition, for example, in the cases Carleson considered in Section 3.

In his corona theorem for the complement of homogeneous sets E, Carleson showed that harmonic measure at a point in $\mathscr{R}$ is given by $dw=h(x)\,dx$ where h satisfies $he^{c(\log h)^{1/2}}\in L^1(dx)$. In [14] he conjectured that $h\in L^p$ for some $p>1$. We prove this fact in Section 3. Finally in Section 4, we give another characterization, in terms of harmonic measure, of sequences in $\mathscr{R}$ whose lifts to $\mathscr{U}$ are interpolating for $H^\infty(\mathscr{U})$. This is based on the work of Garnett, Gehring, and Jones [26] and Lavrentiev's theory of conformal mappings onto domains bounded by chord-arc curves [30]. It provides a simpler test of the interpolating condition in practice. For the reader interested in a short proof of Carleson's corona theorem for the complement of homogeneous sets, we suggest reading Section 2 through Theorem 2.5, then Section 3 through Lemma 3.2. There are several other ways to prove his theorem with the techniques presented herein.

The corona problems treated in this paper can also be studied using $\bar\partial$ methods. This approach will be used in a forthcoming paper by the first author.

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In sorrow we dedicate this paper to the memory of Irving Glicksberg, our mathematical grandfather.

Note: After this paper was first typed the first author and John Garnett succeeded in proving the corona theorem for all Denjoy domains, i.e. domains of the form $\overline{\mathbf{C}}\setminus E$ where $E\subset\mathbf{R}$. The techniques are necessarily quite different from those used in this paper. The proof will appear in Acta Math.

2. Interpolation and the corona theorem

In this section, we will consider Riemann surfaces $\mathcal{R}$ that are *regular* in the sense of potential theory. By this we mean that if $G(\zeta,\zeta')$ is Green's function for $\mathcal{R}$ with pole at ζ', then $G(\zeta,\zeta')$ tends to zero as ζ tends to the ideal boundary of $\mathcal{R}$. We may exhaust $\mathcal{R}$ by surfaces $\mathcal{R}_\varepsilon=\{\zeta\in\mathcal{R}\colon G(\zeta,\zeta')>\varepsilon\}$. Each $\mathcal{R}_\varepsilon$ is the interior of a compact bordered Riemann surface whose boundary consists of finitely many analytic Jordan curves in $\mathcal{R}$. Let $\{\zeta_m\}=\{\zeta\in\mathcal{R}\colon \nabla G(\zeta,\zeta')=0\}$ be the critical points of Green's function. The emphasis in this section will be on solving the corona problem in these latter surfaces, with control on the norms of the corona solutions in terms of δ, n, and the critical points. Since the Green's function for $\mathcal{R}_\varepsilon$ is simply $G(\zeta,\zeta')-\varepsilon$, our results apply to $\mathcal{R}$. Actually most of the results below already contain hypotheses that imply $\mathcal{R}$ is essentially regular. We will comment at the end of this section on how to remove the added hypothesis that $\mathcal{R}$ is regular. With the exception of Lemma 2.4 then, we will maintain the standing assumption that $\mathcal{R}$ is the interior of a compact bordered Riemann surface whose boundary in a larger Riemann surface consists of finitely many analytic curves, with the understanding that all of the theorems extend to regular Riemann surfaces as stated.

Since $\mathcal{R}$ admits non-constant bounded analytic functions, the Blaschke product $B(z)=\prod_{\gamma\in\Gamma}\frac{|\gamma(0)|}{\gamma(0)}\gamma(z)$, defined on $\mathcal{U}$, converges and clearly satisfies $-\log|B(z)|=G(\pi(z),\pi(0))$. Of course, we have adopted the usual convention of setting $|\gamma(0)|/\gamma(0)=1$ when $\gamma(z)\equiv z$. By *assumption*, the boundary of the normal fundamental domain $\mathcal{R}_0$ consists of finitely many arcs of circles orthogonal to $\partial\mathcal{U}$ and finitely many arcs (of positive length) on $\partial\mathcal{U}$. See e.g. Marden [31]. This function $B(z)$ is, then, analytic in a neighborhood of the closure, $\overline{\mathcal{R}}_0$, of $\mathcal{R}_0$.

We shall use the conditional expectation operator invented by Forelli [20], and later made explicit by Earle and Marden [18], [19]. By logarithmic differentiation,

$$B'(z)=\sum_{\gamma\in\Gamma}\frac{\gamma'(z)}{\gamma(z)}B(z).$$

It is easy to verify that this series converges uniformly and absolutely on $\overline{\mathcal{R}}_0$ and hence on $\gamma(\overline{\mathcal{R}}_0)$ for all $\gamma\in\Gamma$. Define, for $f\in H^\infty(\mathcal{U})$,

$$E(f)=\sum_{\gamma\in\Gamma}(f\circ\gamma)\frac{\gamma'}{\gamma}\frac{B}{B'}.$$

Then $E(f)$ is a meromorphic function on $\mathcal{U}$ with $E(f)\circ\gamma=E(f)$ for all $\gamma\in\Gamma$. The poles of $E(f)$ occur at the zeros of B'. These correspond to the critical points $\{\zeta_m\}$ of $G(\zeta,\pi(0))$ under the map π. Note that $E(1)=1$ and $E(fg)=fE(g)$ if

$f \in H_\Gamma^\infty$ and $g \in H^\infty(\mathscr{U})$. This operator has been used by these authors to solve the corona problem on finite bordered Riemann surfaces by finding $g \in H^\infty(\mathscr{U})$ with $E(g)=1$ and $E(gf) \in H_\Gamma^\infty$ for all $f \in H^\infty(\mathscr{U})$. The difficulty in extending this method to infinitely connected surfaces is the lack of control on $\|g\|_\infty$. We will give another method for removing the poles at the critical points that extends more readily to infinitely connected surfaces. For each m, choose one point $z_{m,0} \in \overline{\mathscr{R}}_0$ with $\pi(z_{m,0}) = \zeta_m$ and let $z_{m,k} = \gamma_k(z_{m,0})$ where $\{\gamma_k\}$ is an enumeration of Γ. We call $\{z_{m,k}\}$ the critical points of G on $\mathscr{U}$. We will make the further assumption that $G(\zeta, \pi(0))$ has no critical points on $\partial\mathscr{R}$. In other words, B' has no zeros on $\partial\mathscr{R}_0 \cap \partial\mathscr{U}$. There is no loss of generality in this assumption since any Riemann surface of the type we are considering may be approximated by surfaces with this additional assumption. If $\mathscr{R}$ is a planar domain, then it is a simple consequence of the argument principle that

(2.1) $G(\zeta, \pi(0))$ has $N-1$ critical points (counting multiplicity), where N is the number of closed boundary curves.

See e.g. [33]. More generally, the number of critical points of G is the first Betti number, or the number of generators of the first singular homology group, of $\mathscr{R}$ [46], and hence is finite. See Walsh [44, Chapter VII] for more information concerning the location of the critical points. We shall assume also that these critical points are distinct. Again, by an approximation, there is no loss of generality with this assumption.

Lemma 2.1. *Suppose* $f \in H^\infty(\mathscr{U})$ *and* $f(z_{m,k}) = f(z_{m,0})$ *for all* m *and* k. *Then* $E(f) \in H_\Gamma^\infty$ *and* $\|E(f)\|_\infty \leqq \|f\|_\infty$.

Proof. Since $E(1)=1$, we may write

$$E(f) = f + \sum_{\gamma \in \Gamma} (f \circ \gamma - f) \frac{\gamma'}{\gamma} \frac{B}{B'}.$$

Since by assumption $f \circ \gamma - f = 0$ at the zeros of B' (which are simple), and since B' is analytic in a neighborhood of $\overline{\mathscr{R}}_0$, $E(f)$ is bounded and analytic on $\mathscr{R}_0$. Since $E(f)$ is invariant under the group Γ, $E(f) \in H_\Gamma^\infty$. We recall the pleasant inequality that for $z \in \partial\mathscr{U}$, $z \dfrac{\gamma'(z)}{\gamma(z)} > 0$, and hence for each $z \in \partial\mathscr{R}_0 \cap \partial\mathscr{U}$ we have

$$|E(f)(z)| \leqq \|f\|_\infty \sum_{\gamma \in \Gamma} \left|\frac{\gamma'}{\gamma}\right| \left|\frac{B}{B'}\right| = \|f\|_\infty.$$

Since almost every point on $\partial\mathscr{U}$ is equivalent under the group Γ to some point on $\partial\mathscr{R}_0 \cap \partial\mathscr{U}$, and since $E(f)$ is invariant, $\|E(f)\|_\infty \leqq \|f\|_\infty$.

The next lemma highlights the importance of the critical points of G. It says that it is enough to solve the corona theorem there.

Lemma 2.2 *Suppose*

(i) $1 \geqq \max_{1 \leqq j \leqq n} |f_j(z)| \geqq \delta$ *for all* $z \in \mathscr{R}$

(ii) $g_j \in H^\infty(\mathscr{U})$, $\|g_j\|_\infty \leqq M$

(iii) $g_j \circ \gamma(z) = g_j(z)$ *for* $z \in \{z_{m,k}\}$, $\gamma \in \Gamma$

(iv) $\sum_{j=1}^n f_j(\pi(z)) g_j(z) = 1$ *for* $z \in \{z_{m,k}\}$.

Then there exists $\{h_j\} \in H^\infty(\mathscr{R})$ *such that* $\sum f_j h_j = 1$ *on* $\mathscr{R}$ *and* $\max_j \|h_j\|_\infty$ *is bounded by a constant depending only on n, δ, and M.*

Proof. Let C be the Blaschke product vanishing at $\{z_{m,k}\}$. We may write $\sum (f_j \circ \pi) g_j - 1 = Ch$ for some $h \in H^\infty(\mathscr{U})$. Applying the operator E we obtain $(E(1)=1)$:

$$\sum (f_j \circ \pi) E(g_j) - 1 = E(Ch). \tag{2.2}$$

By Carleson's corona theorem for $\mathscr{U}$, there are $H_j \in H^\infty(\mathscr{U})$ with $\|H_j\|_\infty$ bounded by a constant depending only on δ and n such that $\sum (f_j \circ \pi) H_j = 1$. Multiplying this latter equation by Ch and applying E yields

$$\sum (f_j \circ \pi) E(ChH_j) = E(Ch). \tag{2.3}$$

By (2.2) and (2.3),

$$\sum (f_j \circ \pi) E(g_j - ChH_j) = 1.$$

By Lemma 2.1, $E(g_j - ChH_j) \in H^\infty_\Gamma$ and $\|E(g_j - ChH_j)\|_\infty$ is bounded by a constant depending only upon n, δ, and M. The proof is completed by setting $h_j \circ \pi = E(g_j - ChH_j)$.

We remark here that this method for obtaining (2.3) shows that the canonical function $E(C) \in H^\infty_\Gamma$ is in every ideal generated by corona data. It merits further study. One easy way to now solve the corona problem for finite bordered Riemann surfaces is to, at each ζ_m, choose j such that $|f_j(\zeta_m)| > \delta$, and then pick $g_j \in H^\infty(\mathscr{R})$ such that $g_j(\zeta_m) = \dfrac{1}{f_j(\xi_m)}$ and $g_k(\zeta_m) = 0$ for $k \neq j$. Since there are only finitely many ζ_m, this can be easily done. We need, however, better control on $\|g_j\|_\infty$. The same idea proves our next corollary. The interpolation constant associated with an interpolating sequence $\{w_m\}$ is

$$M = \sup_{\|\{u_m\}\|_{\ell^\infty} \leqq 1} \inf \{\|f\|_\infty : f \in H^\infty(\mathscr{U}) \text{ and } f(w_m) = u_m \text{ for all } m\}.$$

Corollary 2.3. *Suppose the critical points* $\{z_{m,k}\}$ *form an interpolating sequence for* $H^\infty(\mathscr{U})$ *with interpolation constant M. Then the corona theorem is true on* $\mathscr{R}$ *with a bound on the solutions depending only on n, δ, and M.*

For later purposes, it will be necessary to have a concrete property of $\{w_m\}$ which is equivalent to the interpolation condition. Let $\eta=\eta(\{w_m\})=\inf_m \prod_{k:k\neq m} \varrho(w_k, w_m)$. A celebrated theorem of Carleson (see Garnett [25], p. 287) asserts (in its modern form) that

$$\frac{1}{\eta} \leqq M \leqq \left(\frac{C}{\eta}\right)\left(1+\log\frac{1}{\eta}\right), \tag{2.4}$$

where C is some absolute constant and M is the interpolation constant associated with $\{w_m\}$. It is also known (see [25], p. 294) that if $\{w_m\}$ is an interpolating sequence for $H^\infty(\mathcal{U})$ with interpolation constant M, there is a collection $\{f_m\}\subset H^\infty(\mathcal{U})$ with the following properties: $f_m(w_k)=0$ if $m\neq k$, $f_m(w_m)=1$, and $\sum |f_m(z)|\leqq M$ for all $z\in\mathcal{U}$. These functions were discovered by P. Beurling. See Jones [28] for a straightforward formula for them.

If Γ is any Fuchsian group of convergent type, and if $\{z_m\}=\{\gamma_m(a)\}_{m=0}^\infty$ for some $a\in\mathcal{U}$, then $\{z_m\}$ is an interpolating sequence. This follows from the invariance of the Blaschke product vanishing at $\{z_m\}$ and condition (2.4). The next lemma characterizes sequences in $\mathcal{R}$ that lift to interpolating sequences on $\mathcal{U}$ in terms of Green's function on $\mathcal{R}$. Since the proof is general we make *no* assumptions on the Riemann surface $\mathcal{R}$.

Lemma 2.4. *Let $\mathcal{R}$ be a Riemann surface and let Γ be the corresponding Fuchsian group. Suppose $\{z_m\}$ is a Γ - invariant sequence on $\mathcal{U}$ and suppose $\{\zeta_j\}=\pi\{z_m\}$ is the corresponding sequence on $\mathcal{R}$. Then $\{z_m\}$ is an interpolating sequence for $H^\infty(\mathcal{U})$ if and only if Green's function exists and satisfies:*

(2.5) *there is a constant α such that for all j, $\{\zeta\colon G(\zeta,\zeta_j)>\alpha\}$ is simply connected,*

and

(2.6) *there is a constant $N<\infty$ such that for all j*

$$\sum\nolimits_{k:k\neq j} G(\zeta_k,\zeta_j) \leqq N.$$

Proof. If $\{z_m\}$ is interpolating, then clearly Γ is of convergent type and hence G exists. We will estimate the quantity η of condition (2.4). To this end we pick a point $z\in\{z_m\}$ and evaluate the interpolating condition there. For ease of notation, we suppose $z=0$; the proof below can be easily modified otherwise. Let $\{z_{1,m}\}=\{\gamma(0)\colon \gamma\in\Gamma\}\setminus\{0\}$ and let $\{z_{2,m}\}=\{z_m\}\setminus\{\{z_{1,m}\}\cup\{0\}\}$. We must evaluate $\prod_{m:z_m\neq 0}|z_m|=\prod_m |z_{1,m}|\prod_m |z_{2,m}|$. The quantity $\prod_m |z_{2,m}|$ is equal to $\exp\left\{-\sum_{j:\zeta_j\neq\pi(0)} G(\zeta_j,\pi(0))\right\}$ so to prove the lemma, it is only necessary to show that $A\equiv-\log\prod_m |z_{1,m}|$ and α_0 are comparable, where α_0 is the infimum of the set of all α such that $\left\{\zeta\colon G(\zeta,\pi(0))>\alpha\right\}$ is simply connected. As before, let B be the

Blaschke product with $-\log|B(z)|=G(\pi(z), \pi(0))$. Then

$$|B'(0)| = e^{-A} = \prod_{\substack{\gamma\in\Gamma \\ \gamma(0)\neq 0}} |\gamma(0)| < |\tau(0)| \quad \text{for all} \quad \tau\in\Gamma\setminus id.$$

Thus the disk $D=\{z\colon |z|<e^{-A}/2\}$ is contained in the normal fundamental domain. Moreover, by Schwarz's lemma applied to $B(z)/z$, $|B(z)|\geqq e^{-2A}/4$ when $|z|=e^{-A}/2$. (See e.g. [33], p. 167.) Since D contains no two points equivalent under Γ, $\pi\{\zeta\colon G(\zeta, \pi(0))>2A+\log 4\}$ is simply connected. This proves $\alpha_0\leqq 2A+\log 4$. In the other direction, pick a number $\alpha>\alpha_0$ and let $\mathscr{D}=\{\zeta\colon G(\zeta, \pi(0))>\alpha\}$. Let V be the component of $\pi^{-1}(\mathscr{D})$ which contains the origin. Since V contains no $\gamma(0)$ with $\gamma\in\Gamma$, $\gamma\neq$ identity, and since $\pi^{-1}(\mathscr{D})$ is invariant, there is a neighborhood of fixed hyperbolic size about each zero of B, except 0, which does not intersect V. Since $B(z)$ is always an interpolating Blaschke product, $z/B(z)$ is bounded on V (although the bound might not, a priori, be controlled in terms of α). Moreover, $|z/B(z)|\leqq e^{\alpha}$ on $\mathscr{U}\cap\partial V$. Since $|B(z)|\leqq e^{-\alpha}<1$ on V, the harmonic measure of $\partial\mathscr{U}\cap\partial V$ in $\mathscr{U}$ is zero, and hence the harmonic measure of the same set in V is zero. By the maximum principle, $|z/B(z)|\leqq e^{\alpha}$ on V and therefore $e^{-A}=|B'(0)|\geqq e^{-\alpha}$. This shows $A\leqq\alpha_0$ and completes the proof of the lemma.

We remark that (2.5) controls the interpolation constant for each orbit, while (2.6) controls the interactions of the orbits. It is not immediately obvious how one can verify condition (2.6) in practice, but it so happens that there is a relatively straightforward way to do this. Section 4 is devoted to a discussion of this problem. It turns out, however, that condition (2.5) is relatively simple to handle. Suppose, for example, that $\mathscr{R}$ is a planar domain, $\zeta_0\in\mathscr{R}$ and $\operatorname{dist}(\zeta_0, \partial\mathscr{R})=1$. (The problem is dilation invariant, so there is no loss of generality connected with this last assumption.) Let $B(r)=\{\zeta\in\mathbf{C}\colon |\zeta-\zeta_0|\leqq r\}$ and let $\operatorname{cap}(E)$ denote the logarithmic capacity of a compact set $E\subset\mathbf{C}$. Then standard arguments of potential theory show that whenever $\{\zeta\colon G(\zeta, \zeta_0)>\alpha\}$ is simply connected, there is $r=r(\alpha)$ such that

$$\operatorname{cap}(B(r)\cap\partial\mathscr{R}) \geqq r^{-1}.$$

Conversely, if $\operatorname{cap}(B(r)\cap\partial\mathscr{R})\geqq r^{-1}$, there is then $\alpha=\alpha(r)$ such that $\{\zeta\colon G(\zeta, \zeta_0)>\alpha\}$ is simply connected.

It is possible for a sequence $\{\beta_j\}$ in a Riemann surface to be interpolating for $H^\infty(\mathscr{R})$ and yet its lift to $\mathscr{U}$ is not interpolating for $H^\infty(\mathscr{U})$. Indeed, let $\{\beta_j\}$ be an interpolating sequence in $\mathscr{U}$. Then $\{\beta_j\}$ is an interpolating sequence for $H^\infty(\mathscr{R})$ for all $\mathscr{R}\subset\mathscr{U}$. For each j delete a small disk $\Delta_j\subset\{\zeta\in\mathscr{U}\colon G(\zeta, \beta_j)>j\}$ from $\mathscr{U}$. If the radii of the Δ_j are sufficiently small and if $G_{\mathscr{R}}$ is Green's function for the new surface $\mathscr{R}=\mathscr{U}\setminus(\bigcup_j \Delta_j)$, then $\{\zeta\in\mathscr{R}\colon G_{\mathscr{R}}(\zeta, \zeta_j)>N\}$ will be doubly connected for $j>N$. By Lemma 2.4, $\{\beta_j\}$ does not lift to an interpolating sequence for $H^\infty(\mathscr{U})$.

As a consequence of Corollary 2.3 and Lemma 2.4 we obtain the following

Theorem 2.5. *Suppose conditions* (2.5) *and* (2.6) *hold for the critical points of* G. *Then the corona theorem has solutions with bounds depending only on* n, δ, α *and* N.

There are two other versions of this theorem we would like to give. In his original proof of the corona theorem [12, p. 557] Carleson showed that if C is any Blaschke product and if g is bounded and analytic on the (possibly not connected) set $\{z\colon |C(z)|<\varepsilon\}$ then there exists $g_1\in H^\infty(\mathcal{U})$ such that $g_1=g$ on the zeros of C and $\|g_1\|_\infty \leqq K\|g\|_{\infty,\varepsilon}$ where $\|g\|_{\infty,\varepsilon}$ denotes the sup-norm of g on $\{z\colon |C(z)|<\varepsilon\}$ and where K depends only on ε. Letting C denote the Blaschke product vanishing at the critical points of G in $\mathcal{U}$, we obtain the next lemma.

Lemma 2.6. *In Lemma* 2.2 *we may replace hypothesis* (ii) *by the following:*

$$\text{(ii')} \qquad g_j\in H^\infty\{z\in\mathcal{U}\colon |C(z)|<\varepsilon\} \quad \textit{with} \quad \|g_j\|_{\infty,\varepsilon}\leqq M.$$

We obtain the same conclusion except that the bounds on the corona solutions will also depend on ε.

Since $\pi\{z\in\mathcal{U}\colon |C(z)|<\varepsilon\}=\{\zeta\in\mathcal{R}\colon \sum_{k=1}^\infty G(\zeta,\zeta_k)>\log 1/\varepsilon\}$, we deduce the following version of Theorem 2.5, by applying Carleson's corona theorem for simply connected surfaces and Lemma 2.6 to Lemma 2.2.

Theorem 2.7. *If there is an* $N<\infty$ *such that* $\{\zeta\in\mathcal{R}\colon \sum_{k=1}^\infty G(\zeta,\zeta_k)>N\}$ *is a disjoint union of simply connected regions, then the corona theorem has solutions with bounds depending only on* n, δ, *and* N.

We say in this case that C separates the sheets, since the critical points on each sheet are separated from the critical points on the remaining sheets. Conversely, it is easy to see that if distinct, but equivalent, critical points on $\mathcal{U}$ belong to distinct components of $\{z\colon |C(z)|<e^{-N}\}$ then C separates the sheets. Of course we can replace "simply connected regions" in this theorem by regions with connectivity $\leqq m$. For example, let $\mathcal{R}=\mathcal{U}\setminus\bigcup_{j=1}^\infty \Delta_j$ be the Riemann surface obtained by deleting disks Δ_j with centers c_j from the unit disk $\mathcal{U}$. By Walsh's lemma (Lemma 3.1), if each $c_j\in(0,1)$, the critical points of $G(\zeta,0)$ lie in $(0,1)$, intertwined with the centers c_j. If $c_j\to 1$ geometrically then $\{c_j\}$ is an interpolating sequence for $H^\infty(\mathcal{U})$, and hence the set of critical points $\{\zeta_j\}$ is the union of at most two interpolating sequences for $H^\infty(\mathcal{U})$, and therefore for $H^\infty(\mathcal{R})$. Since Green's function for $\mathcal{R}$ is bounded above by Green's function for $\mathcal{U}$, $\{\zeta\colon \sum_{k=1}^\infty G(\zeta,\zeta_k)>N\}$ is at most doubly connected for sufficiently large N. By the proof of Theorem 2.7, we obtain the corona theorem for $\mathcal{R}$. In this regard, we mention that Gamelin [21] has shown that if the corona theorem fails for a plane domain, it fails for a domain of the form $\mathcal{U}\setminus\bigcup_{j=1}^\infty \Delta_j$ where Δ_j are disks with centers $c_j\to 1$. Such domains are regular, by

Wiener's criterion. In contrast to Theorem 2.7, we can also obtain the same result if we separate the orbits of the critical points.

Theorem 2.8. *Suppose there is an* $N<\infty$ *such that all* $\zeta\in\mathscr{R}$ *which satisfy* $G(\zeta,\zeta_k)>N$ *for some* k, *also satisfy*

$$\sum_{j:\, j\neq k} G(\zeta,\zeta_j) < N.$$

Then the corona theorem has solutions on $\mathscr{R}$ *with bounds depending only on* $(n,\delta,\ldots)$ *and* N.

We remark that if the above condition holds for some N_0, it holds for all $N>N_0$.

Proof. Let $\mathscr{D}=\left\{\zeta\in\mathscr{R}\colon \sum_{j=1}^{\infty} G(\zeta,\zeta_j)>3N\right\}$. By the maximum principle, each component of $\mathscr{D}$ must contain a critical point. If $\zeta\in\mathscr{D}$ and $G(\zeta,\zeta_k)>N$ for some N, then $G(\zeta,\zeta_k)>2N$. Thus $\mathscr{D}$ is the disjoint union of regions which contain exactly one critical point. We can easily use functions which are constant on components of $\pi^{-1}(\mathscr{D})$ to solve the corona problem at the critical points. The theorem now follows from Lemma 2.6.

We remark that the components of $\mathscr{D}$ need not be simply connected. If the critical points of G on $\mathscr{U}$ are interpolating, then the hypotheses of both Theorems 2.7 and 2.8 are satisfied. We proved Theorem 2.5 first because it is simpler. It relies only on, say, T. Wolff's proof (see e.g. [23]) of the corona theorem, and not the deep construction of Carleson. Moreover, combined with the results in Section 4, the hypotheses of Theorem 2.5 are easier to verify in practice.

The condition in Theorem 2.8 actually arises in a natural way. Suppose $\{w_k\}\subset\mathscr{R}$ is an interpolating sequence for $H^\infty(\mathscr{R})$. We may find $f_k\in H^\infty(\mathscr{R})$ such that $f_k(w_j)=0$ if $j\neq k$, $f_k(w_k)=1$, and $\|f_k\|_\infty\leqq M$ for all k. Thus

$$1-M\exp\left\{-\sum_{j:\, j\neq k} G(\zeta,w_j)\right\}\leqq 1-|f_k(\zeta)|\leqq |f_k(w_k)-f_k(\zeta)|\leqq 2M\exp\{-G(\zeta,w_k)\}.$$

Taking logarithms we obtain:

(2.7) There is an $N<\infty$ such that $G(\zeta,w_k)>N$ implies $\sum_{j:\, j\neq k} G(\zeta,w_j)<N$.

Theorem 2.9. *Suppose condition* (2.7) *holds for the critical points of* G *with constant* $N=N_0$. *Then a sequence* $\{\beta_j\}$ *in* $\mathscr{R}$ *is interpolating for* $H^\infty(\mathscr{R})$ *if and only if condition* (2.7) *holds for* $\{\beta_j\}$ *with some constant* $N=N_1$. *Moreover, the interpolation constant for* $\{\beta_j\}$ *can be taken to depend only on* N_0 *and* N_1.

We may thus reword Theorem 2.8 to say that if the critical points for Green's function form an interpolating sequence for $H^\infty(\mathscr{R})$ then the corona theorem is true on $\mathscr{R}$. It is easy to see from this theorem that if condition (2.7) holds for the critical points of G and if a sequence $\{\beta_j\}\subset\mathscr{R}$ lifts to an interpolating sequence on $\mathscr{U}$ for $H^\infty(\mathscr{U})$, then $\{\beta_j\}$ is an interpolating sequence for $H^\infty(\mathscr{R})$. However, the conditions in Lemma 2.4 and Theorem 2.9 are indeed different, for it is possible to select the disks Δ_j and points β_j in the example following Lemma 2.4 so that the critical points of G satisfy (2.7). It is also possible for the critical points to satisfy $\sup_k \sum_{j:j\neq k} G(\zeta_j, \zeta_k)<\infty$ yet condition (2.7) fails for ζ_k. Finally, it is easy to give examples of of Riemann surfaces $\mathscr{R}$ and sequences $\{\beta_j\}$ satisfying condition (2.7) yet $\{\beta_j\}$ is *not* interpolating for $H^\infty(\mathscr{R})$. Indeed, it is sufficient to take any two points in a Riemann surface which has a Green's function and for which $H^\infty(\mathscr{R})$ consists only of constants.

Proof of Theorem 2.9. We have already shown the necessity of condition (2.7) for $\{\beta_j\}$ to be interpolating. To prove sufficiency, it is enough to interpolate all sequences of 0's and 1's, with a uniform bound. Let Φ be a subset of $\{\beta_j\}$ and let D_1 and D_2 be the Blaschke products with $|D_1(z)|=\exp\{-\sum_{\beta_j\in\Phi} G(\pi(z), \beta_j)\}$ and $|D_2(z)|=\exp\{-\sum_{\beta_j\notin\Phi} G(\pi(z), \beta_j)\}$. If V is the set of all ζ such that $G(\zeta, \beta_j)<N_1$, for all j, then on the boundary of V, $\sum_{j=1}^\infty G(\zeta, \beta_j)\leqq 2N_1$, by condition (2.7). By the maximum principle $|D_1(z)D_2(z)|\geqq e^{-2N_1}$ for $z\in V$. By condition (2.7) again, we obtain $\max(|D_1(z)|, |D_2(z)|)\geqq e^{-2N_1}$ for all $z\in\mathscr{U}$. If C is the Blaschke product vanishing at the equivalents of the critical points of G, write $C=C_1C_2$ where the zeros of the Blaschke product C_2 consist of all zeros of C where $|D_1(z)|\geqq e^{-2N_1}$. We claim $\{D_1C_1, D_2C_2\}$ form corona data on $\mathscr{U}$. Fix $N>N_0$ to be determined later. Suppose first that z satisfies $G(\pi(z), \zeta_k)>N$ for some k. Note that for $p=1, 2$, Schwarz's lemma yields

$$|D_p(z)| \geqq |D_p(\pi^{-1}(\zeta_k))|-|D_p(z)-D_p(\pi^{-1}(\zeta_k))| \geqq |D_p(\pi^{-1}(\zeta_k))|-2e^{-N}.$$

If $|D_1(\pi^{-1}(\zeta_k))|\geqq e^{-2N_1}$, then by condition (2.7) and this latter inequality, $|D_1(z)C_1(z)|\geqq(e^{-2N_1}-2e^{-N})e^{-N_0}$. If $|D_1(\pi^{-1}(\zeta_k))|<e^{-2N_1}$ then again we conclude $|D_2(z)C_2(z)|\geqq(e^{-2N_1}-2e^{-N})e^{-N_0}$. Finally, if z satisfies $G(\pi(z), \zeta_k)\leqq N$ for all k, then by condition (2.7) and the maximum principle, $|C_j(z)|\geqq|C(z)|\geqq e^{-2N_0}$ for $j=1, 2$. That proves the claim if N is chosen sufficiently large. By Carleson's theorem, we may find $g_1, g_2\in H^\infty(\mathscr{U})$ such that

$$D_1C_1g_1+D_2C_2g_2 = 1$$

with $\|g_j\|_\infty$ bounded by a constant depending only on N_0 and N_1. Applying the expectation operator we obtain $E(D_1C_1g_1)+E(D_2C_2g_2)=1$. Since $D_jC_jg_j$ is invariant on the orbits of the critical points, $E(D_jC_jg_j)\in H_\Gamma^\infty$ by Lemma 2.1, and $\|E(D_jC_jg_j)\|_\infty\leqq\|g_j\|_\infty$. Since the series for E converges absolutely, and since $|D_j|$ is Γ-invariant for $j=1, 2$, $E(D_jC_jg_j)$ vanishes at the zeros of D_j. Thus $E(D_1C_1g_1)\circ\pi^{-1}$ is the desired interpolating function.

We remark here that this method of proof can be used to establish Theorem 2.8 without the use of Carleson's construction. If $\max_{1\leq j\leq n}|f_j(\zeta)|\geqq\delta$ for all $\zeta\in\mathscr{R}$, define sets $\Phi_1, ..., \Phi_n$ as follows. For each ζ_p, let k be the first integer such that $|f_k(\zeta_p)|\geqq\delta$. Then let $\zeta_p\in\Phi_j$ for all $j\neq k$ and $\zeta_p\notin\Phi_k$. Let C_j be the Blaschke product with $-\log|C_j(z)|=\sum_{\zeta_p\in\Phi_j}G(\pi(z),\zeta_p)$. Then as above $\{C_jf_j\}$ form corona data. The proof is completed as in the above proof. This was actually the first proof we discovered.

We mention one more method of solving the corona problem. The previous methods have the drawback that Green's function does not ignore subsets of $\partial\mathscr{R}$ which have zero analytic capacity and positive logarithmic capacity. To avoid this we can use Ahlfors' function, A, instead. Ahlfors' function for a point $\zeta_0\in\mathscr{R}$ is defined by

$$A'_{\zeta_0}(\zeta_0) = \sup\{\operatorname{Re} f'(\zeta_0)\colon f\in H^\infty(\mathscr{R}),\ \|f\|_\infty\leqq 1\}.$$

For plane domains with compact complement, one usually studies the Ahlfors' function for ∞, where $f'(\infty)$ is defined to be $\lim_{z\to\infty} z(f(z)-f(\infty))$. Then $A'_\infty(\infty)$ is called the analytic capacity of the complement of $\mathscr{R}$. For example, if E is a compact subset of $\mathbf{R}$ and $\mathscr{R}=\mathbf{C}\setminus E$ then

$$A_\infty(\zeta) = \frac{\exp\left(\frac{1}{2}\int_E \frac{dt}{t-z}\right)-1}{\exp\left(\frac{1}{2}\int_E \frac{dt}{t-z}\right)+1}.$$

See for example [24, p. 30]. Ahlfors [1], [2] has shown that for our "nice" Riemann surfaces

$$|A_{\zeta_0}(\zeta)| = \exp\{-\textstyle\sum_{j=0}^{n-1} g(\zeta,\zeta_j)\}$$

for some points $\zeta_1, ..., \zeta_{n-1}\in\mathscr{R}$. Thus if $\pi(0)=\zeta_0$ and $\{z_{j,0}\}_{j=0}^{n-1}$ are equivalents of $\{\zeta_j\}_{j=0}^{n-1}$ on $\mathscr{R}_0$, we may write

$$A_{\zeta_0}(\pi(z)) = \textstyle\prod_{j=0}^{n-1}\prod_{\gamma\in\Gamma}\dfrac{|\sigma_{\gamma,j}(0)|}{\sigma_{\gamma,j}(0)}\sigma_{\gamma,j}(z)$$

where $\sigma_{\gamma,j}(z)=\dfrac{\gamma^{-1}(z_{j,0})-z}{1-\overline{\gamma^{-1}(z_{j,0})}z}$. If $\gamma,\tau\in\Gamma$, then $\dfrac{(\sigma'_{\gamma,j}\circ\tau)\tau'}{\sigma_{\gamma,j}\circ\tau}=\dfrac{\sigma'_{\gamma\circ\tau,j}}{\sigma_{\gamma\circ\tau,j}}$. So we define, for $f\in H^\infty(\mathscr{U})$

$$E_1(f) = \textstyle\sum_{\gamma\in\Gamma}\sum_{j=0}^{n-1}(f\circ\gamma)\left(\dfrac{\sigma'_{\gamma,j}}{\sigma_{\gamma,j}}\right)\dfrac{A_{\zeta_0}\circ\pi}{(A_{\zeta_0}\circ\pi)'}.$$

We let $\{w_{j,0}\}_{j=1}^{2(n-1)}$ be equivalents of the critical points of A_{ζ_0} (i.e. where $A'_{\zeta_0}=0$) on $\overline{\mathscr{R}}_0$ and let $w_{j,k}=\gamma_k(w_{j,0})$, where $\{\gamma_k\}$ is an enumeration of Γ. Then all of the results of this section hold for the critical points of Ahlfors' function $\{w_{j,k}\}$ as well as for the critical points of G. One can easily construct Riemann surfaces where

$\sum_k G(\zeta_k, \zeta')=\infty$, so that the methods using the critical points of G will not work, yet this method using Ahlfors' function gives solutions to the corona problem. If $\mathscr{R}=\mathbf{C}\setminus E$ where $E=\bigcup_{i=1}^n [a_i, b_i]$, $a_i \leqq b_i \leqq a_{i+1}$, then the zeros $\{w_k\}_1^{2n}$ of $A'_\infty(z)$ occur exactly when

$$\sum_{i=1}^n \frac{1}{z-a_i} - \frac{1}{z-b_i} = 0.$$

See Walsh [44, p. 139] for one method of locating these zeros. To apply Theorem 2.7, for example, it suffices to show

$$\sum_{k=1}^{2n} G(x, w_k) \leqq M < \infty$$

when $x \in \mathbf{R}\setminus E$, to obtain corona solutions with a bound depending only on M. See Widom [45] for a construction of Green's function for these surfaces and see Rubel and Ryff [37] for an explicit construction of their Fuchsian groups.

We remark that we chose the Ahlfors function here because of its natural association with $H^\infty(\mathscr{R})$, but we could have chosen any function $F \in H^\infty(\mathscr{U})$ with

$$-\log|F(z)| = \sum_{j=1}^m G(\pi(z), \alpha_j), \quad \alpha_j \in \mathscr{R}.$$

Carleson proved the corona theorem for domains $\mathscr{R}=\mathbf{C}\setminus E$ where E is homogeneous (see Section 3 for a definition), by producing projection operators $P\colon H^\infty(\mathscr{U}) \to H^\infty_\Gamma$ such that $P(1)=1$, $P(fg)=fP(g)$ if $f \in H^\infty_\Gamma$, and $\|P(f)\|_\infty \leqq C\|f\|_\infty$. We prove in Section 3 that such surfaces satisfy the hypotheses of Theorem 2.5. The next theorem constructs these projection operators in the more general context of Theorem 2.7.

Theorem 2.10. *Suppose that the hypotheses of Theorem* 2.7 *hold. Then there exists a (linear) projection P of $H^\infty(\mathscr{U})$ onto H^∞_Γ such that*

(i) $P(1) = 1$

(ii) $P(fg) = fP(g)$ *if* $f \in H^\infty_\Gamma$

(iii) $\|P(f)\|_\infty \leqq K\|f\|_\infty$

where K depends only on the constant N appearing in the hypotheses of Theorem 2.7.

Proof. As before, let C be the Blaschke product vanishing at the critical points on $\mathscr{U}$. By hypothesis, we may find a subset $W=\{w_{m,0}\}$ of $\{z_{m,j}\}$ such that $w_{p,0}$ and $w_{q,0}$ are not Γ-equivalent if $p \neq q$, and such that all $z_{n,j}$ in the same component of $\{z\colon |C(z)|<e^{-N}\}$ as some $w_{m,0} \in W$ also belong to W. So W consists of all the critical points on one sheet of $\mathscr{R}$. Let $w_{m,k}=\gamma_k(w_{m,0})$ where $\{\gamma_k\}$ is an enumeration of Γ. Let $A=\{f \in H^\infty(\mathscr{U})\colon f(w_{m,k})=f(w_{1,k})$ for all $k\}$ be the algebra of functions which are constant on the critical points of each sheet. By Carleson's theorem [12,

p. 557] $\{w_{1,k}\}$ is an interpolating sequence for A with interpolating constant K, depending only on N. By a theorem of Varopolous (see Garnett [25, p. 294]), there are functions $F_k \in A$ such that $F_k(w_{m,j})=0$ if $j\neq k$, $F_k(w_{m,k})=1$, and $\sum_{k=1}^{\infty}|F_k(z)|\leqq K^2$ for all $z\in\mathcal{U}$. For $f\in H^{\infty}(\mathcal{U})$, define

$$T(f) = f+\sum_{\gamma_k\in\Gamma}(f\circ\gamma_k^{-1}-f)\,F_k.$$

Then T satisfies (i), (ii), and (iii) (with a different constant K) and is invariant on the orbits of the critical points. By Lemma 2.1, $P\equiv ET$ does the job.

We conclude this section with some remarks on the applicability of our results. As mentioned at the beginning of this section, it is easy to replace the hypothesis that $\mathcal{R}$ is bounded by finitely many analytic curves with the hypothesis that $\mathcal{R}$ is regular. All of the hypotheses of the theorems imply that $\sum_{k=1}^{\infty}G(\zeta_k,\zeta')<\infty$. Unfortunately, this sum is not very stable. Removing countably many points from $\mathcal{R}$ does not affect the corona theorem, nor Green's function. However, if these points happen to be at the critical points, there would be no critical points left in the new surface! For regular Riemann surfaces it is shown in [46] that

$$\sum_{k=1}^{\infty}G(\zeta_k,\zeta') = \int_0^{\infty}\beta(\varepsilon)\,d\varepsilon \tag{2.8}$$

where $\beta(\varepsilon)$ is the first Betti number of $\mathcal{R}_{\varepsilon}=\{\zeta\in\mathcal{R}\colon G(\zeta,\zeta')>\varepsilon\}$. For an arbitrary Riemann surface, the right-hand side of (2.8) is more tractable. In fact, if $\{\mathcal{D}_{\varepsilon}\}_{\varepsilon\to 0}$ is any exhaustion of $\mathcal{R}$, the corresponding integrals are continuous as $\varepsilon\to 0$. Surfaces for which $\int_0^{\infty}\beta(\varepsilon)\,d\varepsilon<\infty$ have been extensively treated, e.g. by Widom [45], [46], [47] and Pommerenke [34], [35] and are called of "Widom-type". It is shown in [32] that Cole's example of a surface where the corona theorem fails can be modified to be of Widom-type, and it is possible to show that if the corona theorem fails for a plane domain, it must fail for a plane domain of Widom-type by modifying the construction in Gamelin [21]. One can replace all our sums of the form $\sum G(\zeta_k,\zeta)$ by integrals of the form $\int_0^{\infty}\beta_{\zeta}(\varepsilon)\,d\varepsilon$. Indeed, it is possible to show using results in [46] and [35] that if a projection operator of the form constructed in Theorem 2.10 exists for a Riemann surface then the surface must be of Widom-type. This approach, however, becomes transparent after observing the following lemma.

Lemma 2.11. *If $\mathcal{R}$ is a Riemann surface of Widom-type, then there is a Riemann surface $\mathcal{S}\supset\mathcal{R}$ such that $\mathcal{S}\setminus\mathcal{R}$ is countable and $\mathcal{S}$ is regular.*

Proof. Let $\mathcal{R}_{\varepsilon}=\{\zeta\in\mathcal{R}\colon G(\zeta,\zeta')>\varepsilon\}$. Since $\mathcal{R}$ is Widom-type, $\beta(\varepsilon)<\infty$. In other words, $\mathcal{R}_{\varepsilon}$ is of finite topological type. By a theorem of Stout [41], $\mathcal{R}_{\varepsilon}$ is conformally equivalent to a finite bordered Riemann surface $\mathcal{S}_{\varepsilon}$ with finitely many isolated points removed. We may take the boundary of $\mathcal{S}_{\varepsilon}$ to consist of finitely many analytic curves in the double of $\mathcal{S}_{\varepsilon}$ (see [4], p. 27 for a definition). It is easy to see then that by adding countably many points to $\mathscr{R}$ we obtain a Riemann surface $\mathscr{S}$ with the same Green's function G such that $\{\zeta\colon G(\zeta,\zeta')>\varepsilon\}$ is conformally equivalent to a finite bordered Riemann surface for each $\varepsilon>0$. If there exists a sequence z_n tending to the ideal boundary of $\mathscr{S}$ such that $G(z_n,\zeta')>\varepsilon_0>0$, then for $\varepsilon_1<\varepsilon_0$, z_n belongs to the regular Riemann surface $\{\zeta\colon G(\zeta,\zeta')>\varepsilon_1\}$ and tends to its boundary. This is a contradiction.

It would, of course, be most natural to replace the assumption that $\mathscr{R}$ is regular by the assumption that $\mathscr{R}$ is maximal in the sense that it is not contained in a large Riemann surface $\mathscr{S}$ with $H^\infty(\mathscr{R})=\{f|_{\mathscr{R}}\colon f\in H^\infty(\mathscr{S})\}$. Unfortunately, one can build such a surface $\mathscr{R}$ such that $\mathscr{R}$ is infinitely connected, $H^\infty(\mathscr{R})\neq$constants, and yet there is a point $\zeta_0\in\mathscr{R}$ such that $\operatorname{grad} G(\zeta,\zeta_0)$ is nonzero for all $\zeta\in\mathscr{R}$. To build such a surface, let $\mathscr{R}_1$ be the annulus $\{\zeta\colon 1/3<|\zeta|<1\}$ and suppose by induction that surfaces $\mathscr{R}_2,\dots,\mathscr{R}_m$ have been constructed, $\mathscr{R}_1\supset\mathscr{R}_2\supset\dots\supset\mathscr{R}_m$, and that $\mathscr{R}_m$ is $\mathscr{R}$ minus a finite number of closed disks. Let G_m be Green's function for $\mathscr{R}_m$ and let $\Delta_1^m,\dots,\Delta_N^m$ be small closed disks which cover $\{\zeta\in\mathscr{R}_m\colon \operatorname{grad} G_m(\zeta,1/2)=0\}$. Now set $\mathscr{R}_{m+1}=\mathscr{R}_m\setminus\bigcup_{j=1}^N\Delta_j^m$ and take $\mathscr{R}$ to be the interior of $\bigcap_m\mathscr{R}_m$. Then if the disks $\{\Delta_j^m\}$ are chosen small enough at each stage, $\mathscr{R}$ will be connected and infinitely connected, $\mathscr{R}$ will be maximal in the required sense, and grad $G(\zeta,1/2)$ will never vanish on $\mathscr{R}$. (This surface $\mathscr{R}$ necessarily has the property that it is not regular in the sense of potential theory.) One could combine the above argument with the argument in [32] to produce such a Riemann surface which has the additional property that the corona theorem fails there. We leave the proofs of these assertions as exercises for the reader.

3. Harmonic measure on homogeneous sets

Recall that a set $E\subset\mathbf{R}$ is homogeneous if there is an $\varepsilon>0$ such that

$$(3.1)\qquad |(x-r,x+r)\cap E|\geqq\varepsilon r\quad\text{for all}\quad r>0\quad\text{and all}\quad x\in E.$$

The emphasis in this section will be on Riemann surfaces $\mathscr{R}$ of the form $\mathbf{C}\setminus E$ where E is homogeneous. We shall first give a proof of Carleson's corona theorem for these surfaces. Without loss of generality, we assume $0\in\mathscr{R}$ and that the universal covering map π satisfies $\pi(0)=0$. For technical ease we assume that $\partial\mathscr{R}=E$ consists of a finite number of closed intervals (some of which will be half-lines). This restriction can be easily removed by a normal families argument. Let $\{L_j\}$ denote the open intervals in $\mathscr{R}\cap\mathbf{R}$ and let $0\in L_0$. Denote a general complex number ζ by $\zeta=\xi+i\eta$, $\xi,\eta\in\mathbf{R}$. Then by symmetry, $\dfrac{\partial}{\partial\eta}G(\zeta,0)=0$ whenever $0\neq\zeta\in\mathbf{R}\cap\mathscr{R}$. Since $G(\zeta,0)\to0$ as $\zeta\to\partial\mathscr{R}$, $G(\zeta,0)$ has a local maximum (with respect to the variable ξ) in each interval L_j, $j\neq 0$. Thus, $G(\zeta, 0)$ has at least one critical point in each L_j, $j\neq 0$. Combining this observation with (2.1) we obtain

Lemma 3.1. *The function* $G(\zeta, 0)$ *has exactly one critical point in each* L_j, $j\neq 0$, *and these are all the critical points.*

(This result is due to Walsh, 1933. See his book [44], Corollary 3, p. 249.)

Carleson's theorem will therefore follow from Theorem 2.5 as soon as we verify

Lemma 3.2. *Suppose we arbitrarily pick exactly one point* ζ_j *from each slit* L_j. *Then there are constants* α *and* N *which depend only on the value of* ε *in* (3.1) *such that* (2.5) *and* (2.6) *hold.*

Proof of (2.5). This is recorded in Carleson's paper [15] in his estimates (14) and (15), p. 355. We recall the proof in a sketched form. Denote by d_j the quantity $\operatorname{dist}(\zeta_j, \partial\mathscr{R})$ (Euclidean distance), and denote by $\mathscr{D}_j$ the disk $\{\zeta\colon |\zeta-\zeta_j|\leqq 1/2\, d_j\}\subset\mathscr{R}$. Then since the logarithmic capacity of $\partial\mathscr{R}\cap\{x\in\mathbf{R}\colon |x-\zeta_j|\leqq 2d_j\}$ is comparable to the capacity of a Euclidean disk of radius d_j (this follows immediately from condition (3.1)), $G(\zeta,\zeta_j)\leqq\alpha=\alpha(\varepsilon)$ whenever $\zeta\in 2\mathscr{D}_j$. Condition (2.5) now follows because $\mathscr{D}_j\subset\mathscr{R}$ and $\mathscr{D}_j$ is obviously simply connected. (See also the discussion in the paragraph which follows the proof of Lemma 2.4.)

Proof of (2.6). For a set Borel $E\subset\partial\mathscr{R}$, let $w(\zeta, E)$ denote the (positive) bounded harmonic function on $\mathscr{R}$ with boundary values 1 a.e. dx on E and 0 a.e. dx on $\partial\mathscr{R}\setminus E$. (In other words, $w(\zeta, E)$ is the harmonic measure of E at ζ.) Suppose we can find sets $E_j\subset\partial\mathscr{R}$ such that

$$E_j \subset I_j \equiv \{x\in\mathbf{R}\colon |x-\zeta_j| \leqq 2d_j\} \tag{3.2}$$

and

$$|E_j| \geqq \varepsilon/2\, d_j,$$

and

$$\|\textstyle\sum_j \chi_{E_j}\|_{L^\infty} \leqq C(\varepsilon). \tag{3.3}$$

Let $f_j(\zeta)$ be the function defined on the upper half-plane $\{\zeta=\xi+i\eta\colon \eta>0\}$ which is bounded and harmonic there and has nontangential boundary values on $\mathbf{R}$ equal to $\chi_{E_j}(x)$. In other words, using the classical formula for the Poisson kernel, we have

$$f_j(\zeta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta}{(x-\xi)^2+\eta^2}\,\chi_{E_j}(x)\,dx. \tag{3.4}$$

Then by the above formula and the assumption (3.2) we see that $f_j(\zeta_j+id_j)\geqq (5\pi d_j)^{-1}|E_j|\geqq(10\pi)^{-1}\varepsilon$. By the maximum principle, $w(\zeta_j+id_j, E_j)\geqq f_j(\zeta_j+id_j)$, so by Harnack's inequality on the strip $\{z=x+iy\colon |x-\zeta_j|<d_j\}$, $w(\zeta, E_j)\geqq A\varepsilon$ for all

$\zeta\in\mathscr{D}_j$, where A is a universal constant independent of $\mathscr{R}$. Recall from the last paragraph that $G(\zeta,\zeta_j)\leqq\alpha(\varepsilon)$ for $\zeta\in\partial\mathscr{D}_j$. By the maximum principle, $G(\zeta,\zeta_j)\leqq\alpha(\varepsilon)(A\varepsilon)^{-1}w(\zeta,E_j)$ for all $\zeta\notin\mathscr{D}_j$. Fix now one point ζ_j and let $dw_j(x)$ denote the element of harmonic measure with respect to ζ_j. Then by our last inequality and assumption (3.3),

$$\sum_{\substack{k\\k\neq j}} G(\zeta_k,\zeta_j)=\sum_{\substack{k\\k\neq j}} G(\zeta_j,\zeta_k)\leqq\alpha(\varepsilon)(A\varepsilon)^{-1}\sum_{\substack{k\\k\neq j}} w(\zeta_j,E_k)$$

$$=C'(\varepsilon)\sum_{\substack{k\\k\neq j}}\int\chi_{E_k}(x)\,dw_j(x)\leqq C'(\varepsilon)\left\|\sum_k\chi_{E_k}\right\|_{L^\infty}\leqq C'(\varepsilon)C(\varepsilon),$$

because dw_j is a probability measure. The first inequality above is valid because $\zeta_j\notin\mathscr{D}_k$ when $k\neq j$.

To finish the proof of the lemma we need now only produce the sets E_j which satisfy (3.2) and (3.3). Let $\tilde{E}_j=I_j\cap\partial\mathscr{R}$. Then by the hypothesis (3.1) on $\mathscr{R}$, $|\tilde{E}_j|\geqq\varepsilon d_j$. Let $b_j(x)=\sum_{\substack{k\\d_k\leqq d_j}}\chi_{\tilde{E}_k}(x)\chi_{I_j}(x)$. Since the intervals $J_k=\{x\in\mathbf{R}:|x-\zeta_k|\leqq d_k\}$ are pairwise disjoint, we have

$$\int b_j(x)\,dx=\sum_{\substack{k\\d_k\leqq d_j}}|\tilde{E}_k\cap I_j|\leqq\sum_k\{|\tilde{E}_k|:\ |\zeta_k-\zeta_j|\leqq 4d_j,\ d_k\leqq d_j\}$$

$$\leqq 2\sum_k\{|J_k|:\ |\zeta_k-\zeta_j|\leqq 4d_j,\ d_k\leqq d_j\}\leqq 2|\{x:\ |x-\zeta_j|\leqq 5d_j\}|\leqq 20d_j.$$

Let $K_j=\left\{x:\ b_j(x)\geqq\dfrac{40}{\varepsilon}\right\}$. By Tchebychev's inequality, $|K_j|\leqq\varepsilon/2d_j$, so that if we define $E_j=\tilde{E}_j\setminus K_j$, $E_j\subset I_j$ and $|E_j|=|\tilde{E}_j|-|K_j|\geqq\varepsilon d_j-\varepsilon/2d_j$, i.e. condition (3.2) holds. To see that condition (3.3) holds, suppose that $\sum\chi_{E_j}(x)\geqq N\geqq\dfrac{40}{\varepsilon}$. By relabeling we may assume that $x\in E_1,\dots,E_N$ where $d_1\leqq d_2\leqq\dots\leqq d_N$. Then by the definition of the function b_N it must be that $b_N(x)\geqq N$. This is a contradiction, for then x would be in K_N and hence x would not be in E_N. This completes the proof of (3.3), and thus the proof of Lemma 3.2 and Carleson's theorem.

We remark here that our proof did not require that there was only one ζ_j in each L_j. For example, if $\zeta_j\in\mathbf{R}\setminus E$ and the intervals $\{x\in\mathbf{R}:\ |x-\zeta_j|\leqq\delta\operatorname{dist}(\zeta_j,E)\}$ are disjoint for some $\delta>0$, then the same proof applies. This gives the corona theorem for some surfaces $\mathbf{C}\setminus E$ where E is not homogeneous. As in the example following Theorem 2.7, if E is homogeneous, we may remove from $\mathscr{R}=\mathbf{C}\setminus E$ intervals centered in such a sequence ζ_j. The resulting critical points form a union of at most two interpolating sequences, and hence the corona theorem holds for such surfaces. By making the removed intervals sufficiently small we obtain a surface whose complement is not homogeneous.

We now turn to a closer study of harmonic measure in domains $\mathscr{R}$ satisfying the hypothesis (3.1) and answer a question raised by Lennart Carleson. Suppose $0\in\mathscr{R}$ and suppose also that $(-1, 1)$ is the largest interval which is both symmetric about 0 and does not intersect $\partial\mathscr{R}$. Let $w_0(E)$ denote the harmonic measure of a set $E\subset\partial\mathscr{R}$, evaluated at 0. Carleson proved in [15], Lemma 4 that there is an $L^1(dx)$ function h on $\partial\mathscr{R}$ such that $w_0(\sigma)=\int_\sigma h\,dx$ for all measurable σ, and such that

$$\int_{\partial\mathscr{R}} h(x)\exp\{c_1(\log^+ h(x))^{1/2}\}\,dx \leqq C(\varepsilon).$$

Carleson asked in [14] if this could be improved to show that $h\in L^p(dx)$ for some $p>1$. The answer is given by

Theorem 3.3. *There are $p=p(\varepsilon)>1$ and $C=C(\varepsilon)<\infty$ such that $\|h\|_{L^p(dx)}\leqq C$.*

For a positive number λ and an interval $I\subset\mathbf{R}$, denote by λI the interval with the same center as I and length $|\lambda I|=\lambda|I|$.

Lemma 3.4. *There is a constant $C(\varepsilon)<\infty$ such that for every interval I of the form $[x_0-\delta, x_0+\delta]$, where $x_0\in\partial\mathscr{R}$ and $0\notin 3I$, there exists a positive function $\tilde{h}(x)=\tilde{h}_I(x)$ with the properties*

(i) $\tilde{h}=h$ *on* $I\cap\partial\mathscr{R}$

(ii) $\int_{I\setminus\partial\mathscr{R}}\tilde{h}\,dx \leqq C(\varepsilon)\int_{I\cap\partial\mathscr{R}} h\,dx$

(iii) *For all finite intervals* $J\subset\mathbf{R}$,

$$\frac{1}{|J|}\int_J \tilde{h}\,dt \leqq C(\varepsilon)\tilde{h}(x)$$

for a.e. $x\in J$.

Once we have established Lemma 3.4, Theorem 3.3 will follow. Indeed, condition (iii) says that $\tilde{h}$ belongs to the Muckenhoupt class A_1 and hence (see e.g. [25]) satisfies a reverse Hölder inequality:

$$\left(\frac{1}{|I|}\int_I \tilde{h}^{1+\beta}\,dx\right)^{1/(1+\beta)} \leqq \frac{C}{|I|}\int_I \tilde{h}\,dx,$$

where $\beta=\beta(\varepsilon)>0$ and $C=C(\varepsilon)$. By conditions (i) and (ii) we conclude

$$\left(\int_{I\cap\partial\mathscr{R}} h^{1+\beta}\,dx\right)^{1/(1+\beta)} \leqq C|I|^{-\beta/(1+\beta)}\int_{I\cap\partial\mathscr{R}} h\,dx.$$

It is an easy exercise to cover $\partial\mathscr{R}$ by a collection of intervals $\{I_j\}$ such that each I_j satisfies the hypotheses of Lemma 3.4, $|I_j|\geqq 2/3$ for all j, and $\sum\chi_{I_j}\leqq 2$. Then by our last inequality,

$$\Big(\int_{\partial\mathcal{R}} h^{1+\beta}\,dx\Big)^{1/(1+\beta)} \leqq \sum_j \Big(\int_{I_j\cap\partial\mathcal{R}} h^{1+\beta}\,dx\Big)^{1/(1+\beta)}$$
$$\leqq C(2/3)^{-\beta/(1+\beta)} \sum_j \int_{I_j\cap\partial\mathcal{R}} h\,dx \leqq 2C(2/3)^{-\beta/(1+\beta)} \int_{\partial\mathcal{R}} h\,dx = 2C(2/3)^{-\beta/(1+\beta)}.$$

It should be noted here that when condition (3.1) holds it is *never* the case that h itself is the restriction to $\partial\mathcal{R}$ of an A_1 weight. This follows from the fact that no A_1 weight lies in $L^1(dx)$. Thus h can be the restriction of an A_1 weight only in a local sense. It is an exercise to show that for general domains of the form $\mathcal{R}=\mathbf{C}\setminus E$, $E\subset\mathbf{R}$, one can have $dw_0=h\,dx$, $h\in L^1(dx)$, $(-1,1)\subset\mathcal{R}$, $1\in E$ and $h|_{E\cap[1,2]}$ is not the restriction of any A_∞ weight (see [25] for a definition) to $E\cap[1,2]$. The reader should consult the recent paper of Wolff [48] for related extension problems with A_p weights. We remark that the following proof is easily modified to give an $\mathbf{R}^n$ version of Lemma 3.4 and hence of Theorem 3.3.

Our first step is to prove a lemma for general domains $\mathcal{D}\subset\mathbf{C}$ of the form $\partial\mathcal{D}\subset\mathbf{R}$, where we assume *nothing* further about the structure of $\partial\mathcal{D}$ except that the logarithmic capacity of $\partial\mathcal{D}$ is positive. Assume that $0\in\mathcal{D}$. Then since the logarithmic capacity of $\partial\mathcal{D}$ is positive, there is for each $\zeta\in\mathcal{D}$ a Borel probability measure, dw_ζ, supported on $\partial\mathcal{D}$ such that when one solves the Dirichlet problem $\Delta u\equiv 0$ on $\mathcal{D}$, $u=f$ on $\partial\mathcal{D}$ (except on a set of null logarithmic capacity), f continuous on $\partial\mathcal{D}$, one has $u(\zeta)=\int_{\partial\mathcal{D}} f\,dw_\zeta$. For a Borel set $E\subset\partial\mathcal{D}$ one then defines the harmonic measure of E at ζ by

$$w(\zeta, E, \mathcal{D}) = \int_E dw_\zeta.$$

It is well-known that, for fixed E, $w(\zeta, E, \mathcal{D})$ is harmonic in ζ. The main idea for the proof of Lemma 3.4 is contained in

Lemma 3.5. *Suppose $x_0\in\mathbf{R}$, $\delta>0$ and suppose $I=[x_0-\delta, x_0+\delta]$ satisfies $0\notin\lambda I$ for some $\lambda>1$. Then if $F\subset I\cap\partial\mathcal{D}$ is a Borel set,*

$$w(0, F, \mathcal{D}) \geqq C(\lambda)\frac{|F|}{|I|}\,w(0, I\cap\partial\mathcal{D}, \mathcal{D})$$

where $C(\lambda)$ is a constant which depends only on λ and not on x_0, δ, or $\mathcal{D}$.

Proof. We give the proof for the case where $\lambda=3$; the general case is virtually identical. To avoid superfluous technical problems we will also assume that every point of $\partial\mathcal{D}$ is regular for the Dirichlet problem. This restriction can easily be removed by the experienced reader. Let $x_1=x_0-2\delta$, $x_2=x_0+2\delta$, and let

$$Q_j = \{\zeta = \xi+i\eta\colon |\xi-x_j|\leqq\delta,\ |\eta|\leqq\delta\}$$
$$\Lambda_j = \{\zeta = \xi+i\eta\colon \xi = x_j,\ |\eta|\leqq\delta\},\qquad j=1,2.$$

We claim

(3.5) $$w(\zeta, I\cap\partial\mathscr{D}, \mathscr{D}) \leqq C\frac{|I|}{|F|}w(\zeta, F, \mathscr{D})$$

$$\zeta\in\Lambda_j, \quad j = 1, 2.$$

We will prove this only for $j=1$; the proof for $j=2$ is identical. We first note that by the Poisson integral formula (3.4) and the argument immediately following it (after an application of Harnack's inequality),

(3.6) $$w(\zeta, F, \mathscr{D}) \geqq C\frac{|F|}{|I|}$$

$$\text{on} \quad \{\zeta\colon |\xi-x_0| \leqq 3\delta, \ |\eta| = \delta\}.$$

Let $F_1=\partial Q_1\cap\{\zeta\colon |\eta|=\delta\}$, let $F_2=\partial Q_1\cap\{\zeta\colon |\xi-x_1|=\delta\}$, and define $W_j(z)=w(z, F_j, Q_1\cap\mathscr{D})$, $j=1, 2$. We will have need of the following inequality:

(3.7) $$W_1(\zeta) \leqq 2W_2(\zeta), \quad \zeta\in\Lambda_1.$$

When $\zeta=x_1$ this last estimate is Lemma 7 in Benedicks [10]. (Benedicks assumes in the statement of his result that $Q_1\cap\mathscr{D}$ has a regular boundary; this can, however, be easily seen to be superfluous.) We note that the proof Benedicks gives is valid *without any changes* for the general case of $\zeta\in\Lambda_1$. By the comparison test-(3.6), and (3.7),

$$w(\zeta, I\cap\partial\mathscr{D}, \mathscr{D}) \leqq 3W_2(\zeta) \leqq C\frac{|I|}{|F|}w(\zeta, F, \mathscr{D})$$

for all $\zeta\in\Lambda_1$. Let $Q_3=\{\zeta\colon |\xi-x_0|\leqq 2\delta, \ |\eta|\leqq\delta\}$. Then by the last inequality plus (3.6),

$$w(\zeta, I\cap\partial\mathscr{D}, \mathscr{D}) \leqq C\frac{|I|}{|F|}w(\zeta, F, \mathscr{D})$$

for all $\zeta\in\partial Q_3$. The lemma now follows from the maximum principle applied on the domain $\mathscr{D}\setminus Q_3$, which by hypothesis contains the point 0.

Lemma 3.5 will be used several times in the proof of Lemma 3.4. For the sake of completeness we first note the following

Corollary 3.6. *If E satisfies*

$$\overline{\lim_{t\to 0}}\, t^{-1}|E\cap(x-t, x+t)| > 0$$

for all $x\in E$, *then* $dw_0=h\,dx$, $h\in L^1(dx)$.

Proof. Suppose not. Then there is a compact set $K_1 \subset E$ such that $|K_1|=0$ and $w_0(K_1)>0$. By taking K_1 a little smaller we may assume there is $\varepsilon>0$ such that for all $x\in K_1$ and for all $\delta>0$ there is $t=t(x,\delta)$ such that $t<\delta$ and $|E\cap(x-t,x+t)|>\varepsilon t$. Now write $w_0=w_1+w_2$ where $w_1=w_0|_{K_1}$ and $w_2=w_0|_{E\setminus K_1}$ are the restrictions of w_0 to K_1 (resp. $E\setminus K_1$). Let $\alpha>0$ be a number to be fixed later, and pick a compact set $K_2\subset E\setminus K_1$ such that $w_2(K_2)\geqq(1-\alpha)w_2(E)$. (So far there is nothing to preclude the possibility that $w_2\equiv 0$.) Let $\delta=\operatorname{dist}(K_1,K_2)>0$ and for each $x\in K_1$ pick $t=t(x)>0$ such that $t<\delta$ and $|E\cap(x-t,x+t)|\geqq\varepsilon t$. Now the Besicovitch covering lemma (see e.g. [39], p. 54) asserts exactly that in such a situation there are a finite number of points, say $x_1,\dots,x_N$, lying in K_1 such that $K_1\subset\bigcup_{j=1}^N(x_j-t_j,x_j+t_j)\equiv\bigcup_{j=1}^N I_j$, where $t_j=t(x_j)$, and such that $\sum_{j=1}^N\chi_{I_j}\leqq 2$. Note that by construction, $I_j\cap K_2=\emptyset$ for all j. From Lemma 3.5 we obtain $w_2(I_j)=w(I_j\setminus K_1)\geqq C|I_j|^{-1}|I_j\cap E|w(I_j)\geqq\frac{C}{2}\varepsilon w_1(I_j)$. Now since $\sum_j\chi_{I_j}\leqq 2$ we also have $w_2\left(\bigcup_1^N I_j\right)\geqq\frac{1}{2}\sum_1^N w_2(I_j)\geqq\frac{C}{4}\varepsilon\sum_1^N w_1(I_j)\geqq\frac{C}{4}\varepsilon w_1(K_1)$, the penultimate inequality following from our previous estimate. On the other hand, $I_j\cap K_2=\emptyset$ for all j and consequently $w_2\left(\bigcup_1^N I_j\right)\leqq w_2(\mathbf{R}\setminus K_2)\leqq\alpha w_2(\mathbf{R})\leqq\alpha$. This implies $\alpha\geqq\frac{C\varepsilon}{4}w_1(K_1)$. Taking $\alpha>0$ small enough gives a contradiction, and therefore Corollary 3.6 holds.

Now let I be an interval in $\mathbf{R}$ with $0\notin 3I$, let $K=I\cap\partial\mathscr{R}$, and let $\{\tilde S_j\}$ denote the Whitney decomposition of K^C (considered as a subset of $\mathbf{R}$) into intervals. Then (see Garnett [25], p. 266) $\operatorname{dist}(\tilde S_j,K)=|\tilde S_j|$. Let $\{S_j\}$ denote the collection of all Whitney intervals $\tilde S_j$ such that $0\notin 24\tilde S_j$. We will have need of the following technical lemma.

Lemma 3.7. *There are sets $E_j\subset 4S_j\cap K$ such that*

$$|E_j|\geqq\frac{\varepsilon}{2}|S_j|\quad\text{and}\quad\left\|\textstyle\sum_j\chi_{E_j}\right\|_{L^\infty}\leqq C(\varepsilon).$$

Proof. By condition (3.1) and the form of the Whitney decomposition, $|4S_j\cap K|\geqq\varepsilon|S_j|$. The proof now follows mutatis mutandis the argument for the proof of (3.2) and (3.3).

We are now ready to produce an extension of h from K to all of $\mathbf{R}$. Define $\tilde h(x)=h(x)$ for $x\in K$, $\tilde h(x)=\frac{1}{|E_j|}\int_{E_j}h\,dt$ for $x\in S_j$, and $\tilde h(x)=\frac{1}{|I|}\int_K h\,dt$ for $x\notin K\cup\left(\bigcup_j S_j\right)$.

Proof of Lemma 3.4. Property (i) follows by definition. To see that property (ii) holds, let $F=\mathbf{R}\setminus(K\cup(\bigcup_j S_j))$ and note that

$$\int_{I\setminus\partial\mathscr{R}}\tilde{h}\,dx \leqq \sum_j\int_{S_j}\left(\frac{1}{|E_j|}\int_{E_j}h\,dt\right)dx+\int_{I\cap F}\left(\frac{1}{|I|}\int_K h\,dt\right)dx$$
$$=\sum_j\frac{|S_j|}{|E_j|}\int_{E_j}h\,dt+\int_K h\,dt\leqq\frac{2}{\varepsilon}C(\varepsilon)\int_K h\,dt+\int_K h\,dt$$

by condition (3.1) and Lemma 3.7.

The proof of Lemma 3.4 (iii) will be split into three cases.

Case 1. J intersects at most two members of the collection $\{S_j\}\cup\{F\}$. If J is contained in one member of this collection, the inequality is trivial, so suppose J meets two members. We show that the values of $\tilde{h}$ on these two members are comparable. If $J\subset S_j\cup S_k$ with S_j and S_k adjacent then

$$\frac{1}{|E_j|}\int_{E_j}h\,dt\leqq\frac{2}{\varepsilon|S_j|}\int_{4S_j\cap K}h\,dt\leqq\frac{4}{\varepsilon|S_k|}\int_{11S_k\cap K}h\,dt.$$

Since $0\notin 11S_k$, by Lemma 3.5 applied to $F=E_k$ we obtain

$$\frac{1}{|E_j|}\int_{E_j}h\,dt\leqq C(\varepsilon)\frac{1}{|E_k|}\int_{E_k}h\,dt.$$

Thus $\max\{\tilde{h}(x)\colon x\in S_j\cup S_k\}\leqq C(\varepsilon)\min\{\tilde{h}(x)\colon x\in S_j\cup S_k\}$. If $J\subset S_j\cup F$ then by Lemma 3.5, since $0\notin 3I$,

$$\frac{1}{|I|}\int_K h\,dt\leqq\frac{C}{|E_j|}\int_{E_j}h\,dt.$$

Moreover, since S_j abuts F, $|S_j|\geqq C|I|$ and so

$$\frac{1}{|E_j|}\int_{E_j}h\,dt\leqq\frac{C(\varepsilon)}{4|S_j|}\int_K h\,dt\leqq\frac{C(\varepsilon)}{|I|}\int_K h\,dt.$$

Again we conclude

$$\max\{\tilde{h}(x)\colon x\in S_j\cup F\}\leqq C(\varepsilon)\min\{\tilde{h}(x)\colon x\in S_j\cup F\}.$$

Case 2. J intersects F and contains some S_j. Then $|J|\geqq C|I|$ and so

$$\frac{1}{|J|}\int_J\tilde{h}\,dt\leqq\frac{1}{|J|}\int_{J\cap F}\tilde{h}\,dt+\frac{1}{C|I|}\int_{F^c}\tilde{h}\,dt$$
$$\leqq\frac{|J\cap F|}{|J|}\frac{1}{|I|}\int_K h\,dt+\frac{1}{C|I|}\left(\int_K h\,dt+\sum_j\int_{S_j}\tilde{h}\,dt\right)$$
$$\leqq\frac{C}{|I|}\int_K h\,dt+\sum_j\frac{|S_j|}{|E_j|}\int_{E_j}h\,dt\leqq\frac{C(\varepsilon)}{|I|}\int_K h\,dt,$$

where the last inequality follows from Lemma 3.7. If $x\in F$ then the above inequalities show $\frac{1}{|J|}\int_J \tilde{h}\,dt \leqq C(\varepsilon)\tilde{h}(x)$. If $x\in S_j$, then since $0\notin 3I$,

$$\frac{1}{|J|}\int_J \tilde{h}\,dt \leqq \frac{C(\varepsilon)}{|I|}\int_K h\,dt \leqq \frac{C'(\varepsilon)}{|E_j|}\int_{E_j} h\,dt = C'(\varepsilon)\tilde{h}(x)$$

by Lemma 3.5. Finally, if $x\in K$ let L be an interval of the form $(x-\delta, x+\delta)$ with $0\notin 3L$. By Lemma 3.5 and condition (3.1),

$$\frac{1}{|J|}\int_J \tilde{h}\,dt \leqq \frac{C(\varepsilon)}{|I|}\int_K h\,dt \leqq \frac{C'(\varepsilon)}{|L\cap\partial\mathscr{R}|}\int_{L\cap\partial\mathscr{R}} h\,dt \leqq \frac{C'(\varepsilon)}{|L|}\int_{L\cap\partial\mathscr{R}} h\,dt.$$

By Lebesgue's differentiation theorem, as $\delta\to 0$ the right-hand side approaches $C'(\varepsilon)h(x)$ for a.e. $x\in\partial\mathscr{R}\cap K$. For such x, $\tilde{h}(x)=h(x)$.

Case 3. J does not intersect F and J contains some S_j. In this case

$$\int_J \tilde{h}\,dt \leqq \int_{J\cap K} h\,dt + \sum_{j:S_j\cap J\neq\emptyset}\int_{S_j}\tilde{h}\,dt$$

$$\leqq \int_{J\cap K} h\,dt + C\sum_{j:S_j\subset 2J}\frac{|S_j|}{|E_j|}\int_{E_j} h\,dt \leqq \int_{J\cap K} h\,dt + C(\varepsilon)\sum_{j:E_j\subset 8J}\int_{E_j} h\,dt.$$

By Lemma 3.7 we obtain

$$\int_J \tilde{h}\,dt \leqq C(\varepsilon)\int_{8J\cap\partial\mathscr{R}} h\,dt.$$

If $0\in 3(8J)$ then apply case 2 to $8J$. If $0\notin 3(8J)$ and if $x\in S_j\subset J$, then $E_j\subset 8J$ so by Lemma 3.5

$$\frac{1}{|J|}\int_J \tilde{h}\,dt \leqq \frac{C(\varepsilon)}{|8J|}\int_{8J} h\,dt \leqq \frac{C'(\varepsilon)}{|E_j|}\int_{E_j} h\,dt = C'(\varepsilon)\tilde{h}(x).$$

If $0\notin 3(8J)$ and $x\in K\cap J$, as in case 2 apply Lemma 3.5 to intervals $L=(x-\delta, x+\delta)$ and let $\delta\to 0$. This completes the proof of Lemma 3.4 (iii) and hence of Theorem 3.3.

4. Harmonic measure and interpolating sequences

In this section we show how estimates on harmonic measure can be used to determine whether a sequence $\{\zeta_j\}$ of points in $\mathscr{R}$ has pullback to the unit disk, $\{\pi^{-1}(\zeta_j)\}$, which is an interpolating sequence. The main result of this section is Theorem 4.1, which provides a necessary and sufficient condition for the interpolating conditions (2.5) and (2.6) to hold. The most important conclusion of Theorem 4.1 (the implication (4.2)$\Rightarrow$(4.3)) is not new. It is merely a rewording of Theorems 2 and 4.1 of [26] in the setting of Riemann surfaces. Since many words *do* have to be changed, we have included a detailed proof. The converse implication ((4.4)⇒(4.5)) is, as far as we know, not recorded in the literature. Our second result, Theorem 4.5, is a minor variant of Theorem 4.1. The only point here is that the hypotheses of Theorem 4.5 are slightly easier to verify in practice. The section ends by showing how Theorem 4.5 can be used to give another short proof of Carleson's theorem.

Theorem 4.1. *Suppose* $\{\zeta_j\}$ *is a sequence of points in* $\mathcal{R}$ *and suppose there is* $N_0>0$ *such that*

(4.1) *the sets* $\{\zeta\colon G(\zeta,\zeta_j)>N_0\}$ *are simply connected and disjoint from each other.*

Let $N_1>N_0$, *let*

$$\mathcal{D}_j = \{\zeta\colon G(\zeta,\zeta_j) > N_1\},$$

and let

$$\mathcal{S}_j = \mathcal{R}\setminus \bigcup_{\substack{k\\ k\neq j}} \mathcal{D}_k$$

If

$$\inf_j w(\zeta_j, \partial\mathcal{R}, \mathcal{S}_j) = a > 0, \tag{4.2}$$

then

$$\sup_j \sum_{k\neq j} G(\zeta_k,\zeta_j) \leqq C(a, N_0, N_1) < \infty. \tag{4.3}$$

Conversely, if (4.1) *holds and for some index* j_0,

$$\sum_{k\neq j_0} G(\zeta_k,\zeta_{j_0}) = A < \infty, \tag{4.4}$$

then

$$w(\zeta_{j_0}, \partial\mathcal{R}, \mathcal{S}_{j_0}) \geqq C(A, N_0, N_1) > 0. \tag{4.5}$$

Proof. We first concentrate on the implication (4.2)⇒(4.3). By the maximum principle, hypothesis (4.2) remains true if $\{\zeta_j\}$ is replaced by a finite subsequence, so it is enough to prove (4.3) for a finite sequence $\{\zeta_j\}$. For such a finite sequence $\{\zeta_j\}$ we now claim that it is sufficient to treat the case where $\mathcal{R}$ is a finite bordered surface all of whose boundary components are analytic curves in the double of $\mathcal{R}$. To see this, let $\{\mathcal{R}_n\}$ be a sequence of such surfaces which exhaust $\mathcal{R}$, and denote by G_n Green's function for $\mathcal{R}_n$. Then if $\zeta,\zeta'\in\mathcal{R}$ are fixed,

$$\lim_n G_n(\zeta,\zeta') = G(\zeta,\zeta')$$

and for each ζ_j,

$$\lim_n w(\zeta_j, \partial\mathcal{R}_n, \mathcal{S}_j) = w(\zeta_j, \partial\mathcal{R}, \mathcal{S}_j).$$

Fix $M>0$ to be determined later, set

$$\Delta_j = \{\zeta\colon G(\zeta, \zeta_j) > N_1+M\},$$

and consider the harmonic measure

$$f_j(\zeta) = w(\zeta, \partial\mathcal{R}, \mathcal{R}\setminus \bigcup_{\substack{k\\ k\neq j}} \Delta_k).$$

Since $\{\zeta_j\}$ has been assumed to be finite, and since the boundary of each Δ_k is an analytic compact curve, Green's theorem applies to $\mathcal{R}\setminus\bigcup_{\substack{k\\ k\neq j}} \Delta_k$ to yield

$$f_j(\zeta_j) = \frac{1}{2\pi}\int_{\partial\mathcal{R}} f_j(\zeta)\frac{\partial G(\zeta, \zeta_j)}{\partial n}\,ds(\zeta) \tag{4.6}$$

$$-\sum_{\substack{k\\ k\neq j}} \frac{1}{2\pi}\int_{\partial\Delta_k} G(\zeta, \zeta_j)\frac{\partial f_j(\zeta)}{\partial n}\,ds(\zeta),$$

where the normal derivatives are taken in the interior direction on $\mathcal{R}\setminus\bigcup_{\substack{k\\ k\neq j}} \Delta_k$. We note that the first integral above is equal to 1; this is because $f_j(\zeta)\equiv 1$ on $\partial\mathcal{R}$. We also note that if we map $\{\zeta\colon G(\zeta,\zeta_k)>N_0\}$ conformally onto the unit disk $\{z\in\mathbf{C}\colon |z|<1\}$, then $\mathcal{D}_k$ maps onto $\left\{z\colon \log\frac{1}{|z|}>N_1-N_0\right\}=\{z\colon |z|\leqq\exp\{(N_0-N_1)\}\}$ Since by assumption $G(\zeta,\zeta_j)$ is harmonic on $\mathcal{D}_k$ when $k\neq j$, and since $\Delta_k\subset\mathcal{D}_k$, Harnack's inequality shows there is $C=C(N_1-N_0)$ such that

$$C^{-1} \leqq \frac{G(\zeta, \zeta_j)}{G(\zeta_k, \zeta_j)} \leqq C, \quad \text{for all} \quad \zeta\in\Delta_k, \quad k\neq j. \tag{4.7}$$

Combining (4.6) and (4.7) we obtain

$$\sum_{\substack{k\\ k\neq j}} G(\zeta_k, \zeta_j)\int_{\partial\Delta_j}\frac{\partial f_j}{\partial n}\frac{ds}{2\pi} \leqq C(1-f_j(\zeta_j)) \leqq C,$$

because $f_j(\zeta_j)\geqq 0$ and $\frac{\partial f_j}{\partial n}\geqq 0$ on $\partial\Delta_k$, $k\neq j$. The implication (4.2)$\Rightarrow$(4.3) of the lemma will therefore be proved as soon as we establish the flux inequality

$$\int_{\partial\Delta_k}\frac{\partial f_j}{\partial n}\,ds \geqq C(a, N_0, N_1) > 0, \quad k\neq j. \tag{4.8}$$

To prove (4.8) we introduce the functions

$$u_{k,j}(\zeta) = w(\zeta, \partial\mathcal{R}, \mathcal{R}\setminus \bigcup_{\substack{\ell\\ \ell\neq k,j}} \Delta_\ell)$$

and

$$v_k(\zeta) = w(\zeta, \partial\Delta_k, \mathscr{R}\setminus\Delta_k) = (N_1+M)^{-1}G(\zeta, \zeta_k).$$

By the maximum principle,

$$f_j(\zeta) \geqq u_{k,j}(\zeta)-v_k(\zeta), \quad \zeta\in \bigcup_{\substack{\ell \\ \ell\neq j}} \Delta_\ell,$$

whenever $k\neq j$. By (4.2) and the maximum principle, $u_{k,j}(\zeta_k)\geqq a$, so by Harnack's inequality (see the argument preceding (4.7)),

$$u_{k,j}(\zeta) \geqq A, \quad \zeta\in\partial\mathscr{D}_k$$

where $A=A(a, N_0, N_1)$. On the other hand, if $M=M(a, N_0, N_1)$ is large enough, inspection of the definition of v_k shows that

$$v_k(\zeta) \leqq A/2, \quad \zeta\in\partial\mathscr{D}_k.$$

Hence we have

(4.9) $$f_j(\zeta) \geqq A/2, \quad \zeta\in\bigcup_k \partial\mathscr{D}_k$$

if M is large enough.

Now (4.9) implies (4.8). To see this, let Ω_k be the "annulus" $\mathscr{D}_k\setminus\Delta_k=\{\zeta\colon N_1<G(\zeta,\zeta_k)<N_1+M\}$ and let

$$w_k(z) = \frac{1}{M}(N_1+M-G(\zeta, \zeta_k)) = w(\zeta, \partial\mathscr{D}_k, \Omega_k).$$

By (4.9),

$$f_j(\zeta) \geqq \frac{A}{2} w_k(\zeta), \quad \zeta\in\Omega_k, \quad k \neq j.$$

Fix $k\neq j$. Then $f_j(\zeta)=w_k(\zeta)=0$, if $\zeta\in\partial\Delta_k$, and consequently $\frac{\partial f_j}{\partial n}\geqq A/2\,\frac{\partial w_k}{\partial n}$ on $\partial\Delta_k$. An application of Green's formula yields

$$\int_{\partial\Delta_k} \frac{\partial f_j}{\partial n}\, ds \geqq A/2 \int_{\partial\Delta_k} \frac{\partial w_k}{\partial n}\, ds = A/2\cdot 2\pi/M.$$

Therefore (4.2) implies (4.3).

We now turn to the proof of the implication (4.4)⇒(4.5). It is possible to give a proof of this which generalizes to higher dimensional settings. However, we know of no such proof which is of reasonable length for a paper such as this. We have therefore opted for giving a much shorter proof based on the theory of conformal mappings. It should be pointed out that if $A/N_0<1$ the proof is *very easy*. Indeed, in that case we have

$$w(\zeta, \bigcup_{\substack{k \\ k\neq j}} \partial\mathscr{D}_k, \mathscr{S}_j) \leqq \frac{1}{N_0} \sum_{k \neq j} G(\zeta, \zeta_k) \quad \text{for all} \quad \zeta\in\mathscr{S}_j$$

and consequently,

$$w(\zeta_j, \partial\mathcal{R}, \mathcal{S}_j) = 1 - w\big(\zeta_j, \bigcup_{\substack{k \\ k\neq j}} \partial\mathcal{D}_k, \mathcal{S}_j\big) \geqq 1 - A/N_0.$$

Unfortunately, this proof does not seem to work when $A/N_0 \geqq 1$.

Let $\{z_m\} = \{\pi^{-1}(\zeta_k)\}$ be the pullback to $\mathcal{U}$, under the covering map π, of the sequence $\{\zeta_k\}$. Let π be such that

$$\pi(0) = \zeta_{j_0}.$$

Our next lemma will be later used to show that it is sufficient to treat the case where $\mathcal{R} = \mathcal{U}$.

Lemma 4.2. $\sum_{\substack{m \\ z_m \neq 0}} \log \frac{1}{|z_m|} \leqq C(A, N_0)$.

Proof. Since $\{\zeta\colon G(\zeta, \zeta_{j_0}) > N_0\}$ is simply connected, Lemma 2.4 when applied to the singleton $\{\zeta_{j_0}\}$ yields

$$\sum \left\{\log \frac{1}{|z_m|}\colon z_m \in \bigcup_{\substack{\gamma\in\Gamma \\ \gamma\neq \mathrm{id}}} \gamma(0)\right\} \leqq C(N_0).$$

(Here Γ is the usual Fuchsian group.) Now let $S = \{z_m\colon z_m \notin \bigcup_{\gamma\in\Gamma} \gamma(0)\}$. Then

$$\sum_{z_m \in S} \log \frac{1}{|z_m|} = \sum_{\substack{k \\ k\neq j_0}} G(\zeta_k, \zeta_{j_0}) = A,$$

and the lemma follows.

Let $\{D_m\}$ be an enumeration of the simply connected components of $\bigcup_j \pi^{-1}(\mathcal{D}_j)$, i.e. $\{D_m\}$ is the pullback to $\mathcal{U}$ of all the "disks" $\mathcal{D}_j$. We label the D_m so that $0 \in D_0$ and $z_m \in D_m$. Consequently, $z_{m_1} \notin D_{m_2}$ if $m_1 \neq m_2$. We need only show that

$$w\big(0, \mathbf{T}, \mathcal{U} \setminus \bigcup_{\substack{m \\ m\neq 0}} D_m\big) \geqq C(A, N_0, N_1) > 0 \tag{4.10}$$

because $w(\zeta_{j_0}, \partial\mathcal{R}, \mathcal{S}_{j_0}) = 1 - w(\zeta_{j_0}, \bigcup_{\substack{k \\ k\neq j_0}} \partial\mathcal{D}_k, \mathcal{S}_{j_0})$ and by the maximum principle

$$w\big(\zeta_{j_0}, \bigcup_{\substack{k \\ k\neq j_0}} \partial\mathcal{D}_k, \mathcal{S}_{j_0}\big) = w\big(0, \cup\{\partial D_m\colon z_m \neq \gamma(0), \gamma\in\Gamma\}, \mathcal{U}\setminus \cup\{D_m\colon z_m \neq \gamma(0), \gamma\in\Gamma\}\big)$$
$$\leqq w\big(0, \bigcup_{\substack{m \\ m\neq 0}} \partial D_m, \mathcal{U} \setminus \bigcup_{\substack{m \\ m\neq 0}} D_m\big).$$

Recall that the pseudo-hyperbolic metric is defined on $\mathcal{U}$ by $\varrho(z, w) = \left|\frac{z-w}{1-z\bar{w}}\right|$.

Lemma 4.3. *There is a constant* $C = C(N_0, N_1) < 1$ *such that*

$$D_m \subset \{z\colon \varrho(z, z_m) \leqq C\}. \tag{4.11}$$

Proof. The lemma follows from standard reasoning and the assumptions which were made: The regions $\{\zeta:\ G(\zeta,\zeta_k)>N_0\}$ are simply connected and $\mathcal{D}_k=\{\zeta:\ G(\zeta,\zeta_k)>N_1\}$. A detailed proof is left to the reader.

Our strategy is now to construct a chord-arc curve Γ such that all sets D_m, $m\neq 0$ are in the unbounded component of Γ^C, and such that $\Gamma\cap\mathbf{T}$ has large Lebesgue measure. Recall that a closed Jordan curve Γ is said to satisfy the chord-arc condition of Lavrentiev if Γ is rectifiable and there is a constant K such that

$$\ell(\widehat{w_1, w_2}) \leqq K|w_1-w_2|$$

for all $w_1, w_2\in\Gamma$. Here $\widehat{w_1, w_2}$ is the shortest subarc of Γ containing w_1 and w_2 as endpoints, and $\ell(\gamma)$ denotes the arclength of a curve γ. The infimum of the values of K for which the above inequality holds is called the *chord arc constant* of Γ. We denote by V the bounded component of Γ^C.

Lemma 4.4. *There is a chord arc curve $\Gamma\subset\overline{\mathcal{U}}$ such that the following conditions hold:*

(4.12) $$\{z:\ |z|<\alpha\}\subset V \quad \textit{where} \quad \alpha=\alpha(N_0, N_1).$$

(4.13) $$D_m\cap\overline{V}=\emptyset, \quad m\neq 0.$$

(4.14) $$|\Gamma\cap\mathbf{T}|\geqq\pi.$$

(4.15) *The chord arc constant of Γ is $\leqq K=K(A, N_0, N_1)$.*

Let us accept the lemma for a moment and see how it can be used to complete the proof of the theorem. Let $f\colon \mathcal{U}\to V$ be univalent and satisfy $f(0)=0$. The properties of f were first studied by Lavrentiev [30]. His work has been extended by others (see e.g. Section 4 of [36] or the introduction to [27]) to show that f' is an outer function and $|f'|\in A_\infty$ on $\mathbf{T}$ with A_∞ bounds depending only on the chord arc constant K. More specifically, the results of the above cited papers contain the following theorem:

Suppose Γ is a chord arc curve with chord arc constant K and length $\ell(\Gamma)=L$. Suppose further that V is the bounded component of Γ^C, $0\in V$, and $\inf_{z\in\Gamma}|z|=r$. If F is univalent on $\mathcal{U}$, $F(\mathcal{U})=V$, and $F(0)=0$, then for all sets $E\subset\Gamma$,

$$|F^{-1}(E)|\geqq C\left(\frac{\ell(E)}{L}\right)^{\eta}$$

where the constants C and η depend only on K and L/r.

Applying the above result and Lemma 4.4 to the set $E=\Gamma\cap\mathbf{T}$ and the conformal mapping f, we see there are constants $C=C(K)$ and $\eta=\eta(K)$ such that

$$w(0,\mathbf{T},\mathscr{U}\setminus\bigcup_{\substack{m\\ m\neq 0}}D_m)\geqq w(0,\Gamma\cap\mathbf{T},V)$$

$$=(2\pi)^{-1}|f^{-1}(\Gamma\cap\mathbf{T})|\geqq(2\pi)^{-1}C\left(\frac{\pi}{4K}\right)^{\eta}$$

which proves (4.10).

Proof of Lemma 4.4. Let $\{\tilde{\mathscr{D}}_m\}=\{\pi^{-1}(\{\zeta\colon G(\zeta,\zeta_k)>N_0\})\}_k$ with the labeling defined so that $D_m\subset\tilde{D}_m$. Then by assumption, $\tilde{D}_{m_1}\cap\tilde{D}_{m_2}=\emptyset$ whenever $m_1\neq m_2$. Using estimate (4.11) it is an easy exercise to construct for each $\beta>0$ a bi-Lipschitz mapping $F_m\colon\tilde{D}_m\to\tilde{D}_m$ such that

$$F_m(D_m)\subset\{z\colon|z-z_m|\leqq\beta^3(1-|z_m|)\},\tag{4.16}$$

$$F_m(z)=z,\quad z\in\partial\tilde{D}_m,\tag{4.17}$$

$$C^{-1}\leqq\frac{|F_m(z)-F_m(w)|}{|z-w|}\leqq C,\quad z,w\in D_m,\tag{4.18}$$

where C is a constant depending only on β, N_0, N_1, and the constant in (4.11). Let $\beta<1$ be a small constant to be fixed later, construct mappings F_m as above and define

$$F(z)=\begin{cases}F_m(z), & z\in\tilde{D}_m,\quad m\neq 0\\ z, & z\in\overline{\mathscr{U}}\setminus\bigcup_{\substack{m\\ m\neq 0}}\tilde{D}_m.\end{cases}$$

For $m\neq 0$ let T_m be the closed tent (inverted cone)

$$T_m=\{z\in\overline{\mathscr{U}}\colon|\arg(z-\tilde{z}_m)-\arg(\tilde{z}_m)|\leqq\beta\}\cup\{\tilde{z}_m\}$$

where $\tilde{z}_m=z_m-\beta(1-|z_m|)z_m$. Then if β is small enough, Lemma 4.2 shows

$$T_m\cap\{z\colon|z|\leqq\beta\}=\emptyset\tag{4.19}$$

and by (4.16)

$$F(D_m)\subset T_m.\tag{4.20}$$

Now let $V_0=\mathscr{U}\setminus\bigcup_{\substack{m\\ m\neq 0}}T_m$ and let $\Gamma_0=\partial V_0$. Then V_0 is a Lipschitz domain and hence Γ_0 is a chord arc curve with chord arc constant $\leqq K(\beta)$. By the definition of T_m, $|T_m\cap\mathbf{T}|\leqq 3\beta(1-|z|)$ as soon as β is small enough, and consequently,

$$|\Gamma_0\cap\mathbf{T}|\geqq 2\pi-\sum_{\substack{m\\ m\neq 0}}|T_m\cap\mathbf{T}|\geqq 2\pi-3\beta\sum_{\substack{m\\ m\neq 0}}(1-|z_m|)\geqq\pi\tag{4.21}$$

as soon as β is small enough, the last inequality above following from Lemma 4.2. Set $V=F^{-1}(V_0)$ and $\Gamma=F^{-1}(\Gamma_0)$. Since bi-Lipschitz mappings take chord arc curves to chord arc curves, condition (4.18) shows Γ is a chord arc curve with chord arc constant bounded by $K(A, N_0, N_1)$. Thus condition (4.15) holds. Since $F(z)=z$, $z\in\mathbf{T}$, condition (4.14) follows from (4.21). Since $F(z)=z$, $z\in D_0$, condition (4.12) follows from (4.19). Finally, condition (4.13) follows from (4.20). This concludes the proof of Lemma 4.4, and hence the proof of Theorem 4.1 is completed.

In the proof of Theorem 4.1 it was not really necessary that the $\mathcal{D}_j$ were topological disks. In practice it is easier to work with continua than with disks. We formulate this result as a theorem.

Theorem 4.5. *Suppose* $\{\zeta_j\}$ *satisfy* (4.1) *and that* $\mathcal{D}_j$ *is as in the statement of Theorem* 4.1. *Suppose further that for each j there is a continuum* $\mathcal{C}_j\subset\overline{\mathcal{D}}_j$ *such that* $\zeta_j\in\mathcal{C}_j$ *and* $\mathcal{C}_j\cap\partial\mathcal{D}_j\neq\emptyset$. *If*

$$\inf_j w\Big(\zeta_j, \partial\mathcal{R}, \mathcal{R}\setminus\bigcup_{\substack{k\\ k\neq j}}\mathcal{C}_k\Big) = a > 0 \tag{4.22}$$

then

$$\sup_j \sum_{\substack{k\\ k\neq j}} G(\zeta_k, \zeta_j) \leqq C(a, N_0, N_1). \tag{4.23}$$

Proof. Let $N_2\geqq N_1$ be a finite number to be fixed later, let $\mathcal{B}_j=\{\zeta: G(\zeta,\zeta_j)>N_2\}$, and let $\mathcal{O}_j=\mathcal{R}\setminus\bigcup_{\substack{k\\ k\neq j}}\mathcal{B}_j$. By Theorem 4.1 it is sufficient to show that if N_2 is large enough, then

$$\inf_j w(\zeta_j, \partial\mathcal{R}, \mathcal{O}_j) \geqq a/2. \tag{4.24}$$

To this end we first map $\mathcal{D}_j$ conformally onto the unit disk $\mathcal{U}\subset\mathbf{C}$ so that ζ_j maps to the origin. With only slight abuse of notation we can then *identify* $\mathcal{D}_j$ with $\mathcal{U}$. We then have for $\zeta\in\mathcal{D}_j$,

$$G(\zeta, \zeta_j) = N_1+\log\frac{1}{|\zeta|}.$$

Let $\beta=\dfrac{1-a}{1-a/2}$. By Beurling's solution of the generalized Carleman—Milloux problem (see e.g. Ahlfors [3], Theorem 3.6) there is $r=r(\beta)>0$ such that

$$w(\zeta, \mathcal{C}_j, \mathcal{D}_j\setminus\mathcal{C}_j) \geqq \beta \tag{4.25}$$

whenever $|\zeta|\leqq r$. Now let $N_2=N_1+\log\dfrac{1}{r}$. Then by (4.25) and by the maximum principle,

$$w\Big(\zeta_j, \bigcup_{\substack{k\\ k\neq j}}\partial\mathcal{B}_k, \mathcal{O}_j\Big) \leqq \beta^{-1}w\Big(\zeta_j, \bigcup_{\substack{k\\ k\neq j}}\mathcal{C}_k, \mathcal{R}\setminus\bigcup_{\substack{k\\ k\neq j}}\mathcal{C}_k\Big)$$

$$= \beta^{-1}\Big(1-w\Big(\zeta_j, \partial\mathcal{R}, \mathcal{R}\setminus\bigcup_{\substack{k\\ k\neq j}}\mathcal{C}_k\Big)\Big) \leqq \beta^{-1}(1-a) = 1-a/2.$$

The proof of the theorem is concluded by noticing that (4.24) follows immediately from the inequality above.

Theorems 4.1 and 4.5 can be used to solve corona problems. We illustrate this by considering planar domains $\mathscr{R}=\mathbf{C}\setminus E$ of the type discussed in Section 3. For such a domain, $\mathscr{R}\cap\mathbf{R}$ can be written as a disjoint union of intervals L_j. For each j pick an arbitrary point $\zeta_j\in L_j$. The only difficulty in Section 3 was to show that

$$\sup_j \sum_{k \neq j} G(\zeta_k, \zeta_j) \leqq C < \infty,$$

i.e. that (4.23) holds. Let $\mathscr{C}_j=L_j$. It was shown just after Lemma 3.2 that there is $N_0>0$ such that (4.1) holds. Thus by Theorem 4.5 it is sufficient to show that (4.22) holds. To this end, fix j and define Ω_j by $(\Omega_j)^C=\mathbf{R}\setminus L_j$. By mapping Ω_j conformally onto the unit disk (there is a *formula* for the conformal mapping) and invoking the hypothesis (3.1) on $\partial\mathscr{R}$ it is easy to see that

$$w(\zeta_j, \partial\mathscr{R}, \Omega_j) \geqq a > 0,$$

i.e. (4.22) holds. Another way to see that this last inequality holds is to use the Poisson integral formula for the upper half plane (3.4) to show that $w(\zeta_j+id_j, \partial\mathscr{R}, \Omega_j)\geqq a'>0$, where $d_j=\operatorname{dist}(\zeta_j, \partial\mathscr{R})$. The desired result then follows from Harnack's inequality.

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References

1. Ahlfors, L. V., Bounded analytic functions, *Duke Math. J.* **14** (1947), 1—11.
2. Ahlfors, L. V., Open Riemann surfaces and extremal problems on compact subregions, *Comment. Math. Helv* **24** (1950), 100—134.
3. Ahlfors, L. V., *Conformal invariants: topics in geometric function theory,* McGraw-Hill, 1973.
4. Ahlfors, L. V. and Sario, L., *Riemann surfaces,* Princeton University Press, 1960.
5. Alling, N. L., A proof of the corona conjecture for finite open Riemann surfaces, *Bull. Amer. Math. Soc.* **70** (1964), 110—112.
6. Alling, N. L., Extensions of meromorphic function rings over non-compact Riemann surfaces, I, *Math. Z.* **89** (1965), 273—299.
7. Behrens, M., On the corona problem for a class of infinitely connected domains, *Bull. Amer. Math. Soc.* **76** (1970), 387—391.
8. Behrens, M., The maximal ideal space of algebras of bounded analytic functions on infinitely connected domains, *Trans. Amer. Math. Soc.* **161** (1971), 358—380.
9. Behrens, M., Interpolation and Gleason parts in L-domains, preprint.
10. Benedicks, M., Positive harmonic functions vanishing on the boundary of certain domains in $\mathbf{R}^n$, *Ark. Mat.* **18** (1980), 53—72.

11. Carleson, L., An interpolation theorem for bounded analytic functions, *Amer. J. Math.* **80** (1958), 921—930.

12. Carleson, L., Interpolation by bounded analytic functions and the corona problem, *Ann. of Math.* **76** (1962), 547—559.

13. Carleson, L., Research on interpolation problems, *Technical report, Department of Mathematics, Uppsala University, Uppsala, 1962.*

14. Carleson, L., Estimates of harmonic measure, *Ann. Acad. Sci. Fenn.* **7** (1982), 25—32.

15. Carleson, L., On H^∞ in multiply connected domains, *Conference on harmonic analysis in honor of Antoni Zygmund,* Vol. II, ed. Beckner, W., et al, Wadsworth, 1983, pp. 349—372.

16. Deeb, W. M., A class of infinitely connected domains and the corona, *Trans. Amer. Math. Soc.* **231** (1977), 101—106.

17. Deeb, W. M. and Wilken, D. R., $\mathscr{D}$ domains and the corona, *Trans. Amer. Math. Soc.* **231** (1977), 107—115.

18. Earle, C. J. and Marden, A., Projections to automorphic functions, *Proc. Amer. Math. Soc.* **19** (1968), 274—278.

19. Earle, C. J. and Marden, A., On Poincaré series with application to H^p spaces on bordered Riemann surfaces, *Illinois J. Math.* **13** (1969), 202—219.

20. Forelli, F., Bounded holomorphic functions and projections, *Illinois J. Math.* **10** (1966), 367—380.

21. Gamelin, T. W., Localization of the corona problem, *Pacific J. Math.* **34** (1970), 73—81.

22. Gamelin, T. W., *Uniform algebras and Jensen measures,* London Math. Soc. Lecture Notes 32, Cambridge University Press, 1978.

23. Gamelin, T. W., Wolff's proof of the corona theorem, *Israel J. Math.* **37** (1980), 113—119.

24. Garnett, J. B., *Analytic capacity and measure,* Lecture Notes in Math. 297, Springer-Verlag, 1972.

25. Garnett, J. B., *Bounded analytic functions,* Academic Press, 1981.

26. Garnett, J. B., Gehring, F. W. and Jones, P. W., Conformally invariant length sums, *Indiana Univ. Math. J.,* to appear.

27. Jerison, D. S. and Kenig, C. E., Hardy spaces, A_∞, and singular integrals on chord-arc domains, *Math. Scand.* **50** (1982), 221—248.

28. Jones, P. W., L^∞ estimates for the $\bar{\partial}$ problem in a halfplane, *Acta Math.* **150** (1983), 137—152.

29. Jones, P. W., Recent advances in the theory of Hardy spaces, in *Proceedings of the International Congress Math., Warsaw,* 1983.

30. Lavrentiev, M., Boundary problems in the theory of univalent functions, *Amer. Math. Soc. Translations, Series* 2, *Vol.* **32**, pp. 1—35. Originally published in *Mat. Sb. (N.S.)* **1** (43) (1936), 815—844 (in Russian).

31. Marden, A., On finitely generated Fuchsian groups, *Comment. Math. Helv.* **42** (1967), 81—85.

32. Nakai, M., Corona problem for Riemann surfaces of Parreau—Widom type, *Pacific J. Math.* **103** (1982), 103—109.

33. Nehari, Z., *Conformal mapping,* McGraw-Hill, 1952.

34. Pommerenke, Ch., On automorphic functions and Carleson sets, *Michigan Math. J.* **23** (1976), 129—136.

35. Pommerenke, Ch., On the Green's function of Fuchsian groups, *Ann. Acad. Sci. Fenn.* **2** (1976), 409—427.

36. Pommerenke, Ch., Schlichte funktionen und analytische funktionen von beschränkter mittler oszillation, *Comment. Math. Helv.* **52** (1977), 591—602.

37. RUBEL, L. A. and RYFF, J. V., The bounded weak-star topology and the bounded analytic functions, *J. Funct. Anal.* **5** (1970), 167—183.
38. SCHEINBERG, S., thesis, Princeton University, 1962.
39. STEIN, E. M. and WEISS, G., *An introduction to Fourier analysis on Euclidean Spaces,* Princeton University Press, 1971.
40. STOUT, E. L., Two theorems concerning functions holomorphic on multiply connected domains, *Bull. Amer. Math. Soc.* **69** (1963), 527—530.
41. STOUT, E. L., Bounded holomorphic functions on finite Riemann surfaces, *Trans. Amer. Math. Soc.* **120** (1965), 255—285.
42. STOUT, E. L., On some algebras of analytic functions on finite open Riemann surfaces, *Math. Z.* **92** (1966), 366—379. Corrections to: On some algebras of analytic functions on finite open Riemann surfaces, *Math. Z.* **95** (1967), 403—404.
43. STOUT, E. L., Interpolation on finite open Riemann surfaces, *Proc. Amer. Math. Soc.* **18** (1967), 274—278.
44. WALSH, J. L., *The location of critical points,* Amer. Math. Soc. Colloq. Publ. Vol. XXXIV, 1950.
45. WIDOM, H., Extremal polynomials associated with a system of curves in the complex plane, *Adv. in Math.* **2** (1969), 127—132.
46. WIDOM, H., H^p sections of vector bundles over Riemann surfaces, *Ann. of Math.* **94** (1971), 304—324.
47. WIDOM, H., The maximum principle for multiple-valued analytic functions, *Acta Math.* **126** (1971), 63—81.
48. WOLFF, T., Restrictions of A_∞ weights, preprint.

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