The local Π_1 of the complement of a hypersurface with normal crossings in codimension 1 is abelian

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0. Introduction

Let $(X, 0) \subset (\mathbf{C}^{n+1}, 0)$ be a germ of reduced analytic hypersurface in $(\mathbf{C}^{n+1}, 0)$ defined by $f=0$, where $f \in \mathcal{O}_{\mathbf{C}^{n+1}, 0}$ is a germ of analytic function in $\mathbf{C}^{n+1}$ at 0. We shall prove the following:

Main Theorem. *Assume that outside an analytic subgerm $(Y, 0)$ of $(X, 0)$ of dimension at most $n-2$ the only singularities of $(X, 0)$ are normal crossings then the local fundamental group of the complement of $(X, 0)$ in $(\mathbf{C}^{n+1}, 0)$ is abelian.*

Remark. Using Milnor fibration theorem ([M] theorems 4.8. and 5.11.) this theorem implies that under its hypothesis the Milnor fiber of $(X, 0)$ has a fundamental group which is free abelian of rank the number of analytic components of X at 0 minus one. In particular, if $(X, 0)$ is analytically irreducible, the Milnor fiber is simply connected. This result extends a result of M. Kato and Y. Matsumoto ([K—M]) which says that if the singular locus of $(X, 0)$ has codimension 2, the Milnor fiber of $(X, 0)$ is simply-connected.

We shall still denote by X and Y representants of $(X, 0)$ and $(Y, 0)$ in a sufficiently small neighbourhood of 0 in $\mathbf{C}^{n+1}$.

We notice that, if $\varepsilon > 0$ is small enough, the balls B_ε^{2n+2} of $\mathbf{C}^{n+1}$ centered at 0 with radius $\varepsilon > 0$:

$$B_\varepsilon^{2n+2} := \{z \in \mathbf{C}^{n+1}, \|z\| \leq \varepsilon\}$$

— make a fundamental system of good neighbourhoods of 0 in $\mathbf{C}^{n+1}$ with regard to both X and Y in the sense of D. Prill (cf. [P] definition 1) by using the local conic structure of an analytic set (cf. [B—V] lemma (3.2.)).

Thus the local fundamental group of the complement of $(X, 0)$ in $(\mathbf{C}^{n+1}, 0)$ (cf. [G] exposé XIII p. 15 and commentaires p. 26) is $\pi_1(B_\varepsilon^{2n+2}-X, x)$, when $\varepsilon>0$ is small enough and $x\in B_\varepsilon^{2n+2}-X$ (cf. [*P*] proposition 2).

By using a theorem of [H—L$_1$] (theorem (0.2.1.)) it is enough to prove our theorem in the case $n=2$.

Our proof is strongly inspired by [D$_1$] (see [D$_2$] too) and, of course, the theorem above will imply the fact proved by W. Fulton and P. Deligne (cf. [F] and [D$_2$]) that the fundamental group of the complement of a complex projective plane curve with only nodes as singularities is abelian, as we shall see below.

Our main theorem above gives a positive answer to a question of [S] (2.14).

1. Proof of the main theorem in the irreducible case

In this paragraph we shall assume $(X, 0)$ to be irreducible.

(1.1.) As we have said above, *we may suppose* $n=2$.

(1.1.1.) We notice that, because of the local conic structure of analytic sets (cf. [B—V] lemma (3.2.)), B_ε^6-X has the homotopy type of S_ε^5-X when $\varepsilon>0$ is small enough and $S_\varepsilon^5=\partial B_\varepsilon^6$ is the sphere boundary of B_ε^6. Actually one proves that B_ε^6-X is: diffeomorphic to $(S_\varepsilon^5-X)\times[0, 1[$.

(1.1.2.) Let Σ be the singular locus of X. Now let us choose $\varepsilon>0$ such that for any ε', $\varepsilon\geqq\varepsilon'>0$, $S_{\varepsilon'}^5$ is transverse to $X-\Sigma$ and Σ and hence, as above, $S_{\varepsilon'}^5-X$ is a deformation retract of $B_{\varepsilon'}^6-X$.

(1.1.3.) We call $K_\varepsilon:=X\cap S_\varepsilon^5$. Then K_ε is a manifold outside $\partial\Sigma_\varepsilon:=\Sigma\cap S_\varepsilon^5$. Now $\partial\Sigma_\varepsilon$ is a 1-dimensional compact submanifold of S_ε^5 which is the union of embedded circles in S_ε^5. Thus $S_\varepsilon^5-\partial\Sigma_\varepsilon$ is simply connected because $\dim S_\varepsilon^5=5$ and S_ε^5 is simply connected.

Locally at every point of $\partial\Sigma_\varepsilon$ the space K_ε looks like two embedded 3-dimensional manifolds cutting transversally along $\partial\Sigma_\varepsilon$. (Fig 1).

We call $K_\varepsilon^*:=K_\varepsilon-\partial\Sigma_\varepsilon$. Let $T(K_\varepsilon^*)$ be a tubular neighbourhood of K_ε^* in S_ε^5.

Because $(X, 0)$ is irreducible, K_ε^* is connected. We can apply Van Kampen's theorem to $S_\varepsilon^5-\partial\Sigma_\varepsilon=(S_\varepsilon^5-X)\cup T(K_\varepsilon^*)$. As $(S_\varepsilon^5-X)\cap T(K_\varepsilon^*)=T(K_\varepsilon^*)-K_\varepsilon^*$, we have the cocartesian diagramm:

$$\begin{tikzcd}[column sep=small] & \pi_1(S_\varepsilon^5-X, x) \arrow[dr] & \\ \pi_1(T(K_\varepsilon^*)-K_\varepsilon^*, x) \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \pi_1(S_\varepsilon^5-\partial\Sigma_\varepsilon, x) = \{1\} \\ & \pi_1(T(K_\varepsilon^*), x) \arrow[ur] & \end{tikzcd}$$

with $x\in T(K_\varepsilon^*)-K_\varepsilon^*$.

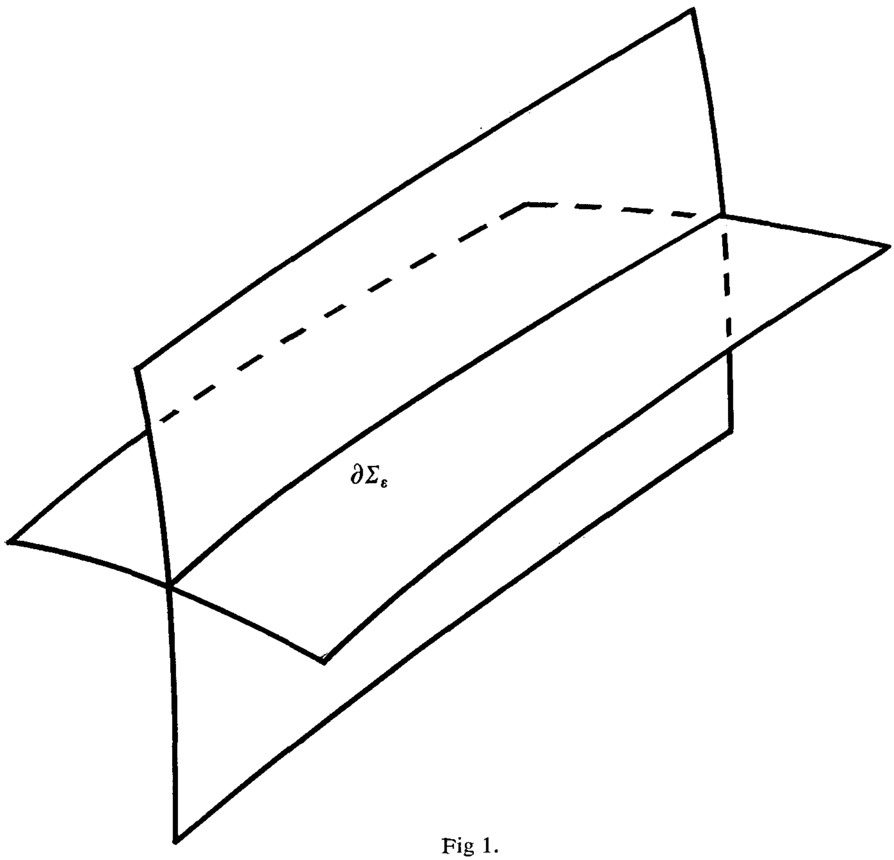


Fig 1.

(1.1.4.) Notice that $T(K_\varepsilon^*)$ retracts on K_ε^* and in the above diagram the homomorphism β is induced by the punctured discbundle $T(K_\varepsilon^*)-K_\varepsilon^* \to K_\varepsilon^*$. Thus the kernel Ker β of β is a quotient of $\mathbf{Z}$ and β is surjective. Actually we shall prove below that this kernel is isomorphic to $\mathbf{Z}$, but what we shall need now is that it is an abelian group.

By diagramm chasing one finds easily that the normal subgroup generated by the image $\alpha(\operatorname{Ker}\beta)$ in $\pi_1(S_\varepsilon^5-X, x)$ is $\pi_1(S_\varepsilon^5-X, x)$ itself. Thus, if one proves that α is surjective, as Ker β is a normal subgroup of $\pi_1(T(K_\varepsilon^*)-K_\varepsilon^*, x)$, its image by α is normal. Then the surjectivity of α would imply that the restriction of α to Ker β is already surjective and this implies that $\pi_1(S_\varepsilon^5-X, x)$ is abelian as announced.

Now, if $\pi_1(S_\varepsilon^5-X, x)$ is abelian, using the fibration theorem of Milnor ([M] Theorem 4.8.) *in this case when* $(X, 0)$ *is irreducible* we obtain that $\pi_1(S_\varepsilon-X, x)$ is isomorphic to $\mathbf{Z}$ and the Milnor fiber of $(X, 0)$ is simply connected. This shows altogether that Ker β is isomorphic to $\mathbf{Z}$.

(1.2.) In this case where $(X, 0)$ is analytically irreducible, it remains to prove that the homomorphism α of the cocartesian diagramm of (1.1.3.) is surjective.

To show this surjectivity we shall proceed analogously as in [D1].

(1.2.1.) Let $n\colon \bar{X} \to X$ be the normalization of X.

Consider $Z := (B_\varepsilon^6 - X) \times ((X - \Sigma) \cap B_\varepsilon^6)$ and $Z_1 := (B_\varepsilon^6 - X) \times n^{-1}(X \cap B_\varepsilon^6 - \{0\})$. Let Δ be the diagonal of $B_\varepsilon^6 \times B_\varepsilon^6$, and we denote by $\Delta^* := \Delta - \{0\}$.

Let us define:

$$V_\delta(\Delta) := \{(x, y) \in B_\varepsilon^6 \times B_\varepsilon^6, \|x - y\| < \delta\}$$

with $0 < \delta \ll \varepsilon$, where $\| \ \|$ is the Euclidean norm of $\mathbf{C}^3$. We shall denote:

$$\bar{V}_\delta(\Delta) := \{(x, y) \in B_\varepsilon^6 \times B_\varepsilon^6, \|x - y\| \leqq \delta\}.$$

For any $x \in B_\varepsilon^6 - \{0\}$, we define $V_{\delta, x}$ by:

$$V_{\delta, x} := \{y \in B_\varepsilon^6, (x, y) \in V_\delta(\Delta)\}.$$

These form a fundamental system of neighbourhoods of x as $\delta > 0$ and $\delta \ll \varepsilon$. Denote $T_\delta(X) := \bigcup_{x \in X} V_{\delta, x}$.

(1.2.2.) Now we define a complex non-Hausdorff 3-dimensional analytic manifold by $T_\delta(\bar{X})$ as an union of charts $V_{\delta, n(x)}$, with $x \in n^{-1}(B_\varepsilon^6 \cap X)$, where two points $p \in V_{\delta, n(x)}$ and $q \in V_{\delta, n(y)}$ are identified when $p = q$ as points in B_ε^6 and x and y belong to the same connected component of $n^{-1}(X \cap V_{\delta, p})$.

The space $n^{-1}(B_\varepsilon^6 \cap X)$ is naturally immersed in $T_\delta(\bar{X})$ by the correspondence $i\colon x \in n^{-1}(B_\varepsilon \cap X) \mapsto x' \in T_\delta(\bar{X})$, where x' is represented by $n(x)$ in the chart $V_{\delta, n(x)}$.

The natural correspondence of charts from the chart $V_{\delta, n(x)}$ of $T_\delta(\bar{X})$ to the chart $V_{\delta, n(x)}$ of B_ε induces an étale map:

$$\varphi\colon T_\delta(\bar{X}) \to T_\delta(X)$$

such that the composition of φ with the above immersion i induces the map $n^{-1}(B_\varepsilon^6 \cap X) \to B_\varepsilon^6 \cap X$ defined by the normalization.

(1.2.3.) Let $\varepsilon_1, \varepsilon_2$ such that $\varepsilon \geqq \varepsilon_2 > \varepsilon_1 > 0$. We denote by·

$$B_{\varepsilon_1, \varepsilon_2} := \{z \in \mathbf{C}^3, \varepsilon_1 \leqq \|z\| \leqq \varepsilon_2\}$$

$$X_{\varepsilon_1, \varepsilon_2} := X \cap B_{\varepsilon_1, \varepsilon_2}, \quad \bar{X}_{\varepsilon_1, \varepsilon_2} := n^{-1}(X_{\varepsilon_1, \varepsilon_2})$$

$$T_\delta(X_{\varepsilon_1, \varepsilon_2}) := T_\delta(X) \cap B_{\varepsilon_1, \varepsilon_2}$$

$$T_\delta(\bar{X}_{\varepsilon_1, \varepsilon_2}) := \varphi^{-1}(T_\delta(X_{\varepsilon_1, \varepsilon_2}))$$

(1.2.4.) Lemma. *Let $\varepsilon_1, \varepsilon_2$ such that $\varepsilon \geqq \varepsilon_2 > \varepsilon_1 > 0$.* There exists $\delta_0 > 0$ *such that, for any δ, $\delta_0 \geqq \delta > 0$:*

1) *The space $T_\delta(\bar{X}_{\varepsilon_1, \varepsilon_2})$ is a Hausdorff manifold;*

2) *The immersion i induces an embedding of $\bar{X}_{\varepsilon_1, \varepsilon_2}$ into $T_\delta(\bar{X}_{\varepsilon_1, \varepsilon_2})$;*

3) *The image* $i(\bar{X}_{\varepsilon_1,\varepsilon_2})$ *of* $\bar{X}_{\varepsilon_1,\varepsilon_2}$ *by this embedding is a deformation retract of* $T_\delta(\bar{X}_{\varepsilon_1,\varepsilon_2})$;
4) *The space* $X_{\varepsilon_1,\varepsilon_2}$ *is a deformation retract of* $T_\delta(X_{\varepsilon_1,\varepsilon_2})$;
5) *There is a subspace* $\tilde{T}_\delta$ *of* $T_\delta(\bar{X}_{\varepsilon_1,\varepsilon_2})$ *such that:*
   α) φ *induces a homeomorphism of* $\tilde{T}_\delta$ *onto a tubular neighbourhood of* $X_{\varepsilon_1,\varepsilon_2}-\Sigma$,
   β) $\tilde{T}_\delta - i(\bar{X}_{\varepsilon_1,\varepsilon_2})$ *is a deformation retract of* $T_\delta(\bar{X}_{\varepsilon_1,\varepsilon_2})-\varphi^{-1}(X_{\varepsilon_1,\varepsilon_2})$.

Proof. Let V be a neighbourhood of the singular locus $\Sigma\cap X_{\varepsilon_1,\varepsilon_2}$ in B_ε such that, for any connected component V_i for V, the space $V_i\cap\Sigma\cap X$ is connected and $n^{-1}(V_i\cap X)$ has exactly two components. As $B_{\varepsilon_1,\varepsilon_2}$ is compact, if $\delta>0$ is small enough, for any $t\in T_\delta(X_{\varepsilon_1,\varepsilon_2})$, there is a neighbourhood U_t of t such that, for any

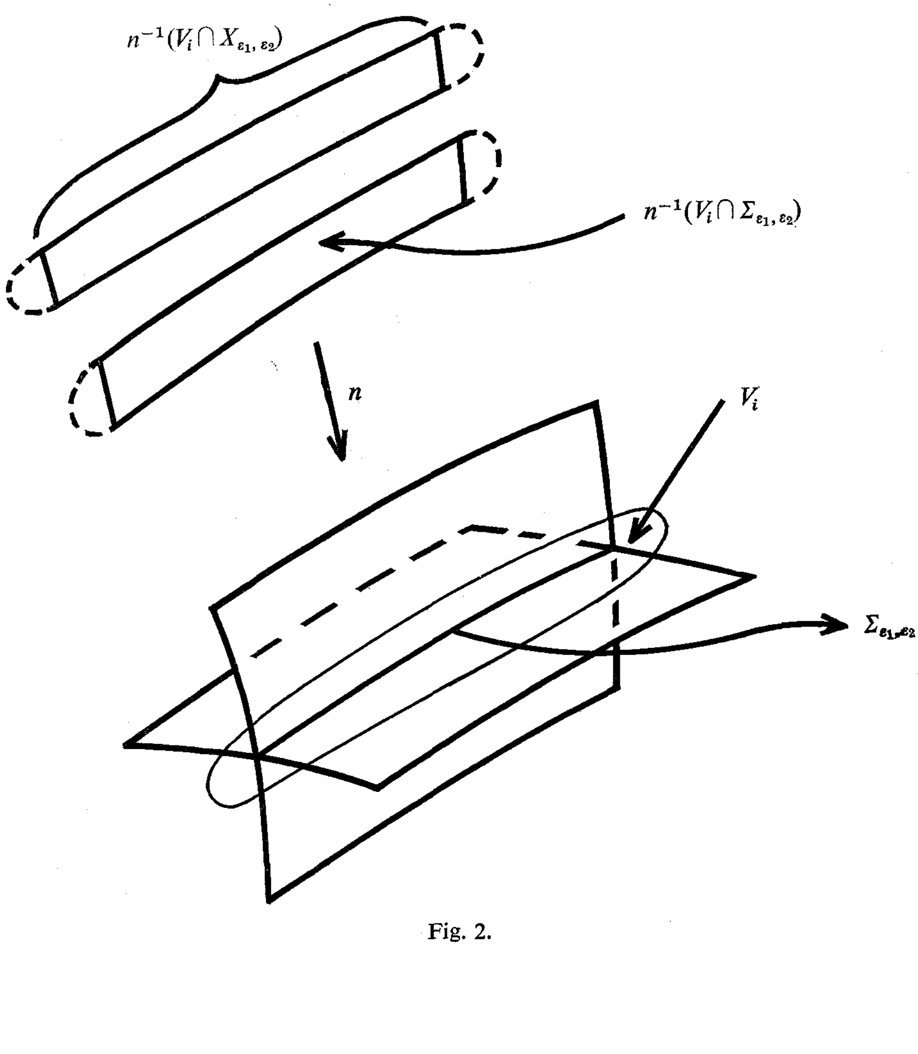


Fig. 2.

$t' \in U_t$, $n^{-1}(V_{\delta,t'} \cap X)$ has the same number of components which is equal to one or two, and, if it is two, $V_{\delta,t'}$ is contained in some component V_i of V and each of the components of $n^{-1}(V_{\delta,t'} \cap X)$ is contained in each of the components of $n^{-1}(V_i \cap X)$.

Let x_1, x_2 be two distinct points of $T_\delta(\bar{X}_{\varepsilon_1,\varepsilon_2})$. If their images in $T_\delta(X_{\varepsilon_1,\varepsilon_2})$ are distinct, because $T_\delta(X_{\varepsilon_1,\varepsilon_2})$ is a subspace of $\mathbf{C}^3$, it is Hausdorff and one finds easily neighbourhoods of x_1 and x_2 in $T_\delta(\bar{X}_{\varepsilon_1,\varepsilon_2})$ to separate x_1 and x_2 there. Thus the only case to be really considered is when $x=\varphi(x_1)=\varphi(x_2)$. By definition, as $x_1 \neq x_2$, the points x_1 and x_2 belong respectively to the charts $V_{\delta,n(y_1)}$ and $V_{\delta,n(y_2)}$ and, y_1 and y_2 belong to different components of $n^{-1}(V_{\delta,x} \cap X)$. Now let U_1 and U_2 be the neighbourhoods of x_1 and x_2 in $T_\delta(\bar{X}_{\varepsilon_1,\varepsilon_2})$ the images of which are both equal to a neighbourhood U of x in $T_\delta(X_{\varepsilon_1,\varepsilon_2})$ contained in $V_{\delta,n(y_1)} \cap V_{\delta,n(y_2)} \cap U_x$. Let $x' \in U$ and x'_1, x'_2 the corresponding points in U_1 and U_2. If one proves $x'_1 \neq x'_2$, then $U_1 \cap U_2 = \emptyset$. As $x'_1 \in V_{\delta,n(y_1)}$ and $x'_2 \in V_{\delta,n(y_2)}$ it is enough to prove that y_1 and y_2 lie on different components of $n^{-1}(V_{\delta,x'} \cap X)$. Because U is contained in U_x, for any x' in the open set U, the number of components does not change and is equal to two, as it is two at x. As y_1 and y_2 belong to different components of $n^{-1}(V_{\delta,x} \cap X)$, they lie on different components of $n^{-1}(V_i \cap X)$ if $V_{\delta,x} \subset V_i$. Thus they must lie on different components of $n^{-1}(V_{\delta,x'} \cap X)$ for any $x' \in U$, because it is clear that $V_{\delta,x'} \subset V_i$ and by choice of δ each of the components of $n^{-1}(V_{\delta,x'} \cap X)$ is contained in each of the components of $n^{-1}(V_i \cap X)$.

To prove the second assertion we consider the following commutative diagramm:

$$\begin{array}{ccc} n^{-1}(B^6_\varepsilon \cap X) & \xrightarrow{i} & T_\delta(\bar{X}) \\ \downarrow{\scriptstyle n_\varepsilon} & & \downarrow{\scriptstyle \varphi} \\ B^6_\varepsilon \cap X & \subset & T_\delta(X) \end{array}$$

where n_ε is induced by the normalization n. As n_ε is an isomorphism outside Σ, the immersion i restricted to $n^{-1}(B^6_\varepsilon \cap X - \Sigma)$ is an embedding. Thus it remains to prove that the restriction of i to $\bar{X}_{\varepsilon_1,\varepsilon_2} \cap n^{-1}(\Sigma)$ is an injective map when δ is sufficiently small. As $\bar{X}_{\varepsilon_1,\varepsilon_2} \cap n^{-1}(\Sigma)$ is compact, for any δ small enough and any $x \in \bar{X}_{\varepsilon_1,\varepsilon_2} \cap n^{-1}(\Sigma)$, $n^{-1}(X \cap V_{\delta,n(x)})$ has two connected components and this proves our second assertion.

Let σ be the function defined on $T_\delta(\bar{X}_{\varepsilon_1,\varepsilon_2})$ by:

$$\forall z \in T_\delta(\bar{X}_{\varepsilon_1,\varepsilon_2}), \quad \sigma(z) = \inf_{x \in \bar{X} \cap i^{-1}(V_{\delta,\varphi(z)})} (\|\varphi(z) - \varphi(x)\|^2).$$

Actually one can prove that this function is real analytic in a neighbourhood of $i(\bar{X}_{\varepsilon_1,\varepsilon_2})$. As $\sigma=0$ coincides with $i(\bar{X}_{\varepsilon_1,\varepsilon_2})$ and the boundary of $\bar{X}_{\varepsilon_1,\varepsilon_2}$ is real analytic, there is δ_1 small enough such that for any δ, $0<\delta \leqq \delta_1$, the space $\sigma=\delta$ is smooth and cuts the boundary of $\bar{X}_{\varepsilon_1,\varepsilon_2}$ transversally. This shows that for any δ, δ', $0<\delta'<\delta \leqq \delta_1$, $T_\delta(\bar{X}_{\varepsilon_1,\varepsilon_2})$ retracts by deformation on $T_{\delta'}(\bar{X}_{\varepsilon_1,\varepsilon_2})$. For $0<\delta \leqq \delta_1$, the $T_\delta(\bar{X}_{\varepsilon_1,\varepsilon_2})$ define a fundamental system of good neighbourhoods of $i(\bar{X}_{\varepsilon_1,\varepsilon_2})$, because $i(\bar{X}_{\varepsilon_1,\varepsilon_2})$ is compact; thus they define good neighbourhoods of $i(\bar{X}_{\varepsilon_1,\varepsilon_2})$ in the sense of D.

Prill ([P]) and retract on it by deformation. This proves our third assertion. Actually one may obtain that $T_\delta(\overline{X}_{\varepsilon_1,\varepsilon_2})$ is diffeomorphic to $\overline{X}_{\varepsilon_1,\varepsilon_2}\times\mathring{D}$ where $\mathring{D}$ is an open 2-disc.

As φ is étale, for any δ, $0<\delta\leqq\delta_1$, the restriction of φ to the boundary of $T_\delta(\overline{X}_{\varepsilon_1,\varepsilon_2})$, defined by $\sigma=\delta$, is an immersion. As the singularities of $X-\{0\}$ are normal crossings, the self intersections of φ $(\sigma=\delta)$ are normal crossings if $0<\delta\leqq\delta_2$ for δ_2 small enough. If δ_2 is well chosen, there is a submanifold with corners of $T_\delta(X_{\varepsilon_1,\varepsilon_2})$ the corners of which are the self intersections of $\varphi(\sigma=\delta)$ and which is a neighbourhood of $\Sigma\cap X_{\varepsilon_1,\varepsilon_2}$.

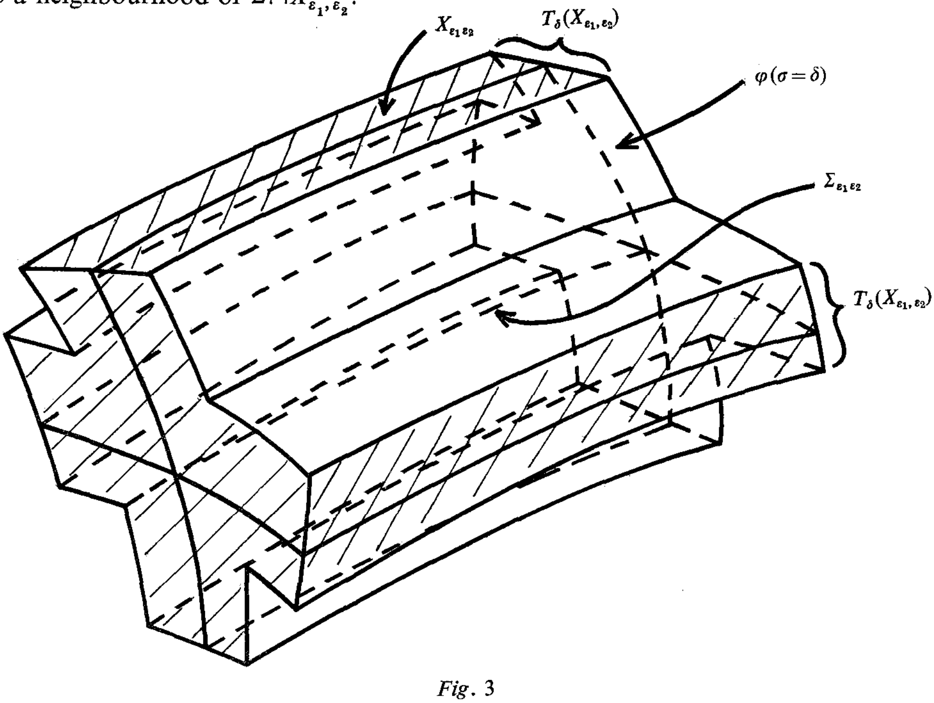


Fig. 3

Because the spaces $T_\delta(\overline{X}_{\varepsilon_1,\varepsilon_2})$ define a fundamental system of neighbourhoods of $i(\overline{X}_{\varepsilon_1,\varepsilon_2})$ when $\delta>0$ is small enough, one obtains that the $T_\delta(X_{\varepsilon_1,\varepsilon_2})$ define a fundamental system of neighbourhoods of $X_{\varepsilon_1,\varepsilon_2}$. As $X_{\varepsilon_1,\varepsilon_2}$ is a CW-complex, to prove the fourth assertion it is enough to prove that the $T_\delta(X_{\varepsilon_1,\varepsilon_2})$ define a fundamental system of good neighbourhoods of $X_{\varepsilon_1,\varepsilon_2}$ in the sense of D. Prill ([P]), i.e. if δ,δ' are small enough, $\delta>\delta'>0$, then $T_{\delta'}(X_{\varepsilon_1,\varepsilon_2})$ is a deformation retract of $T_\delta(X_{\varepsilon_1,\varepsilon_2})$. To prove this fact one builds up a vector field on $B_{\varepsilon_1,\varepsilon_2}$ the integration of which gives the deformation retraction: actually one obtains a diffeomorphism of manifolds with corners. Precisely we have :

As φ induces an étale mapping from $T_\delta(\overline{X}_{\varepsilon_1,\varepsilon_2})$ onto $T_\delta(X_{\varepsilon_1,\varepsilon_2})$ when $\delta>0$

is small enough, we may find a covering $(U_\alpha)_{\alpha\in A}$ of $T_\delta(X_{\varepsilon_1,\varepsilon_2})$ such that either $U_\alpha\cap\Sigma_{\varepsilon_1,\varepsilon_2}=\emptyset$ or $U_\alpha\cap\Sigma_{\varepsilon_1,\varepsilon_2}\neq\emptyset$ and:

i) if $U_\alpha\cap\Sigma_{\varepsilon_1,\varepsilon_2}=\emptyset$, there is a real analytic function $\sigma_\alpha\colon U_\alpha\to\mathbf{R}$ such that $T_\delta(X_{\varepsilon_1,\varepsilon_2})\cap U_\alpha=\{\sigma_\alpha<\delta\}$, and $T_{\delta'}(X_{\varepsilon_1,\varepsilon_2})\cap U_\alpha=(\sigma_\alpha<\delta'\}$ for any $\delta\geqq\delta'>0$ and $X_{\varepsilon_1,\varepsilon_2}\cap U_\alpha:=\{\sigma_\alpha=0\}$;

ii) if $U_\alpha\cap\Sigma_{\varepsilon_1,\varepsilon_2}\neq\emptyset$ there are two analytic functions $\sigma_\alpha\colon U_\alpha\to\mathbf{R}$ and $\sigma'_\alpha\colon U_\alpha\to\mathbf{R}$ such that: $T_{\delta'}(X_{\varepsilon_1,\varepsilon_2})\cap U_\alpha=\{\sigma_\alpha<\delta'\}\cup\{\sigma'_\alpha<\delta'\}$ for any $\delta\geqq\delta'>0$, $X_{\varepsilon_1,\varepsilon_2}\cap U_\alpha:=\{\sigma_\alpha=0\}\cup\{\sigma'_\alpha=0\}$ and for any $\delta\geqq\delta'\geqq0$, the spaces $\{\sigma_\alpha=\delta'\}$ and $\{\sigma'_\alpha=\delta'\}$ are submanifolds of U_α which cut transversally.

Let us fix $\delta>0$ small enough such that properties above are satisfied. Now consider δ', $\delta>\delta'>0$. In each U_α we build up a smooth vector field v_α such that:

i) For any $x\in U_\alpha\cap(T_{\delta_1}(X_{\varepsilon_1,\varepsilon_2})-T_{\delta_2}(X_{\varepsilon_1,\varepsilon_2}))$, $v_\alpha(x)\neq0$ for some δ_1,δ_2 such that $\delta_1>\delta>\delta'>\delta_2>0$, and $v_\alpha(x)=0$ in $T_{\delta_3}(X_{\varepsilon_1,\varepsilon_2})$ for some $0<\delta_3<\delta_2$.

ii) If $U_\alpha\cap\Sigma_{\varepsilon_1,\varepsilon_2}=\emptyset$, for any $x\in U$ such that $v_\alpha(x)\neq0$ one has $d\sigma_\alpha(v_\alpha(x))<0$, and for any $x\in U_\alpha\cap\partial T_\delta(X_{\varepsilon_1,\varepsilon_2})$, $v_\alpha(x)$ is tangent to $\partial T_\delta(X_{\varepsilon_1,\varepsilon_2})$ where $\partial T_\delta(X_{\varepsilon_1,\varepsilon_2})$ is the image of the boundary of $T_\delta(\bar{X}_{\varepsilon_1,\varepsilon_2})$, inverse image of $\partial\bar{X}_{\varepsilon_1,\varepsilon_2}\times\mathring{D}$ by the diffeomorphism of $T_\delta(\bar{X}_{\varepsilon_1,\varepsilon_2})$ onto $\bar{X}_{\varepsilon_1,\varepsilon_2}\times\mathring{D}$ where $\mathring{D}$ is an open 2-disc (notice that the boundary $\partial T_\delta(X_{\varepsilon_1,\varepsilon_2})$ lies on $S_{\varepsilon_1}\cup S_{\varepsilon_2}$).

iii) If $U_\alpha\cap\Sigma_{\varepsilon_1,\varepsilon_2}\neq\emptyset$, for any $x\in U_\alpha$ such that $v_\alpha(x)\neq0$, we have $d\sigma_\alpha(v_\alpha(\mathrm{x}))<0$ and $d\sigma'_\alpha(v_\alpha(x))<0$, moreover at any point $x\in U_\alpha$ such that $\sigma_\alpha(x)=\sigma'_\alpha(x)$ we have $d\sigma_\alpha(v_\alpha(x))=d\sigma'_\alpha(v_\alpha(x))$ and at any point x of $U_\alpha\cap\varphi(\partial T_\delta(\bar{X}_{\varepsilon_1,\varepsilon_2}))$, $v_\alpha(x)$ is tangent to $\varphi(\partial T_\delta(\bar{X}_{\varepsilon_1,\varepsilon_2}))=\partial T_\delta(X_{\varepsilon_1,\varepsilon_2})$.

Now let $(\varphi_\alpha)_{\alpha\in A}$ be a partition of unity associated to the covering $(U_\alpha)_{\alpha\in A}$ of $T_\delta(X_{\varepsilon_1,\varepsilon_2})$, we have a vector field $v=\sum_{\alpha\in A}\varphi_\alpha v_\alpha$ on $T_\delta(X_{\varepsilon_1,\varepsilon_2})$ the integration of which gives the diffeomorphism of $T_\delta(X_{\varepsilon_1,\varepsilon_2})$ onto $T_{\delta'}(X_{\varepsilon_1,\varepsilon_2})$ for any $\delta\geqq\delta'>0$. This proves the fourth assertion of lemma (1.2.4.).

The space $T_\delta(\bar{X}_{\varepsilon_1,\varepsilon_2})-\varphi^{-1}(X_{\varepsilon_1,\varepsilon_2})$ is readily a tubular neighborhood of $\bar{X}_{\varepsilon_1,\varepsilon_2}-n^{-1}(\Sigma)$ (wich is diffeomorphic to $X_{\varepsilon_1,\varepsilon_2}-\Sigma$ by φ) minus its center, i.e. $\bar{X}_{\varepsilon_1,\varepsilon_2}-n^1(\Sigma)$ itself. Besides of it its projection into $T_\delta(X_{\varepsilon_1,\varepsilon_2})$ by φ has overlappings near the singularities Σ. Therefore for the fifth assertion, one may take, $\tilde{T}_\delta$ to be the interior of the space $\{z\in T_\delta(\bar{X}_{\varepsilon_1,\varepsilon_2})\colon\sigma(z)=\inf\{\sigma(x)\colon x\in\varphi^{-1}(\varphi(z))\}\}$. To prove α) and β) we proceed as above, using vector fields.

Actually we have a more precise result than the one of 5) α) of Lemma (1.2.4.):

(1.2.5.) Lemma. *Let ε', $\varepsilon_2\geqq\varepsilon'\geqq\varepsilon_1$. The space $\varphi(\tilde{T}_\delta)\cap S^5_{\varepsilon'}=T(K^*_{\varepsilon'})$ is a tubular neighbourhood of $(X-\Sigma)\cap S^5_{\varepsilon'}=K^*_{\varepsilon'}$ in $S^5_{\varepsilon'}$. Moreover $\varphi(\tilde{T}_\delta)$ is diffeomorphic to $[\varepsilon_1,\varepsilon_2]\times T(K^*_{\varepsilon'})$.*

This lemma comes from the fact $S_{\varepsilon'}$ is transverse to $\varphi(\tilde{T}_\delta)$ and Lemma (1.2.4.).

(1.2.6.) Let $P_{\varepsilon_1,\varepsilon_2}:=B^6_{\varepsilon_2}\times B^6_{\varepsilon_2}-\overset{\circ}{\widehat{B^6_{\varepsilon_1}\times B^6_{\varepsilon_1}}}$ for $\varepsilon_2>\varepsilon_1>0$ and $\overset{\circ}{\widehat{B^6_{\varepsilon_1}\times B^6_{\varepsilon_1}}}$ being the interior of $B^6_{\varepsilon_1}\times B^6_{\varepsilon_1}$.

Let δ_0 be the constant defined in the lemma (1.2.4.) for $\varepsilon=\varepsilon_2>\varepsilon_1$. Then let us choose ε_0 and δ such that $\delta_0 \geqq \delta$ and $\varepsilon_2 \geqq \varepsilon_0 \geqq \varepsilon_1+\delta$.

Let W_δ be the inverse image of $V_\delta(\Delta^*)$ by $\psi\colon B_\varepsilon^6 \times (n^{-1}(X \cap B_\varepsilon^6)) \to B_\varepsilon^6 \times B_\varepsilon^6$ where the first component is the projection and the other is induced by the composition of the normalization and the inclusion of $X \cap B_\varepsilon^6$ into B_ε^6. We denote $\overline{W}_\delta := \psi^{-1}(\overline{V}_\delta(\Delta^*))$.

We have the inclusion $u\colon Z_1 \cap W_\delta \cap \psi^{-1}(P_{\varepsilon_0,\varepsilon_2}) \subset Z_1 \cap \psi^{-1}(P_{\varepsilon_0,\varepsilon_2})$ and the mapping $q\colon Z_1 \cap \psi^{-1}(P_{\varepsilon_0,\varepsilon_2}) \to B_\varepsilon - X$ induced by the first projection.

(1.2.7.) Lemma. *The composition $q \circ u$ factorizes through $T_\delta(\overline{X}_{\varepsilon_0,\varepsilon_2}) - \varphi^{-1}(X)$:*

$$\begin{tikzcd} Z_1 \cap W_\delta \cap \psi^{-1}(P_{\varepsilon_0,\varepsilon_2}) \arrow[rr, "h"] \arrow[dr, "q\circ u"'] & & T_\delta(\overline{X}_{\varepsilon_1,\varepsilon_2}) - \varphi^{-1}((X) \arrow[dl, "\varphi_0"] \\ & B_\varepsilon - X & \end{tikzcd}$$

where φ_0 is induced by φ.

(1.2.8.) We have the following commutative diagram:

$$\begin{tikzcd} Z_1 \cap W_\delta \cap \psi^{-1}(P_{\varepsilon_0,\varepsilon_2}) \arrow[r, "h"] \arrow[d, hook, "u"] \arrow[dr, "q\circ u"] & T_\delta(\overline{X}_{\varepsilon_1,\varepsilon_2}) - \varphi^{-1}(X) \arrow[d, "\varphi_0"] & \tilde{T}_\delta - i(\overline{X}_{\varepsilon_1,\varepsilon_2}) \arrow[l, hook', "k"'] \arrow[d, "\varphi_1"] \\ Z_1 \cap \psi^{-1}(P_{\varepsilon_0,\varepsilon_2}) \arrow[r, "q"'] & B_\varepsilon^6 - X & \varphi(\tilde{T}_\delta - i(\overline{X}_{\varepsilon_1,\varepsilon_2})) \arrow[l, hook'] \\ & S_\varepsilon^5 - X \arrow[u, hook, "j"] & T(K_\varepsilon^*) - K_\varepsilon^* \arrow[l, hook', "a"'] \arrow[u, hook, "i"'] \end{tikzcd}$$

where $\tilde{T}_\delta$ is defined by Lemma (1.2.4.) 5-α) where in this case $\varepsilon=\varepsilon_2$ φ_1 is induced by φ and, i, j, a, k, u are inclusion.

All the spaces above are connected and denoting by $*$ the adequate base point, the above diagramm gives for the fundamental groups the following commutative diagramm of groups:

$$\begin{tikzcd} \pi_1(Z_1 \cap W_\delta \cap \psi^{-1}(P_{\varepsilon_0,\varepsilon_2}), *) \arrow[r, "h_*"] \arrow[d, "u_*"] \arrow[dr, "(q\circ u)"] & \pi_1(T_\delta(\overline{X}_{\varepsilon_1,\varepsilon_2}) - \varphi^{-1}(X), *) \arrow[d, "(\varphi_0)_*"] & \pi_1(\tilde{T}_\delta - i(\overline{X}_{\varepsilon_1,\varepsilon_2}), *) \arrow[l, "k_*"'] \arrow[d, "(\varphi_1)_*"] \\ \pi_1(Z_1 \cap \psi^{-1}(P_{\varepsilon_0,\varepsilon_2}), *) \arrow[r, "q_*"'] & \pi_1(B_\varepsilon^6 - X, *) & \pi_1(\varphi(\tilde{T}_\delta - i(X_{\varepsilon_1,\varepsilon_2}), *) \arrow[l] \\ & \pi_1(S_\varepsilon^5 - X, *) \arrow[u, "j_*"] & \pi_1(T(K_\varepsilon^*) - K_\varepsilon^*, *) \arrow[l, "\alpha = a_*"'] \arrow[u, "i_*"'] \end{tikzcd}$$

As j_* is an isomorphism because of the local conic structure of X at 0 (cf. 1.2.2.)), i_* is an isomorphism because of lemma (1.2.5.), $(\varphi_1)_*$ is an isomorphism because of 5)—α) of lemma (1.2.4.) and k_* is an isomorphism because of the 5)—β) of lemma (1.2.4.) we obtain that α is an epimorphism if and only if $(\varphi_0)_*$ is an epimorphism. From (1.1.5.) to prove our main theorem it remains to prove the surjectivity of $(\varphi_0)_*$. Obviously if u_* is surjective, $(\varphi_0)_*$ is surjective, because, $Z_1 \cap \psi^{-1}(P_{\varepsilon_0, \varepsilon_2})$ containing $(B^6_\varepsilon - X) \times \{y\}$ for any $y \in n^{-1}(X \cap B_{\varepsilon_0, \varepsilon_2})$, and $Z_1 := (B^6_\varepsilon - X) \times n^{-1}(X \cap B^6_\varepsilon - \{0\})$, q_* is already surjective.

(1.2.9.) To prove the surjectivity of u_*, one considers the following commutative diagramm:

$$\begin{tikzcd}
Z \cap L' \cap S^{11}_{\varepsilon'} \arrow[r, hook] \arrow[dr, "w_0"'] & Z \cap V_\delta \cap S^{11}_{\varepsilon'} \arrow[r, hook] \arrow[d, hook, "w"] & Z \cap V_\delta \cap P_{\varepsilon_0, \varepsilon_2} \arrow[r] \arrow[d, hook, "v"] & Z_1 \cap W_\delta \cap \psi^{-1}(P_{\varepsilon_0, \varepsilon_2}) \arrow[d, hook, "u"] \\
& Z \cap S^{11}_{\varepsilon'} \arrow[r, hook, "s"'] & Z \cap P_{\varepsilon_0, \varepsilon_2} \arrow[r, "r"'] & Z_1 \cap \psi^{-1}(P_{\varepsilon_0, \varepsilon_2})
\end{tikzcd}$$

where r is induced by the identity on $B_\varepsilon - X$ and the natural section of n over $X - \Sigma$ and u, v, w, w_0, s are inclusions. Moreover ε' and ε_2 are chosen such that $S^{11}_{\varepsilon'} \subset P_{\varepsilon_0, \varepsilon_2}$ and $(Z \cap S^{11}_{\varepsilon'}) \times]0, 1]$ is homeomorphic to $Z \cap B_{\varepsilon'}$. Then L' is a linear 3-space of $\mathbf{C}^3 \times \mathbf{C}^3$ going through 0 such that $Z \cap L' \cap S^{11}_{\varepsilon'}$ is contained in $Z \cap V_\delta \cap S^{11}_{\varepsilon'}$.

Now r_* is surjective because the complement of $r(Z \cap P_{\varepsilon, \varepsilon_1})$ has real codimension two in $Z_1 \cap \psi^{-1}(P_{\varepsilon_0, \varepsilon_2})$. On the other hand s will be shown to be a homotopy equivalence (cf. (1.2.12.)). Thus to prove the surjectivity of u_* it will be enough to get the surjectivity of $(w_0)_*$.

(1.2.10.) Now let ε'' be small enough, $\varepsilon' \gg \varepsilon'' > 0$ such that the inclusions:

$$Z \cap B^{12}_{\varepsilon'} \supset Z \cap B^{12}_{\varepsilon''} \supset Z \cap S^{11}_{\varepsilon''}$$

$$Z \cap L' \cap B^{12}_{\varepsilon''} \supset Z \cap L' \cap S^{11}_{\varepsilon''}$$

— are homotopy equivalences. From [H—L$_2$] (compare to H—L$_1$ (théorème (0.2.1.)) and [D$_2$] (Assertion 1.3.)) if L' is sufficiently general for $x \in Z \cap L' \cap B^{12}_{\varepsilon''}$, the homomorphism:

$$(w_1)_* \colon \pi_1(Z \cap L' \cap B^{12}_{\varepsilon''}, x) \to \pi_1(Z \cap B^{12}_{\varepsilon''}, x)$$

induced by the inclusion is an epimorphism because Z is non singular and $\dim Z \cap L' = 2$.

Moreover one can show that the inclusion of $Z \cap L' \cap S^{11}_{\varepsilon'}$ into $Z \cap L' \cap B^{12}_{\varepsilon'}$ induces an epimorphism:

$$(c_1)_* \colon \pi_1(Z \cap L' \cap S^{11}_{\varepsilon'}, x) \to \pi_1(Z \cap L' \cap B^{12}_{\varepsilon'}, x)$$

for any $x \in Z \cap L' \cap S^{11}_{\varepsilon'}$.

To prove this last fact first notice that the closure of $\overline{Z \cap L'}$ is a surface and

$\overline{Z\cap L'}-Z\cap L'$ is the intersection of $B^6_\varepsilon\times(\Sigma\cap B^6_\varepsilon)\cup X\times X$ and $L'\cap B^6_\varepsilon\times B^6_\varepsilon$. By a genericity argument one sees that, if L' is sufficiently general the intersections of $X\times X$ and $B^6_\varepsilon\times(\Sigma\cap B^6_\varepsilon)$ with L' are transverse at any point in $B^{12}_{\varepsilon'}-\{0\}$. Thus inside $\mathring{B}^{12}_{\varepsilon'}$ this intersection is a non singular curve outside 0. Actually outside of 0 the components of the intersection of L' with $X\times X$ are composed of non singular points of $Z\cap L'$ and the other components of $\overline{Z\cap L'}-Z\cap L'$ are self intersections of $\overline{Z\cap L'}$ which are normal crossing points of two non singular components outside of 0. Notice moreover that this curve $(\overline{Z\cap L'}-Z\cap L')\cap B^{12}_{\varepsilon'}$ is defined by complex analytic equations on a neighbourhood of $B^{12}_{\varepsilon'}$. Finally, for a general 3-plane L', $S^{11}_{\varepsilon'}$ cuts $Z\cap L'$ transversally, because $S^{11}_{\varepsilon'}$ cuts Z transversally and the space of complex hyperplanes which are not transverse to $Z\cap S^{11}_{\varepsilon'}$ has a real dimension at most equal to 9. For the same reason $S^{11}_{\varepsilon'}$ cuts $(X\times X)\cap L'$ and $(\mathbf{C}^3\times\Sigma)\cap L'$ transversally if L' is sufficiently general.

Now our assertion will be a consequence of:

(1.2.11.) Lemma. *Let V be a complex analytic surface closed in an open neighbourhood U of 0 in $\mathbf{C}^n$ and $0\in V$. Suppose that the singular points of V are normal crossings with two non singular components outside 0. Let Γ be a curve contained in V which contains the singular locus of V, which is non singular outside 0 and which is defined by complex analytic equations in U. Suppose B is a closed ball centered at 0 contained in U, such that its boundary S cuts Γ and $V-\Gamma$ transversally. Then the inclusion $S\cap(V-\Gamma)\subset B\cap(V-\Gamma)$ gives an epimorphism:*

$$\pi_1(S\cap(V-\Gamma),x)\to\pi_1(B\cap(V-\Gamma),x)$$

for $x\in S\cap(V-\Gamma)$.

Proof. We essentially use Morse theory on a manifold with boundary to prove this lemma.

First notice that if B' is a sufficiently small ball the inclusion of $(B-\mathring{B}')\cap(V-\Gamma)$ into $B\cap(V-\Gamma)$ gives an isomorphism of the fundamental groups: if 0 does not belong to Γ, $B\cap(V-\Gamma)$ is obtained from $(B-\mathring{B}')\cap(V-\Gamma)$ by adding the 4-cell $B'\cap V$ if B' is sufficiently small, because $B'\cap\Gamma=\emptyset$; if $0\in\Gamma$, we use the conic structure theorem already quoted (cf. [B—V] loc. cit.) to show that the above inclusion is a homotopy equivalence (actually in that case $(B-B')\cap(V-\Gamma)$ and $B\cap(V-\Gamma)$ are homeomorphic).

We fix such a ball B': notice that its boundary S' cuts Γ and $V-\Gamma$ transversally as any other spheres centered at 0 contained in it.

Now let us consider the restriction of the distance function to 0 on Γ. By eventually replacing it by the distance function σ to a point near to 0, we may assume that the restrictions of σ to Γ and $V-\Gamma$ are Morse functions. If r and r' were the radii of B and B' for a point chosen near enough to 0 the sets $\sigma^{-1}(r)\cap(V-\Gamma)$,

$\{\sigma \leqq r\} \cap (V-\Gamma)$, $\{r' \leqq \sigma \leqq r\} \cap V-\Gamma$ are respectively homeomorphic to $S \cap (V-\Gamma)$, $B \cap (V-\Gamma)$ and $(B-\mathring{B}') \cap (V-\Gamma)$.

Let $f_i=0$ $i=1, \dots, k$ be the equations of Γ in U. Let φ be the restriction of $\sum_{i=1}^k |f_i|^2$ to V. We shall denote by $T_\alpha (\alpha \in \mathbf{R}^+)$ the sets $\{\varphi \leqq \alpha\}$ and $\partial T_\alpha = \{\varphi = \alpha\}$. Thus $T_0 = \Gamma$. One sees that for $\alpha > 0$ small enough, ∂T_α is transverse to S and S'. Moreover if α is small enough the inclusion $(V-\mathring{T}_\alpha) \cap (B-\mathring{B}') \subset (V-\Gamma) \cap (B-\mathring{B}')$ is a homotopy equivalence, as well as $(V-\mathring{T}_\alpha) \cap S \subset (V-\Gamma) \cap S$.

We are going to do Morse theory on the manifold with corners $(V-\mathring{T}_\alpha) \cap B$ (compare to [H—L$_1$] (§ 3)). Actually, because of the above remark, one must consider $(V-\mathring{T}_\alpha) \cap \{\sigma \leqq r\}$ instead. To be simple and avoid cumbersome notations we shall assume that the function σ is actually defined by the distance function to 0. The Morse function considered is then $-\sigma$.

We shall prove that $(V-\mathring{T}_\alpha) \cap (B-\mathring{B}')$ is obtained from $(V-\mathring{T}_\alpha) \cap S$ by adding cells of real dimension $\sqrt{2}$ at least and this will prove our lemma.

Actually in our proof the restriction of the Morse function $-\sigma$ has no critical point on the corners. Thus these corners do not bother. The restriction of $-\sigma$ to $V-\Gamma$ has critical points of indexes at least 2 because of the well-known result of A. Andreotti—T. Frankel (cf. [A—F]). It remains to consider the critical points on the boundary. It is known that in that case the only critical points giving cells are the ones for which the gradient of $-\sigma$ is entering our manifold (compare to [H—L$_1$] (§ 3)). We shall show that precisely for these points the indexes are at least 2 (for that purpose the method indicated is very similar to [H—L$_1$] (§ 5)).

Let x be a non-degenerate critical point of the restriction of $-\sigma$ to Γ. For α small enough the critical points of the restriction of $-\sigma$ on ∂T_α can only appear near to a critical point x of the restriction of $-\sigma$ on Γ. Because if $-\sigma$ is transverse to Γ at some point $y \in \Gamma$ then for α small enough $-\sigma$ is transverse to ∂T_α in a neighbourhood of $y \in \Gamma$ in V.

Actually, in the case x is a non singular point of V, the non-degeneracy of the critical point x of the restriction of $-\sigma$ on Γ makes that, for any α small enough, it appears two critical points x_α and x'_α of the restriction of $-\sigma$ on ∂T_α which tend to x as α tends to zero. By using the implicit function theorem one shows that, for $\alpha > 0$ small enough, x_α and x'_α lie on a real non singular curve going through x.

Proceeding as in [H—L$_1$] (§ 5) (compare to [H—L$_2$]) the index of the restriction of $-\sigma$ on ∂T_α at x_α or x'_α is equal to the index i at x of the restriction of $-\sigma$ on Γ or to $i+1$ according to the fact the gradient of $-\sigma$ is going out or entering $(V-\mathring{T}_\alpha) \cap (B-\mathring{B}')$. Using again the quoted result of A. Andreotti and T. Frankel (cf. [A—F]), we obtain that $i \geqq 1$ and at critical points of the restriction of $-\sigma$ on ∂T_α, when $\alpha > 0$ is small enough, and with entering gradient into $(V-\mathring{T}_\alpha) \cap (B-\mathring{B}')$, the indexes will be at least 2.

In the case the critical point x of the restriction of $-\sigma$ to Γ lies on the singular locus of V, as, at x there are two non-singular components having distinct tangent planes, we make the above argument on each of the components. In that case there will be four critical points of the restriction of $-\sigma$ to ∂T_α which tend to x when α tends to 0 and at two of them the gradient of $-\sigma$ is entering $(V-\mathring{T}_\alpha)\cap(B-\mathring{B}')$ and the indexes are at least equal to 2.

This ends the proof of lemma (1.2.11.).

From the commutative diagramm of inclusions:

$$\begin{tikzcd}
Z\cap L'\cap S^{11}_{\varepsilon'} \arrow[r, hook, "w_0"] \arrow[d, hook, "c_1"] & Z\cap S^{11}_{\varepsilon'} \arrow[d, hook, "b_1"] \\
Z\cap L'\cap B^{12}_{\varepsilon'} \arrow[r, hook, "w_2"'] & Z\cap B^{12}_{\varepsilon'} \\
Z\cap L'\cap B^{12}_{\varepsilon''} \arrow[r, hook, "w_1"'] \arrow[u, hook, "c_2"] & Z\cap B^{12}_{\varepsilon''} \arrow[u, hook, "b_2"]
\end{tikzcd}$$

we obtain the commutative diagramm of fundamental groups:

$$\begin{tikzcd}
\pi_1(Z\cap L'\cap S^{11}_{\varepsilon'}, *) \arrow[r, "(w_0)_*"] \arrow[d, "(c_1)_*"] & \pi_1(Z\cap S^{11}_{\varepsilon'}, *) \arrow[d, "(b_1)_*"] \\
\pi_1(Z\cap L'\cap B^{12}_{\varepsilon'}, *) \arrow[r, "(w_2)_*"'] & \pi_1(Z\cap B^{12}_{\varepsilon'}, *) \\
\pi_1(Z\cap L'\cap B^{12}_{\varepsilon''}, *) \arrow[r, "(w_1)_*"'] \arrow[u, "(c_2)_*"] & \pi_1(Z\cap B^{12}_{\varepsilon''}, *) \arrow[u, "(b_2)_*"]
\end{tikzcd}$$

As (w_1) is an epimorphism and $(b_1)_*$, $(b_2)_*$ are isomorphism, $(w_2)_*$ is an epimorphism, thus $(w_0)_*$ is an epimorphism, because we saw that $(c_1)_*$ is an epimorphism.

This implies the surjectivity of α and proves the main theorem when X is irreducible at 0.

(1.2.12.) To complete the proof of the main theorem in the case X is irreducible at 0, it remains to prove that s: $Z\cap S_{\varepsilon'}\hookrightarrow Z\cap P_{\varepsilon_0,\varepsilon_2}$ is a homotopy equivalence.

To prove this last fact we notice that, for $\varepsilon>0$ small enough, the following stratification of $X\cap\mathring{B}_\varepsilon$ obviously satisfies the Whitney condition:

$$X\cap\mathring{B}_\varepsilon=((X-\Sigma)\cap\mathring{B}_\varepsilon)\cup((\Sigma-(0))\cap\mathring{B}_\varepsilon)\cup\{0\}.$$

From this results immediately that the stratification of $\bar{Z}\cap(\mathring{B}_\varepsilon\times\mathring{B}_\varepsilon)$ given by the following 12 stratas satisfies the Whitney condition (to avoid too many notations we assume X and Σ to be closed in $\mathring{B}_\varepsilon$):

$$(\mathring{B}_\varepsilon-X)\times(X-\Sigma),\quad (X-\Sigma)\times(X-\Sigma),\quad (\mathring{B}_\varepsilon-X)\times(\Sigma-\{0\}),$$
$$(X-\Sigma)\times(\Sigma-\{0\}),\quad (\Sigma-\{0\})\times(\Sigma-\{0\}),\quad (\mathring{B}_\varepsilon-X)\times\{0\},\quad (X-\Sigma)\times\{0\},$$
$$(\Sigma-\{0\})\times\{0),\quad (\Sigma-\{0\})\times(X-\Sigma),\quad \{0\}\times(X-\Sigma),\quad \{0\}\times(\Sigma-\{0\}),\quad \{0\}.$$

Now if one chooses $\varepsilon_0 < \varepsilon$, for any ε' $0 < \varepsilon' < \varepsilon_0$ we have that the boundary and the corners of $B^{12}_{\varepsilon'} \times B^{12}_{\varepsilon'}$ cut this stratification of $\bar{Z}$ transversally. Thus as it is shown in [B—V], the polydiscs $B^6_{\varepsilon'} \times B^6_{\varepsilon'}$, for $0 < \varepsilon' < \varepsilon_0$, will define a fundamental system of neighbourhoods of 0 in $\bar{Z}$ relatively to $\bar{Z} - Z$. As the balls $B^{12}_{\varepsilon''}$ define already good neighbourhoods of 0 in $\bar{Z}$ relatively to $\bar{Z} - Z$, this shows, using again the conic structure of the singularity (cf. [B—V] loc. cit.) that for ε_0 small enough the inclusion $Z \cap S' \subset Z \cap P_{\varepsilon_0, \varepsilon_2}$ is a homotopy equivalence.

2. The reducible case

In this paragraph we show how to get the main theorem when $(X, 0)$ has several analytic components from the case when there is only one component.

(2.1.) We could have proceeded as P. Deligne in [D_2], but actually we prove a result similar to the one obtained by M. Oka and K. Sakamoto in [O—S]. In fact this last result was proved in the projective case. To adapt it to our situation which is local, we shall see we need to be much more technical. Thus our main theorem, as stated in the introduction, will be a consequence of the result already obtained in the first paragraph for irreducible hypersurfaces and the following lemma:

(2.1.1.) Lemma.. *Let X be an analytic hypersurface in an open neighbourhood U of 0 in $\mathbf{C}^3$ and $0 \in X$. Suppose it decomposes locally at 0 into two analytic hypersurfaces, i.e. if U is chosen small enough:*

$$X = X_1 \cup X_2$$

moreover suppose that at any point $x \in X_1 \cap X_2 - 0$ both X_1 and X_2 are non singular and intersect transversally. Then for $\varepsilon > 0$ small enough there is a canonical isomorphism:

$$\pi_1(B_\varepsilon - X, x) \xrightarrow{\sim} \pi_1(B_\varepsilon - X_1, x) \times \pi_1(B_\varepsilon - X_2, x) \quad \textit{with} \quad x \in B_\varepsilon - X.$$

(2.1.2.) Remark. Notice that X_1 and X_2 might be non-irreducible and have "bad" singularities. Moreover using the quoted result of [H—L_1] (théorème (0.2.1.)) there is a similar lemma for hypersurfaces in $\mathbf{C}^n$.

Proof. Suppose X_1 and X_2 are defined in U by the analytic equations $f_1 = 0$ and $f_2 = 0$:

$$X_i := \{(x, y, z) \in U,\ f_i(x, y, z) = 0\} \quad i = 1, 2.$$

For the sake of simplicity in the proof we may suppose U to be a ball centered at 0.

(2.1.3.) We recall that from the work of [H—L_1] if H_0 is a sufficiently general plane of $\mathbf{C}^3$ going through 0, for $\varepsilon > 0$ small enough and for affine planes H_c parallel to H_0 and near to H_0, $H_0 \neq H_c$, the fundamental groups of $(B_\varepsilon - X) \cap H_c$ and $(B_\varepsilon - X_i) \cap H_c$ $(i = 1, 2)$ are respectively isomorphic to the ones of $B_\varepsilon - X$ and $B_\varepsilon - X_i$

$(i=1, 2)$. From [L_1] for instance, one finds it is enough to assume $H_0 \cap X$ and $H_0 \cap X_i$ $(i=1, 2)$ to be reduced to get the preceding property.

(2.1.4.) Let us choose coordinates in a way such that $f_1(x, 0, 0) \not\equiv 0$ and $f_2(0, y, 0) \not\equiv 0$ and moreover the topological type of the plane curve $f_1(x, 0, z)=0$ at 0 is the one of a general plane section of X_1 at 0 and such that the topological type of the plane curve $f_2(0, y, z)=0$ at 0 is the one of a general plane section of X_2 at 0. (Recall that a general plane section of X_i at 0 $(i=1, 2)$ has a well defined equisingular type by using the fact that reduced plane sections of X_i define a family of plane curves for which there is a well defined general equisingular type at 0 (cf. [Z_1])).

Following M. Oka and K. Sakamoto in [O—S] we consider the hypersurface $\mathscr{X}$ of $\bar{D} \times U$, where $\bar{D}=\{t \in \mathbf{C}, |t| \leqq 1\}$, defined by:

$$\mathscr{X} = \{(t, x, y, z) \in \bar{D} \times U,\ f_1(x, ty, z) f_2(tx, y, z) = 0\}.$$

Notice that we have a natural mapping $\mathscr{X} \xrightarrow{p} \bar{D}$ induced by the projection. Then $p^{-1}(0)$ is given by the union of two cylinders $f_1(x, 0, z)=0$ and $f_2(0, y, z)=0$, and $p^{-1}(1)$ corresponds to our original hypersurface X. We shall denote by $X_1(t)$ and $X_2(t)$ the hypersurfaces of U defined by $f_1(x, ty, z)=0$ and $f_2(tx, y, z)=0$ respectively and $\mathrm{X}(t)=X_1(t) \cup X_2(t)$.

(2.1.5.) Now let H_0 be a general hyperplane going through 0 defined by $z=ax+by$, with a and b sufficiently general and both non zero, such that $H_0 \cap X$ has at 0 the topological type of a general plane section at 0.

We shall denote $C_i(t)=X_i(t) \cap H_0$.

According to (2.1.3.), for $\varepsilon>0$ small enough, if $H_c:=\{z=ax+by+c\}$, with $0<|c| \ll \varepsilon$, the fundamental group of $(B_\varepsilon - X) \cap H_c$ is isomorphic to the one of $B_\varepsilon - X$.

The intersection of $p^{-1}(0)$ with $\{0\} \times H_0$ is given by two curves C_1 and C_2 in general position, i.e. with tangent cones without common components if a and b have been conveniently chosen, and C_i has at 0 the topological type of a section of X_i by a general plane $(i=1, 2)$.

As the intersection of $p^{-1}(t)$ with $\{t\} \times H_0$ is given, for all t except a finite number, by two curves having respectively at 0 the topological types of general plane sections of X_1 and X_2 at 0, we shall prove that for all t except a finite number, the topological type at 0 of $p^{-1}(t) \cap (\{t\} \times H_0)$ will be the one of $p^{-1}(0) \cap (\{0\} \times H_0)$. Actually we notice that the Milnor number $\mu(t)$, of $p^{-1}(t) \cap (\{t\} \times H_0)$ at $(t, 0)$ equals $\mu_1+\mu_2+2\big(C_1(t) \cdot C_2(t)\big)_0 - 1$, where μ_i is the Milnor number of $(C_i, 0)$ or $\big(C_i(t), 0\big)$ $(i=1, 2)$ and $(\ \cdot\ )_0$ denotes the intersection multiplicity at 0. As the lowest value of $\mu(t)$ is obtained when $t=0$, using a known result proved by [L_2] (Corollaire 7) (cf. [L—R] (§ 3) too), we obtain that, for all t except a finite number, the equisingular type of $p^{-1}(t)$ at $(t, 0)$ is the one of $p^{-1}(0)$ at $(0, 0)$. In particular for these values of t necessarily the tangent cones of $C_1(t)$ and $C_2(t)$ have no common components.

Moreover notice that in $\{0\} \times U$, the plane $\{0\} \times H_0$ cuts the singular locus of

$p^{-1}(0)$ transversally. Thus for t small enough, $\{t\}\times H_0$ cuts the tangent cone of the singular locus of $p^{-1}(t)$ transversally. This implies that for t small enough, there is $\varepsilon_t>0$ such that for any c, $0<|c|\ll\varepsilon_t$, the singular points of $H_c\cap X_i(t)$ $(i=1,2)$ are the ones where H_c cuts the singular locus of $X_i(t)$ and for $t\neq 0$ their corresponding equisingular type is the one of $X_i(t)$ along the corresponding branch of the singular locus.

On the other hand, if $\varepsilon_0>0$ is small enough, because of the assumptions in (2.1.4.) that $f_1(x,0,0)\not\equiv 0$ and $f_2(0,y,0)\not\equiv 0$, for any $c\neq 0$ small enough, $X_1(0)\cap H_c$ and $X_2(0)\cap H_c$ cut transversally inside B_{ε_0} at $(C_1(0)\cdot C_2(0))_0$ points. Thus for $t\neq 0$ small enough, for any $c\neq 0$ small enough, $X_1(t)\cap H_c$ and $X_2(t)\cap H_c$ cut transversally at $(C_1(0)\cdot C_2(0))_0=(C_1(t)\cdot C_2(t))_0$ points inside B_{ε_0}.

(2.1.6.) From the preceding observation we find a real analytic path α: $[0,1]\to\bar{D}$ such that:

1° $\alpha(0)=0$ and $\alpha(1)=1$;

2° For $\tau\neq 1$ the equisingular type of $p^{-1}(\alpha(\tau))\cap(\{\alpha(\tau)\}\times H_0)$ at $(\alpha(\tau),0)$ is the one of $p^{-1}(0)\cap(\{0\}\times H_0)$ at $(0,0)$;

3° For $\tau\in[0,1]$ the plane H_0 cuts transversally at 0 the tangent cones of the singular loci of $X_1(\alpha(\tau))$ and $X_2(\alpha(\tau))$;

4° For $\tau\in[0,1]$, there is $\varepsilon_\tau>0$ small enough, such that for any c, $0<|c|\ll\varepsilon_\tau$, the curves $H_c\cap X_1(\alpha(\tau))$ and $H_c\cap X_2(\alpha(\tau))$ cut transversally inside B_{ε_τ} at $(C_1(\alpha(\tau))\cdot C_2(\alpha(\tau)))_0$ points.

(2.1.7.) Now the coordinates are fixed as it was described in (2.1.4.), the hyperplane $H_0:=\{z=ax+by\}$ is given by (2.1.5.) and the real analytic path is defined as in (2.1.6.).

According to (2.1.3.) there is $\varepsilon>0$ such that:

1° The local fundamental group at 0 of the complement of X in $\mathbf{C}^3$ is given by the fundamental group of $B_\varepsilon-X$;

2° There is $\gamma_1>0$ such that for any c, $0<|c|\leqq\gamma_1$, the fundamental group of $H_c\cap(B_\varepsilon-X)$ is isomorphic to the one of $B_\varepsilon-X$, and moreover the curve $X\cap H_c$ cuts S_ε transversally;

3° The singularities of the curve $X\cap H_c$ are the ones of $X_1\cap H_c$ and $X_2\cap H_c$ and the intersections points of $X_1\cap H_c$ and $X_2\cap H_c$, the number of which is $(X_1\cap H_0\cdot X_2\cap H_0)_0$ are ordinary double points.

Let us first fix such an $\varepsilon>0$. There is $\tau_0\neq 1$ near enough to 1 such that:

1° For any $\tau_0\leqq\tau\leqq 1$, the curve $C(\alpha(\tau))=C_1(\alpha(\tau))\cup C_2(\alpha(\tau))$ (where $(i=1,2)$: $C_i(\alpha(\tau))=X_i(\alpha(\tau))\cap H_0)$ cuts transversally S_ε and there is γ_1' such that for any c, $0<|c|\leqq\gamma_1'$, the curve $(X_1(\alpha(\tau))\cap X_2(\alpha(\tau)))\cap H_c$ cuts S_ε transversally.

2° As $X_1\cap H_c$ and $X_2\cap H_c$ cut transversally each other in $H_c\cap B_\varepsilon$, we may ask that for any τ, $\tau_0\leqq\tau\leqq 1$, the curves $X_1(\alpha(\tau))\cap H_c$ and $X_2(\alpha(\tau))\cap H_c$ cut transversally each other in $H_c\cap B_\varepsilon$ at $(X_1\cap H_0\cdot X_2\cap H_0)_0$ points.

Then we have:

(2.1.8.) For any τ, $\tau_0 \leqq \tau \leqq 1$, and any c, $0 < |c| \leqq \inf(\gamma_1, \gamma_1')$ the fundamental groups of $(B - X(\alpha(\tau))) \cap H_c$ are isomorphic to the local fundamental group at 0 of the complement of X in $\mathbf{C}^3$.

Actually one proves that the pairs $(B_\varepsilon \cap H_c, B_\varepsilon \cap H_c \cap X(\alpha(\tau))$, are homeomorphic for any τ, $\tau_0 \leqq \tau \leqq 1$, and any c, $0 < |c| \leqq \inf(\gamma_1, \gamma_1')$.

This result comes from the fact that, for any τ and c as considered, the curves $H_c \cap X(\alpha(\tau))$ cut S_ε transversally and the singularities of these curves inside B_ε are equisingularly the same for the considered τ and c.

(2.1.9.) A standard argument of equisingularity theory allows us to build-up explicitly a Lipschitz vector field in $\alpha([\tau_0, 1]) \times (B_\varepsilon \cap H_c)$ which realizes by integration the searched homeomorphisms.

(2.1.10.) According to (2.1.3.) again there is $\varepsilon' > 0$, $\varepsilon' \leqq \varepsilon$, such that $B_{\varepsilon'} - X(\alpha(\tau_0))$ has its fundamental group isomorphic to the local fundamental group at 0 of the complement of $X(\alpha(\tau_0))$ in $\mathbf{C}^3$ and there is $\gamma_2 > 0$, such that, for any c, $0 < |c| \leqq \gamma_2$, the fundamental group of $(B_{\varepsilon'} - X(\alpha(\tau_0))) \cap H_c$ is isomorphic to $B_{\varepsilon'} - X(\alpha(\tau_0))$.

Now to end our proof of lemma (2.1.1.) we need a more precise description of our fundamental groups by generators and relations.

(2.1.11.) One knows the following fact (cf. [H—L$_1$] Lemme (4.2.3.)):

If L_0 is a sufficiently general line going through 0 defined by $l_1 = l_2 = 0$ (where l_i are linear forms), for $\varepsilon > 0$ small enough and $0 < \eta \ll \varepsilon$, if $T_\eta := \{|l_1|^2 + |l_2|^2 \leqq \eta\}$ the inclusion:

$$T_\eta \cap B_\varepsilon - X \subset B_\varepsilon - X$$

is a homotopy equivalence (actually one may prove these spaces are homeomorphic).

One may choose L_0 in H_0 (and $l_1 = z - ax - by$) and then if $\varepsilon > 0$ is small enough, $0 < \eta \ll \varepsilon$, $0 < \gamma_1'' \ll \eta$, $0 < |c| \leqq \gamma_1''$ the inclusion:

$$H_c \cap (T_\eta \cap B_\varepsilon - X) \subset H_c \cap (B_\varepsilon - X)$$

is a homotopy equivalence (again here, one may prove these spaces are homeomorphic).

(2.1.12.) One sees from this last statement how one can obtain the fundamental group of $H_c \cap (B_\varepsilon - X)$ by generators and relations, because if c is small enough, l_2 induces a mapping $\lambda\colon H_c \cap (T_\eta \cap B_\varepsilon - X) \to D_1$ where $D_1 := \{z \in \mathbf{C}, |z|^2 \leqq \eta - |c|^2\}$ which is a C^∞ fibration outside a finite number of "critical" points. Thus by an argument due to Van Kampen in the projective case (cf. [V—K]) the generators of the fundamental group of $H_c \cap (T_\eta \cap B_\varepsilon - X)$ at a point x are given by a base of the free group $\pi_1(\lambda^{-1}(\lambda(x)), x)$ if $\lambda(x)$ is not one of the quoted "critical" points, and the relations are obtained by considering the monodromy of the above generators around the "critical" points (compare with [C—L] theorem (1.2)). Notice that if c

is small enough, among these critical points are the values of l_2 at the singular points of $X \cap H_c$ in B_ε.

(2.13.) Now let us choose L_0 and H_0, $\varepsilon > 0$ and $\varepsilon' > 0$ such that the conditions of (2.1.7.) and (2.1.10.) are fulfilled and such that we obtain these results simultaneously for X and $X(\alpha(\tau_0))$. Moreover we may assume that for any τ, $\tau_0 \leqq \tau \leqq 1$, $(L_0 \cdot X(\alpha(\tau)))_0$ equals the multiplicity of $X(\alpha(\tau))$ at 0 which does not depend on τ. This last condition implies that, if $\varepsilon > 0$, $\varepsilon \geqq \varepsilon' > 0$ are conveniently chosen, and $0 < \eta \ll \varepsilon'$, $0 < \gamma \ll \eta$ there is L defined by $l_1 = c$, $l_2 = c'$ such that:

1° $0 < |c| \leqq \gamma$, $|c'|^2 \leqq \eta - |c|^2$;

2° For any τ, $\tau_0 \leqq \tau \leqq 1$, L cuts $X(\alpha(\tau))$ transversally in B_ε at $(L_0 \cdot X(\alpha(\tau)))_0$ points and $B_\varepsilon \cap L \cap X(\alpha(\tau_0)) \subset B_{\varepsilon'}$;

3° The inclusion $H_c \cap (T_\eta \cap B_{\varepsilon'} - X) \subset H_c \cap (B_\varepsilon - X)$ is a homotopy equivalence;

4° The inclusion $H_c \cap (T_\eta \cap B_{\varepsilon'} - X(\alpha(\tau_0))) \subset H_c \cap (B_\varepsilon - X(\alpha(\tau_0)))$ is a homotopy equivalence.

Because of 2° above the vector field constructed in (2.1.9.) can be built in such a way the image of $B_\varepsilon \cap L - X(\alpha(\tau_0))$ is $B_\varepsilon \cap L - X(\alpha(\tau))$ for any $\tau_0 \leqq \tau \leqq 1$ by the family of homeomorphisms obtained by integration.

As the inclusion $B_{\varepsilon'} \cap L - X(\alpha(\tau_0)) \subset B_\varepsilon \cap L - X(\alpha(\tau_0))$ is obviously an homotopy equivalence because $L \cap X(\alpha(\tau_0)) \cap B_\varepsilon$ is contained in $B_{\varepsilon'}$, the loops, which generate $\pi_1(H_c \cap (T_\eta \cap B_{\varepsilon'} - X(\alpha(\tau_0))), x)$ and thus $\pi_1(B_{\varepsilon'} - X(\alpha(\tau_0)), x)$, give a base of the free group, fundamental group of $L \cap B_\varepsilon - X$ and thus generators of the fundamental group of $B_\varepsilon - X$.

(2.1.14.) Let us suppose that we can choose a base of the free group $\pi_1(B_{\varepsilon'} \cap L - X(\alpha(\tau_0)), x)$ defined by loops at x contained in $B_{\varepsilon'} \cap L - X(\alpha(\tau_0))$ turning simply around the points of $B_{\varepsilon'} \cap L \cap X(\alpha(\tau_0))$ such that the loops turning aroung points of $B_{\varepsilon'} \cap L \cap X_1(\alpha(\tau_0))$ give generators which commute in $\pi_1(B_{\varepsilon'} - X(\alpha(\tau_0)), x)$ with the ones defined by other loops at x turning simply around the points of $B_{\varepsilon'} \cap L \cap X_2(\alpha(\tau_0))$.

Now using (2.1.12.) we see that our lemma (2.1.1.) is a consequence of the hypothesis (2.1.14.).

To be more precise, let g_i $(i = 1, \ldots, r)$ and h_j $(j = 1, \ldots, s)$ be the elements of the base of $\pi_1(L \cap B_\varepsilon - X, x)$ coming from the loops having the property in (2.1.14.). The g_i will come from the loops turning around the points of $L \cap X_1(\alpha(\tau_0)) \cap B_\varepsilon$ and the h_j will be the ones coming from the loops turning around the points of $L \cap X_2(\alpha(\tau_0)) \cap B_\varepsilon$. As the inclusion $B_{\varepsilon'} - X(\alpha(\tau_0)) \subset B_\varepsilon - X(\alpha(\tau_0))$ gives an epimorphism of the fundamental group and as $B_\varepsilon - X(\alpha(\tau_0))$ is homeomorphic to $B_\varepsilon - X$ by an homeomorphism which sends $(L - X(\alpha(\tau_0))) \cap B_\varepsilon$ onto $(L - X) \cap B_\varepsilon$, the images $\tilde{g}_i$ and $\tilde{h}_j$ of g_i and h_j $(i = 1, \ldots, r,\ j = 1, \ldots, s)$ in the fundamental group of $B_\varepsilon - X$ commute. According to the hypothesis in (2.1.14.) we have $\tilde{g}_i \tilde{h}_j = \tilde{h}_j \tilde{g}_i$ $(i = 1, \ldots, r,$ $j = 1, \ldots, s)$ in the fundamental group $B_\varepsilon - X$. Now from (2.1.12.) the group

$\pi_1(B_\varepsilon - X, x)$ is obtained from the free group $\pi_1(L\cap B_\varepsilon - X, x)$ by dividing by the normal subgroup generated by:

$$\begin{array}{ll} g_i^{-1}\varphi_{ik}g_{\sigma_k(i)}\varphi_{ik}^{-1} & i=1,\dots,r \\ & j=1,\dots,s \\ h_j^{-1}\psi_{jk}h_{\sigma'_k(j)}\psi_{jk}^{-1} & k=1,\dots,v \end{array}$$

where $\sigma_k(i)$ and $\sigma'_k(j)$ are the indexes obtained from i and j by the monodromy around the k-th "critical" point described in (2.1.12.) and φ_{ik} and ψ_{jk} are words in $g_1, \dots, g_r, h_1, \dots, h_s$ which describe the monodromy.

As $\tilde{g}_i$ and $\tilde{h}_j$ commute in $\pi_1(B_\varepsilon - X, x)$, this last group is the quotient of the direct product $G\times H$ of the free groups generated respectively by the g_i $(i=1,\dots,r)$ and the h_j $(j=1,\dots,s)$ by the image in $G\times H$ of the above normal subgroup of $\pi_1(L\cap B_\varepsilon - X, x) = G * H$ (free product of G and H). Now the image $\bar{\varphi}_{ik}$ of φ_{ik} in $G\times H$ can be written in $G\times H$:

$$\bar{\varphi}_{ik} = \bar{\varphi}_{ik}^{(1)}\bar{\varphi}_{ik}^{(2)}$$

where $\bar{\varphi}_{ik}^{(1)}$ is the image of a word $\varphi_{ik}^{(1)}$ in G and $\bar{\varphi}_{ik}^{(2)}$ is the image of a word $\varphi_{ik}^{(2)}$ in H.

Similarly: $\bar{\psi}_{jk} = \bar{\psi}_{jk}^{(1)}\bar{\psi}_{jk}^{(2)}$ in $G\times H$.

Thus $\pi_1(B_\varepsilon - X, x)$ is isomorphic to the quotient of $G\times H$ by the normal subgroup generated by:

$$\begin{array}{l} \bar{g}_i^{-1}\bar{\varphi}_{ik}^{(1)}\bar{g}_{\sigma_k(i)}\bar{\varphi}_{ik}^{(1)\,-1} \\ \bar{h}_j^{-1}\bar{\psi}_{ik}^{(2)}\bar{h}_{\sigma'_k(j)}\bar{\psi}_{jk}^{(2)\,-1}. \end{array}$$

It is now an exercise to see that the fundamental group of $B_\varepsilon - X_1$ is isomorphic to the quotient of G by the normal subgroup generated by:

$$g_i^{-1}\varphi_{ik}^{(1)}g_{\sigma_k(i)}\varphi_{ik}^{(1)\,-1}$$

and the fundamental group of $B_\varepsilon - X_2$ is isomorphic to the quotient of H by the normal subgroup generated by:

$$h_j^{-1}\psi_{jk}^{(2)}h_{\sigma'_k(j)}\psi_{jk}^{(2)\,-1}.$$

This obviously gives our lemma (2.1.1.).

It remains to prove that we may assume (2.1.14.). For this purpose we shall investigate the geometry of $\mathscr{X} \xrightarrow{p} \bar{D}$ as we go to 0 along the path α.

(2.2.) We fix H_0 as in (2.1.5.).

(2.2.1.) According to the definition of α in (2.1.6.) the germs of curves $(X(\alpha(\tau))\cap H_0, 0)$ are equisingular when $0\leqq\tau\leqq\tau_0$.

Thus we can choose $\varepsilon''>0$ such that $\varepsilon''\leqq\varepsilon'$ and the spheres $S_{\varepsilon''_1}$ are transverse to $X(\alpha(\tau))\cap H_0$ for any ε''_1, $0<\varepsilon''_1\leqq\varepsilon''$ and $0\leqq\tau\leqq\tau_0$ (cf. [Z_1] or [L_2]).

Let us fix such an ε''. By a compacity argument there is $\gamma_3>0$ such that for any τ, $0\leqq\tau\leqq\tau_0$ and for any c, $0\leqq|c|\leqq\gamma_3$ the curves $H_c\cap X(\alpha(\tau))$ cut $S_{\varepsilon''}$ transversally.

(2.2.2.) By a construction similar to the one quoted in (2.1.9.) we find that $\gamma_3>0$ can be chosen in such a way that, for any τ, $0<\tau\leqq\tau_0$, and any c, $0<|c|\leqq\gamma_3$ the pairs $(B_{\varepsilon''}\cap H_c, B_{\varepsilon''}\cap H_c\cap X(\alpha(\tau)))$ are homeomorphic. In particular the $B_{\varepsilon''}\cap H_c-X(\alpha(\tau))$ are homeomorphic when $0<\tau\leqq\tau_0$.

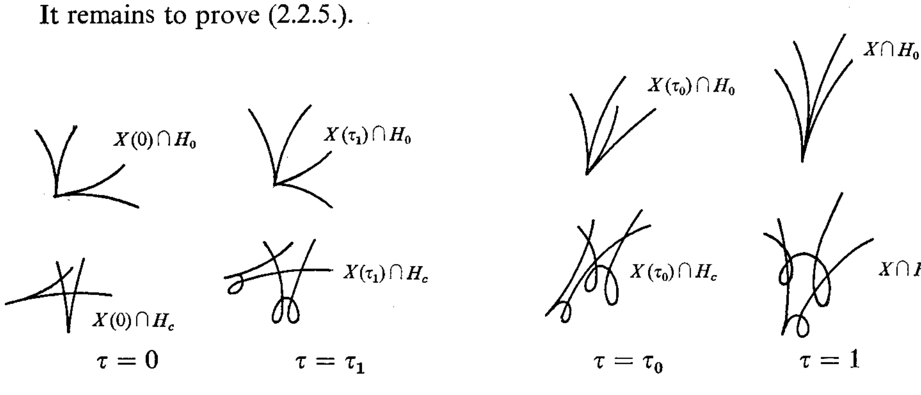
(2.2.3.) The same argument as in [L_1] shows that there is a natural epimorphism of the fundamental group of $B_{\varepsilon''}\cap H_c-X(0)$ onto the one of $B_{\varepsilon''}\cap H_c-X(\alpha(\tau_1))$ when $0<|c|\leqq\gamma_3'$ and $\tau_1\neq 0$ is small enough.

More precisely one can find a line L_0 in H_0 such that $(L_0\cdot X(\alpha(\tau)))_0$ is equal to the multiplicity of X at 0 for any τ, $0\leqq\tau\leqq 1$. Then choosing ε'' accordingly to L_0 and (2.2.1.) as in (2.1.13.) for $\varepsilon>0$, one may find L in H_c such that, in the above epimorphism, the generators of the fundamental group of $B_{\varepsilon''}\cap H_c-X(0)$ coming from loops in $L\cap B_{\varepsilon''}-X(0)$ which give a base of its fundamental group are sent on generators of the fundamental group of $B_{\varepsilon''}\cap H_c-X(\alpha(\tau_1))$ which are represented by loops in $L\cap B_{\varepsilon''}-X(\alpha(\tau_1))$.

(2.2.4.) As in (2.1.13.), we notice that the homeomorphisms of (2.2.2.) built similarly as in (2.1.9.) may be constructed such that the image of $(L\cap H_c-X(\alpha(\tau_1)))\cap B_{\varepsilon''}$ is $(L\cap H_c-X(\alpha(\tau_0)))\cap B_{\varepsilon''}$. Thus to find a base of the free group, fundamental group of $(L\cap H_c-X(\alpha(\tau_0)))\cap B_{\varepsilon''}$, which satisfies the hypothesis (2.1.14.), the inclusion of $B_{\varepsilon''}-X(\alpha(\tau_0))$ into $B_\varepsilon-X(\alpha(\tau_0))$ for $0<\varepsilon''\leqq\varepsilon'$ being a homotopy equivalence, it is enough to prove:

(2.2.5.) We can choose a base of $\pi_1(B_{\varepsilon''}\cap H_c-X(\alpha(\tau_1)), x)$ defined by loops at x contained in $B_{\varepsilon''}\cap L-X(\alpha(\tau_1))$ turning simply around the points of $B_{\varepsilon''}\cap L\cap X(\alpha(\tau_1))$ such that the loops turning around points of $B_{\varepsilon''}\cap L\cap X_1(\alpha(\tau_1))$ give generators which commute in $\pi_1(B_{\varepsilon''}-X(\alpha(\tau_1)), x)$ with the ones defined by the other loops at x turning simply around the points of $B_{\varepsilon''}\cap L\cap X_2(\alpha(\tau_1))$.

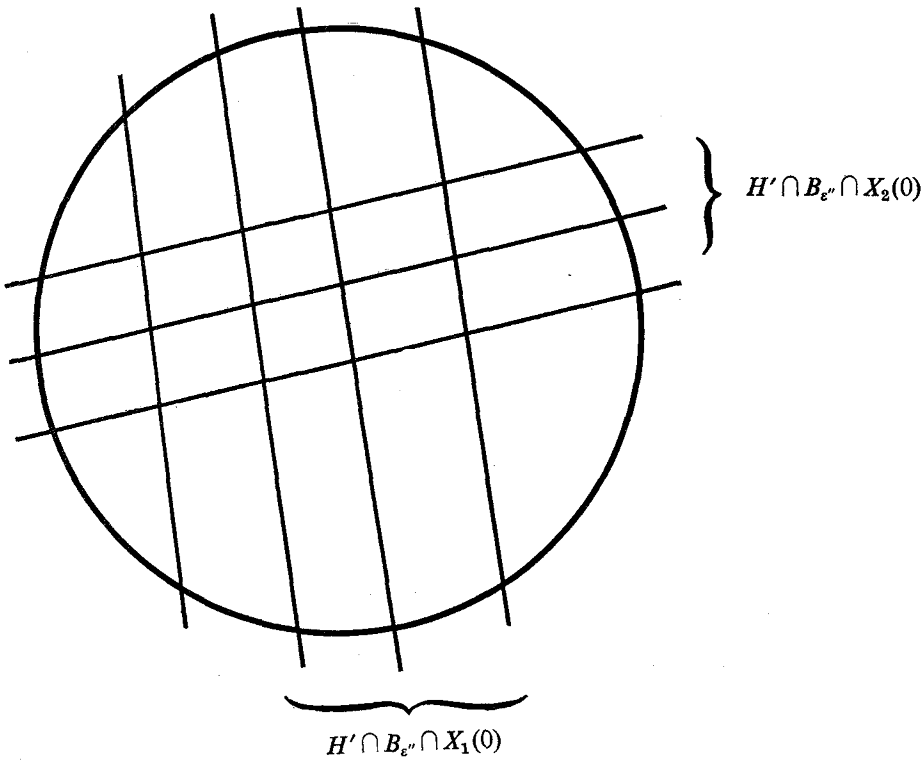
It remains to prove (2.2.5.).



(2.3.) Actually notice that, $X(0)$ being the union of two cylinders on $X_1(0)\cap H_0$ and $X_2(0)\cap H_0$, if $\varepsilon''>0$, is such that $S_{\varepsilon''}$ cuts $X(0)\cap H_0$ transversally, it cuts the non singular part of $X(0)$ and its singular part transversally. Thus as, for any ε_1'',

$0<\varepsilon_1''\leqq\varepsilon''$, $S_{\varepsilon_1''}$ cuts $X(0)\cap H_0$ transversally, the fundamental group of $B_{\varepsilon''}-X(0)$ is isomorphic to the local fundamental group at 0 of the complement of $X(0)$ in $\mathbf{C}^3$.

Let L_0 chosen as in (2.2.3.) and let L_0' be a line in the plane $z=0$ such that $(L_0'\cdot X(0))_0=(L_0\cdot X(0))_0$ which is the multiplicity $m(X, 0)$ of X at 0. One can show that there $\varepsilon''>0$ such that, for any L and L' near and parallel to L_0 and L_0' respectively and cutting $X(0)$ in $m(X, 0)$ points there is an homeomorphism of $B_{\varepsilon''}$ onto itself which induces a homeomorphism of $B_{\varepsilon''}-X(0)$ onto itself and a homeomorphism of $B_{\varepsilon''}\cap L-X(0)$ onto $B_{\varepsilon''}\cap L'-X(0)$. Thus, if we can build up the loops in $B_{\varepsilon''}\cap L'-X(0)$ (instead of $B_{\varepsilon''}\cap L-X(0)$) to obtain similar properties as (2.2.5.), we bring these loops into $B_{\varepsilon''}\cap L-X(0)$ by the above homeomorphism to obtain (2.2.5.). But L' is contained in a plane H' parallel to the plane $z=0$ and, if L' is sufficiently near to L_0', we see that $H'\cap B_{\varepsilon''}\cap X_1(0)$ and $H'\cap B_{\varepsilon''}\cap X_2(0)$ are given by parallel lines which cut transversally one another. In this situation the construction of the loops in $L'\cap B_{\varepsilon''}-X(0)$ to obtain the desired property is obvious.



This ends the proof of Lemma (2.1.1.).

(2.4.) As it was quoted in the remark (2.1.2.), using the result of [H—L1] (Theorem (0.2.1.)) we have:

(2.4.1.) Corollary. *Let X be an analytic hypersurface in an open neighbourhood U of 0 in $\mathbf{C}^n$ and $0 \in X$. Suppose it decomposes locally at 0 into two analytic hypersurfaces, i.e. if U is chosen small enough:*

$$X = X_1 \cup X_2.$$

Suppose that there is an analytic subspace Y of codimension 2 in X such that at any point $x \in X_1 \cap X_2 - Y$ both X_1 and X_2 are non singular and intersect transversally. Then the local fundamental group at 0 of the complement of X in $\mathbf{C}^n$ is isomorphic to the direct product of the local fundamental group at 0 of the complement of X_1 in $\mathbf{C}^n$ and the one of the complement of X_2 in $\mathbf{C}^n$.

3. Some consequences

(3.1.) Application to projective hypersurfaces. The following result was first stated by O. Zariski in [Z_2] and proved by P. Deligne and W. Fulton (cf. [D_2]):

(3.1.1.) Theorem. *Let C be a complex projective plane curve in $\mathbf{P}^2$. Suppose that C has only nodes as singular points. Then the complement of C in $\mathbf{P}^2$ has an abelian fundamental group.*

Proof. One can consider this theorem as a consequence of our main theorem. In fact one considers the cone C_1 in $\mathbf{C}^3$ which lies over C. Then the local fundamental group at 0 of the complement of C_1 in $\mathbf{C}^3$ is isomorphic to the fundamental group of the complement of C_1 in $\mathbf{C}^3$. But the complement of C_1 in $\mathbf{C}^3$ is a locally trivial $\mathbf{C}^*$-fibration over the complement of C in $\mathbf{P}^2$. Now using the exact homotopy sequence of this fibration and our main theorem we obtain the above Theorem (3.1.1.), because obviously C_1 satisfies the conditions of our main theorem.

Using another result stated by O. Zariski in [Z_3] and proved by H. Hamm and Lê Dũng Tráng in [H—L_1] (0.3.) the theorem above can be stated for hypersurfaces:

(3.1.2.) Theorem. *Let H be a complex projective hypersurface in $\mathbf{P}^n$. Suppose that there is a codimension 2 subvariety Y in H such that the singularities of $H-Y$ are normal crossings, then the complement of H in $\mathbf{P}^n$ has an abelian fundamental group.*

Using again the result stated by O. Zariski in [Z_2] (cf. [H$-$$L_1$] (0.3.)) and proceeding as in the proof of (3.1.1.), our Lemma (2.1.1.) give the following consequence:

(3.1.3.) Theorem. *Let H_1 and H_2 be two complex projective hypersurfaces in $\mathbf{P}^n$. Suppose there is a codimension 2 subvariety Y in $H_1 \cup H_2$ such that for any point $x \in H_1 \cap H_2 - Y$ the hypersurfaces H_1 and H_2 cut transversally, then if L is a sufficiently general hyperplane of $\mathbf{P}^n$ the fundamental group of the complement of $H_1 \cup H_2 \cup L$ in $\mathbf{P}^n$ is isomorphic to the product of the fundamental groups of the complements of $H_1 \cup L$ and $H_2 \cup L$ in $\mathbf{P}^n$.*

(3.2.) Another application of our main theorem is to give a partial answer to a question of K. Saito in [S] ((2.14.) Note).

Let us formulate the question again here:

Let U be a domain in $\mathbf{C}^n$ and X be a closed hypersurface in U. One defines the sheaf of logarithmic 1-forms $\Omega_U^1(\log X)$ on U along X and the residue map res: $\Omega_U^1(\log X)\to\pi_*(\mathfrak{M}_{\tilde{X}})$ where $\pi\colon \tilde{X}\to X$ is the normalization of X and $\mathfrak{M}_{\tilde{X}}$ is the sheaf of meromorphic functions on $\tilde{X}$ (cf. [S] (1.2.), (2.2.)).

Then the following question was asked in [S] ((2.14) Note and (2.13)):

(3.2.1.) *Question:* The following conditions on X are equivalent:

i) The local fundamental groups at any $x\in X$ of the complement of X in U are abelian;

ii) There is a subvariety Y of codimension 2 in X such that $X-Y$ has at most normal crossings;

iii) The residues of any elements of $\Omega_U^1(\log X)$ are holomorphic on $\tilde{X}$ (i.e. weakly holomorphic on X).

The implication: "i) implies ii)" is coming from the fact that along any codimension one component of the singular locus at any general point of this component X is equisingular (cf. [Z_1]) and a computation of O. Zariski which shows that the only germs of plane curves which have a local complement with abelian fundamental groups are the non singular germs and the germs of an ordinary singularity. Our main theorem shows that ii) implies i).

The implication ii)$\Rightarrow$iii) comes from an easy computation. The implication iii)$\Rightarrow$ii) is still open. We only know a weaker assertion.

(3.2.2.) Theorem. (Cf. [S] (2.11.)) *Let C be a germ of plane curve in $\mathbf{C}^2$ at* 0. *Then the residues of the elements of Ω^1 $(\log C)$ at* 0 *are holomorphic on the normalization of C if and only if C is either non singular at* 0 *or with an ordinary double point at* 0.

(3.2.3.) It is an interesting problem to find a direct relation between the conditions i) and iii) of (3.2.1.).

Bibliography

[A—F] A. Andreotti—T. Frankel, The Lefschetz theorem on hypersurface sections, *Ann. Math.*, **69**, p. 713—717, 1959.

[B—V] D. Burghelea—A. Verona, Local homological properties of analytic sets, *Manusc. Math.* **7**, p. 55—66, 1972.

[C—L] D. Cheniot—Lê Dũng Tráng, Remarques sur les deux exposés précédents, *in Singularités à Cargèse, Astérisque* **7—8**, 1973.

[D_1] P. Deligne, Lettre à W. Fulton du 3. 07. 1979.

[D_2] P. DELIGNE, Le groupe fondamental du complément d'une courbe plane n'ayant que des points doubles ordinaires est abélien (d'après W. Fulton) *Sem. Bourbaki* n° 543, 1979/1980.

[F] W. FULTON, On the fundamental group of the complement of a node curve, *Ann. Math.* **111**, p. 407—409, 1980.

[G] A. GROTHENDIECK, SGA 2, exposé XIII, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux*, Masson-North Holland Pub., 1968.

[H—L_1] H. HAMM—LÊ DŨNG TRÁNG, Un théorème de Zariski du type de Lefschetz, *Ann. Sc. Ec. Norm. Sup.*, **6**, p. 317—366, 1973.

[H—L_2] H. HAMM—LÊ DŨNG TRÁNG, Théorèmes du type de Lefschetz, en préparation.

[K—M] M. KATO—Y. MATSUMOTO, *On the connectivity of the Milnor fiber of a holomorphic function at a critical point, in manifolds*, Tokyo 1973, Kinokunya.

[L_1] LÊ DŨNG TRÁNG, Une application du théorème de Zariski, *Note aux C.R.A.S.*, **275**, p. 57—59, 1972.

[L_2] LÊ DŨNG TRÁNG, Sur un critère d'Equisingularité, *Note aux C.R.A.S.*, **272**, p. 138—140, 1971.

[L—R] LÊ DŨNG TRÁNG—C. P. RAMANUJAM, The invariance of Milnor's number implies the invariance of the topological type, *Amer. J. Math.* **98**, p. 67—78, 1976.

[M] J. MILNOR, Singular points of complex hypersurfaces, *Ann. Math. Stud.* **61**, Princeton Un. 1968.

[O—S] M. OKA—K. SAKAMOTO, Product theorem of the fundamental group of a reducible curve, *J. Math. Soc. Jap*, **39**, p. 599—602, 1978.

[P] D. PRILL, Local classification of quotients of complex manifolds by discontinuous groups, *Duke Math. J.*, **34**, p. 375—386, 1967.

[S] K. SAITO, Theory of logarithmic differential and and logarithmic vector fields, *Journ. Fac. Sc. Un. Tokyo, Sc. IA*, **27**, p. 265—291, 1980.

[V—K] E. VAN KAMPEN, On the fundamental group of an algebraic curve, Amer. J. Math., Vol. **55**, 1933.

[Z_1] O. ZARISKI, Studies in Equisingularity I, II, III, *Amer. J. Math.*, **87**, p. 507—536 and p. 972—1006, 1965, and **90**, p. 961—1023, 1968.

[Z_2] O. ZARISKI, On the problem of existence of algebraic functions of two variables possessing a given branch curve, *Amer. J. Math.*, **51**, p. 305—328, 1929.

[Z_3] O. ZARISKI, On the Poincaré group of a projective hypersurface, *Ann. Math.* **38**, p. 131—141, 1937.

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