XVII. Forcing Axioms

§0. Introduction

This chapter reports various researches done at different times in the later eighties. In Sect. 1, 2 we represent [Sh:263] which deals with the relationship of various forcing axioms, mainly SPFA = MM, SPFA $\nvdash$ PFA$^+$ (=Ax$_1$[proper]) but SPFA implies some weaker such axioms (Ax$_1$[$\aleph_1$-complete], see 2.14, and more in 2.15, 2.16). See references in each section.

In sections 3, 4 we deal with the canonical functions (from ω_1 to ω_1) modulo normal filters on ω_1. We show in §3 that even PFA$^+$ does not imply Chang's conjecture [even is consistent with the existence of $g \in {}^{\omega_1}\omega_1$ such that for no $\alpha < \aleph_2$ is g smaller (modulo $\mathcal{D}_{\omega_1}$) than the α-th function]. Then we present a proof that Ax[α-proper] $\nvdash$ Ax [β-proper] where $\alpha < \beta < \omega_1$, β is additively indecomposable (and state that any CS iteration of c.c.c. and $\aleph_1$-complete forcing notions is α-proper for every α).

In the fourth section we get models of CH + "ω_1 is a canonical function" without $0^{\#}$, using iteration not adding reals, and some variation (say ω_1 is the α-th function, CH + $2^{\aleph_1} = \aleph_3$ $|\alpha| = \aleph_2$ (see 4.7(3)). The proof is in line of the various iteration theorems in this book, so here we deal with using large cardinals consistent with $V = L$.

Historical comments are introduced in each section as they are not so strongly related.

We recall definition VII 2.10: If φ is a property of forcing notions, $\alpha \leq \omega_1$ then we write $\mathrm{Ax}_\alpha[\varphi]$ for the statement:
whenever P is a forcing notion satisfying φ, $\langle \mathcal{I}_i : i < \omega_1 \rangle$ are pre-dense subsets of P, $\langle \underset{\sim}{S}_\beta : \beta < \alpha \rangle$ are P-names of stationary subsets of ω_1,
then there is a directed, downward closed set $G \subseteq P$ such that for all $i < \omega_1$, $\mathcal{I}_i \cap G \neq \emptyset$ and for all $\beta < \alpha$ the set $\underset{\sim}{S}_\beta[G]$ is stationary.
We write $\mathrm{Ax}[\varphi]$ for $\mathrm{Ax}_0[\varphi]$ and $\mathrm{Ax}^+[\varphi]$ for $\mathrm{Ax}_1[\varphi]$, PFA for Ax[proper], SPFA for Ax[semiproper], similarly PFA$^+$ and SPFA$^+$.

§1. Semiproper Forcing Axiom Implies Martin's Maximum

We prove that Ax[preserving every stationarity of $S \subseteq \omega_1$] = MM (= Martin maximum) is equivalent (in ZFC) to the older axiom Ax[semiproper] = SPFA (= semiproper forcing axiom).

1.1 Lemma. If $\mathrm{Ax}_1[\aleph_1\text{-complete}]$, P is a forcing notion satisfying $(*)_1$ (below) *then* P is semiproper, where
$(*)_1 \overset{\text{def}}{=}$ "the forcing notion P preserves stationary subsets of ω_1".

1.1A Remark. 1) This is from Foreman, Magidor and Shelah [FMSh:240].
2) It follows that SPFA$^+$ = Ax_1[semiproper] is equivalent to MM$^+$ (compare [FMSh:240]). The conclusion is superseded by 1.2, but not the lemma.
3) The proof is very similar to III 4.2.
4) Of course every semiproper forcing preserves stationarity of subsets of ω_1 (see X 2.3(8)).

Proof. Clearly $\mathrm{Ax}_1[\aleph_1\text{-complete}]$ implies $\mathrm{Rss}(\aleph_1, \kappa)$ for any κ (see Defefinition XIII 1.5(1).). By XIII 1.7(3) "forcing with P does not destroy semi-stationarity of subsets of $\mathcal{S}_{<\aleph_1}(2^{|P|})$" implies P is semiproper. (So by 1.1A(4) these two properties are equivalent). $\square_{1.1}$

1.2 Theorem.
Ax [not destroying stationarity of subsets of ω_1] $\equiv$Ax [semiproper], i.e. MM (= Martin Maximum) $\equiv$ SPFA (i.e., proved in ZFC).

Proof. As every semiproper forcing preserves stationary subsets of ω_1 (X 2.3(8)), clearly MM $\Rightarrow$ SPFA. So it suffices to prove:

1.3 Lemma. [SPFA.]
Every forcing notion P satisfying $(*)_1$ is semiproper, where
$(*)_1 \overset{\text{def}}{=}$ "the forcing notion P preserves stationarity of subsets of ω_1".

Proof. We assume $(*)_1$. Without loss of generality the set of members (= conditions) of P is a cardinal $\lambda_0 = \lambda(0)$. Too generously, for $\ell = 0, 1, 2, 3$, let $\lambda_{\ell+1} = \lambda(\ell+1) = (2^{|H(\lambda_\ell)|})^+$. Let $<^*_{\lambda_\ell}$ be a well ordering of $H(\lambda_\ell)$, end extending $<^*_{\lambda_m}$ for $m < \ell$. Let

$$K^{\text{neg}}_P \overset{\text{def}}{=} \{N : N \prec (H(\lambda_2), \in, <^*_{\lambda_2}), \|N\| = \aleph_0, P \in N \text{ (hence } \lambda_0, \lambda_1 \in N) \text{ and}$$
$$\neg(\forall p \in P \cap N)(\exists q)[p \leq q \in P \text{ and } q \text{ is semi generic for } (N, P)]\}$$

and

$$K^{\text{pos}}_P \overset{\text{def}}{=} \{N : N \prec (H(\lambda_2), \in, <^*_{\lambda_2})), \|N\| = \aleph_0, P \in N \text{ (hence } \lambda_0, \lambda_1 \in N)$$
$$\text{and } \neg(\exists N')[N \prec N' \in K^{\text{neg}}_P \text{ and } N \cap \omega_1 = N' \cap \omega_1]\}.$$

We now define a forcing notion Q

$$Q \overset{\text{def}}{=} \{\langle N_i : i \leq \alpha\rangle : \alpha < \omega_1, N_i \in K^{\text{neg}}_P \cup K^{\text{pos}}_P,$$
$$N_i \in N_{i+1}, \text{ and } N_i \text{ is increasing continuous in } i\}.$$

The order on Q is being an initial segment.
The rest of the proof of Lemma 1.3 is broken to facts 1.4 — 1.11.

1.4 Fact. If $P \in M_0 \prec (H(\lambda_3), \in, <^*_{\lambda_3}), \|M_0\| = \aleph_0$, *then* there is M_1 such that $M_0 \prec M_1 \prec (H(\lambda_3), \in, <^*_{\lambda_3}), \|M_1\| = \aleph_0, M_0 \cap \omega_1 = M_1 \cap \omega_1$ and $M_1 \restriction H(\lambda_2) \in K^{\text{neg}}_P \cup K^{\text{pos}}_P$.

Proof. As $P \in M_0$, clearly $\lambda_0 \in M_0$; hence $\lambda_1, \lambda_2 \in M_0$ hence $(H(\lambda_\ell), \in, <^*_{\lambda_\ell})$ belong to M_0 for $\ell = 0, 1, 2$, so $K_P^{\rm pos} \in M_0$ and $K_P^{\rm neg} \in M_0$. We can assume $M_0 \restriction H(\lambda_2) \notin K_P^{\rm pos}$, so by the definition of $K_P^{\rm pos}$ there is N' such that (abusing our notation) $M_0 \cap H(\lambda_2) = M_0 \restriction H(\lambda_2) \prec N' \in K_P^{\rm neg}$, $\|N'\| = \aleph_0$ and $N' \cap \omega_1 = (M_0 \restriction H(\lambda_0)) \cap \omega_1$; hence $N' \cap \omega_1 = M_0 \cap \omega_1$.
Let M_1 be the Skolem Hull of $M_0 \cup (N' \cap H(\lambda_1))$ in $(H(\lambda_3), \in, <^*_{\lambda_3})$. So

$$\begin{array}{llcl} H(\lambda_3): & M_0 & \longrightarrow & M_1 \\ H(\lambda_2): & M_0 \cap H(\lambda_2) \prec N' & & \uparrow \\ H(\lambda_1): & & & N' \cap H(\lambda_1) \end{array}$$

We claim that $M_1 \cap H(\lambda_1) = N' \cap H(\lambda_1)$. To prove this claim, let x be an arbitrary element of $M_1 \cap H(\lambda_1)$. Now x must be of the form $f(y)$, where f is a Skolem function of $(H(\lambda_3), \in, <^*_{\lambda_3})$ with parameters in M_0, and $y \in N' \cap H(\lambda_1)$ (note that $N' \cap H(\lambda_1)$ is closed under taking finite sequences). Note that f's definition may use parameters outside $H(\lambda_2)$, but $f' \stackrel{\rm def}{=} f \cap (H(\lambda_1) \times H(\lambda_1))$ belongs to $H(\lambda_2)$, so $f' \in M_0 \cap H(\lambda_2) \subseteq N'$, so also $x = f(y) = f'(y) \in N'$.
So we have

$$M_1 \cap \omega_1 = N' \cap \omega_1 = M_0 \cap \omega_1,$$

$$M_0 \prec M_1 \prec (H(\lambda_3), \in, <^*_{\lambda_3}),$$

$$||M_1|| = \aleph_0 \text{ (as } ||M_0||, ||N'|| = \aleph_0).$$

$$M_1 \cap H(\lambda_1) = N' \cap H(\lambda_1)$$

We can conclude by 1.5(1) below that $M_1 \restriction H(\lambda_2) \in K_P^{\rm neg}$, thus finishing the proof of Fact 1.4, as:

1.5 Subfact. 1) Suppose for $\ell = 0, 1$, N^ℓ is countable, $P \in N^\ell \prec (H(\lambda_2), \in, <^*_{\lambda_2})$ and $N^0 \cap H(\lambda_1) = N^1 \cap H(\lambda_1)$, *then* $N^1 \in K_P^{\rm neg} \Leftrightarrow N^2 \in K_P^{\rm neg}$.
2) Really, even $N^1 \cap \omega_1 \subseteq N^0 \subseteq N^1 \prec (H(\lambda_2), \in, <^*_{\lambda_2}), N^0 \in K_P^{\rm neg}$ implies $N^1 \in K_P^{\rm neg}$ (we can also fix the P in the definition of "$N \in K_P^{\rm neg}$").

Proof. 1) Because in "q is (N, P)-semi generic", not "whole N" is meaningful, just $N \cap \omega_1$, the set $N \cap P$ and the set of P-names of countable ordinals which

belong to N, hence (for "reasonably closed N") this depends only on $N \cap 2^{|P|}$ (even $|P|^{<\kappa}$, when $P \models \kappa$-c.c.).
2) Assume $N^1 \notin K_P^{\rm neg}$. If $p \in P \cap N^0$ then $p \in P \cap N^1$, hence there is $q \in P$ which is (N^1, P)-semi generic, $q \geq p$. But as $N^0 \prec N^1$ have the same countable ordinals, q is also (N^0, P)-semi generic. $\square_{1.5,1.4}$

1.6 Fact. Q is a semiproper forcing.

Proof. Let $Q, P \in M \prec (H(\lambda_3), \in, <^*_{\lambda_3}), M$ countable. Let $p \in Q \cap M$. It is enough to prove that there is a q such that $p \leq q \in Q$ and q is semi generic for (M, Q).
Let $\delta = M \cap \omega_1$. By Fact 1.4 there is M_1, with $M \prec M_1 \prec (H(\lambda_3), \in, <^*_{\lambda_3})$, $||M_1|| = \aleph_0, M_1 \cap \omega_1 = \delta$ and $M_1 \restriction H(\lambda_2) \in K_P^{\rm neg} \cup K_P^{\rm pos}$. We can find by induction on n a condition $q_n = \langle N_i : i \leq \delta_n \rangle \in Q \cap M_1, q_n \leq q_{n+1}, q_0 = p$, such that: *for every* Q-name $\underset{\sim}{\gamma}$ of an ordinal which belongs to M_1 for some natural number $n = n(\underset{\sim}{\gamma})$ and ordinal $\alpha(\underset{\sim}{\gamma}) \in M_1$ we have $q_n \Vdash_Q$ "$\underset{\sim}{\gamma} = \alpha(\underset{\sim}{\gamma})$" and for every dense subset $\mathcal{I}$ of Q which belongs to M_1, for some n, $q_n \in \mathcal{J}$. Now $q \stackrel{\rm def}{=} \langle N_i : i \leq \delta^* \rangle$ with $\delta^* = \bigcup\limits_{n<\omega} \delta_n$ and $N_{\delta^*} \stackrel{\rm def}{=} \bigcup_{i<\delta^*} N_i$ will be (M_1, Q)-generic if it is a condition in Q at all, as for this the least obvious part is $N_{\delta^*} \in K_P^{\rm neg} \cup K_P^{\rm pos}$. Clearly (by 1.4) for each $x \in H(\lambda_2)$, $\mathcal{I}_x = \{\langle M'_i : i \leq j \rangle \in Q : x \in \bigcup_{i \leq j} M'_i\}$ is a dense subset of Q and $[x \in M_1 \cap H(\lambda_2) \Rightarrow \mathcal{I}_x \in M_1]$ and $\langle M'_i : i \leq j \rangle \in Q \cap M_1 \Rightarrow \bigcup_{i \leq j} M'_i \subseteq M_1$ (as M'_i, j are countable), and so $\bigcup_{i<\delta^*} N_i = M_1 \restriction H(\lambda_2)$, which belongs to $K_P^{\rm neg} \cup K_P^{\rm pos}$ by the choice of M_1. Now $q \geq q_0 = p$; and, as q is (M_1, Q)-generic it is (M_1, Q)-semi generic hence as in the proof of 1.5 (or see X2.3(9)), as $M \prec M_1, M \cap \omega_1 = M_1 \cap \omega_1$, we know q is also (M, Q)-semi generic, as required. By the way, necessarily $\delta^* = \delta$. $\square_{1.6}$

1.7 Conclusion. [SPFA] There is a sequence $\langle N^*_i : i < \omega_1 \rangle$ such that

$$(\forall \alpha < \omega_1)[\langle N^*_i : i \leq \alpha \rangle \in Q].$$

Proof. By Fact 1.6 and SPFA (and as $\mathcal{I}_{\alpha_0} = \{\langle N_i : i \leq \alpha \rangle : \alpha \geq \alpha_0\}$ is a dense subset of Q for every $\alpha_0 < \omega_1$; which can be proved by induction on α_0 : for

$\alpha_0 = 0$ or $\alpha_0 = \beta + 1$ by Fact 1.4, for limit α_0 by the proof of Fact 1.6 or simpler).

$\square_{1.7}$

1.8 Observation. $i \subseteq N_i^*$ for $i < \omega_1$.

Proof. As $[i < j \Rightarrow N_i \subseteq N_j]$ and as $N_i^* \in N_{i+1}^*$ (see the definition of Q), we can prove this statement by induction on i. $\square_{1.8}$

1.9 Definition. $S \stackrel{\text{def}}{=} \{i < \omega_1 : N_i^* \in K_P^{\text{neg}}\}$.

1.10 Fact. S is not stationary.

Proof. Suppose it is; then for every $i \in S$ for some $p_i \in N_i^* \cap P$ there is no (N_i^*, P)-semi-generic q such that $p_i \leq q \in P$. By Fodor's lemma (as N_i^* is increasing continuous and each N_i^* is countable), for some $p \in \bigcup_{i<\omega_1} N_i^* \cap P$ the set $S_p \stackrel{\text{def}}{=} \{i \in S : p_i = p\}$ is stationary.

If $p \in G \subseteq P$ and G generic over V, then in $V[G]$ we can find an increasing continuous sequence $\langle N_i : i < \omega_1 \rangle$ of countable elementary submodels of $\left(H^V(\lambda_2), \in, <^*_{\lambda_2}, G\right)$ (with G as a predicate), $N_i^* \subseteq N_i$. As P preserves stationarity of subsets of ω_1, and $E = \{i : N_i^* \cap \omega_1 = N_i \cap \omega_1 = i\}$ is a club of ω_1 (in $V[G]$), and $S_p \subseteq \omega_1$ is stationary (in V, hence in $V[G]$), it follows that there is $\delta \in S_p$ with $N_\delta^* \cap \omega_1 = N_\delta \cap \omega_1 = \delta$. As this holds in $V[G], p \in G$, clearly there is $q \in G, q \geq p$, such that $q \Vdash$ "δ and $\langle \underset{\sim}{N}_i : i < \omega_1 \rangle$ are as above". As $q \Vdash$ "$N_\delta^* \subseteq N_\delta^*[\underset{\sim}{G}] \subseteq \underset{\sim}{N}_\delta$ and $\delta \in E$", also $q \Vdash$ "$N_\delta^* \cap \omega_1 = N_\delta^*[\underset{\sim}{G}] \cap \omega_1$", so q is (N_δ^*, P)-semi generic, contradiction to the definition of S and K_P^{neg} and the choice of $p_\delta = p$. $\square_{1.10}$

1.11 Fact. P is semiproper.

Proof. As S is not stationary, for some club $C \subseteq \omega_1, (\forall \delta \in C) N_\delta^* \in K_P^{\text{pos}}$. Now if $M \prec \left(H(\lambda_3), \in, <^*_{\lambda_3}\right)$ is countable, and $P, \langle N_i^* : i < \omega_1 \rangle, C$ belong to M, then $M \cap \bigcup_{i<\omega_1} N_i^* = N_\delta^*$ for some $\delta \in C$; hence $N_\delta^* \subseteq M \restriction H(\lambda_2)$; as both N_δ^*

and $M\restriction H(\lambda_2)$ are elementary submodels of $\left(H(\lambda_2), \in, <^*_{\lambda_2}\right)$ we get

$$N^*_\delta \prec M\restriction H(\lambda_2) \prec \left(H(\lambda_2), \in, <^*_{\lambda_2}\right).$$

Clearly $N^*_\delta \cap \omega_1 = \delta = M \cap \omega_1$. As $M\restriction H(\lambda_2)$ is countable and by the meaning of "$N^*_\delta \in K^{\text{pos}}_P$" we have $M\restriction H(\lambda_2) \notin K^{\text{neg}}_P$, i.e., for every $p \in P \cap M(= P \cap (M\restriction H(\lambda_2))))$ there is an $(M\restriction H(\lambda_2), P)$-semi-generic $q, p \leq q \in P$. Necessarily q is (M, P)-semi-generic (as in the proof of 1.5(1)); this is enough. $\square_{1.11,1.3,1.2}$

1.12 Conclusion. SPFA implies $\mathcal{P}(\omega_1)/\mathcal{D}_{\omega_1}$ is $\aleph_2$-saturated i.e. satisfies the $\aleph_2$-c.c.

Proof. Actually it follows by Foreman Magidor Shelah [FMSh:240], and Theorem 1.2, but as this is a book we give a proof.

Let $\Xi \subseteq \mathcal{P}(\omega_1)$ be a maximal antichain modulo $\mathcal{D}_{\omega_1}$. Remember seal(Ξ) = $\{\langle(\gamma_i, a_i) : i \leq \alpha\rangle : \alpha < \omega_1,\ a_i$ is a countable subset of Ξ, non empty for simplicity, $\gamma_i < \omega_1, a_i$ and γ_i are strictly increasing continuous in i, and for limit $\delta \leq \alpha$ we have $\gamma_\delta \in \bigcup_{i<\delta} \bigcup_{A\in a_i} A\}$. This forcing is S-complete for every $S \in \Xi$ (see XIII 2.8) hence does not destroy the stationarity of subsets of ω_1. Hence by 1.3 seal(Ξ) is semiproper.

Now $\mathcal{I}_i = \{\bar{a} \in \text{seal}(\Xi) : \ell g(\bar{a}) \geq i\}$ is a dense subset of seal(Ξ). So by SPFA there is a directed $G \subseteq \text{seal}(\Xi)$ satisfying $\bigwedge_{i<\omega_1} G \cap \mathcal{I}_i \neq \emptyset$. Let $\bigcup G$ be $\langle(\gamma_i, a_i) : i < \omega_1\rangle$. We claim $\Xi = \bigcup\{a_i : i < \omega_1\}$. Let $C \stackrel{\text{def}}{=} \{\gamma_i : i = \gamma_i = \omega i$ is a limit$\}$, $a_i = \{A_\alpha : \alpha < \omega i\}$, $A \stackrel{\text{def}}{=} \{\delta < \omega_1 : (\exists i < \delta)(\delta \in A_i)\}$. Now if $S \in \Xi \setminus \{A_i : i < \omega_1\}$, then for all $i < \omega_1$, $S \cap A_i$ is nonstationary, so also $S \cap A$ is nonstationary, which is impossible as $C \subseteq A$ and C is a club. $\square_{1.12}$

§2. SPFA Does Not Imply PFA$^+$

It is folklore that in the usual forcing for PFA(=Ax[proper]) (or SPFA= Ax[semiproper]) any subsequent reasonably closed forcing preserves PFA (or

SPFA). Magidor and Beaudoin refine this, showing that starting from a model of PFA, forcing a stationary subset of $\{\delta < \omega_2 : \mathrm{cf}(\delta) = \aleph_0\}$ by

$$P = \{h : h \text{ a function from some } \alpha < \omega_2 \text{ to } \{0,1\} \text{ such that :}$$
$$\text{for all } \delta \in S^2_1 \text{ we have : } h^{-1}(\{1\}) \cap \delta \text{ is not stationary in } \delta\}$$

(ordered by inclusion) produces a stationary subset of $\{\delta < \omega_2 : \mathrm{cf}(\delta) = \aleph_0\}$ which does not reflect, and this still preserves PFA but easily makes PFA$^+$ (and SPFA) fail.

We can also start with $V \models$PFA, and force $h : \omega_2 \to \omega_1$ such that no $h^{-1}(\{\alpha\}) \cap \delta$ is stationary in δ, where $\alpha < \omega_1, \delta < \omega_2$, and $\mathrm{cf}(\delta) = \aleph_1$.

It had remained open whether SPFA $\vdash$ SPFA$^+$ and we present here the solution, first starting with a supercompact limit of supercompacts and then only from one supercompact. I thank Todorcevic and Magidor for asking me this question.

2.1 Theorem. Suppose κ is a supercompact limit of supercompacts. *Then*, in some generic extension, SPFA holds but PFA$^+$ fails.

The proof is presented in 2.3 - 2.9.

Overview of the Proof. Let f^* be a Laver diamond for κ (see Definition VII 2.8, as Laver shows w.l.o.g. it exists). Our proof will unfold as follows. We shall first define a semiproper iteration $\bar{Q}^\kappa$. Now $\Vdash_{P_\kappa}$ "SPFA" is as in the proof of X 2.8. We then define in V^{P_κ} a proper forcing notion R and an R-name $\underset{\sim}{S}$, $\Vdash_R$ "$\underset{\sim}{S} \subseteq \omega_1$ is stationary". We then show, that for no directed $G \subseteq R$ in V^{P_κ} is $\underset{\sim}{S}[G]$ well defined (i.e., $(\forall i < \omega_1)(\exists p \in G)[p \Vdash_R$ "$i \in \underset{\sim}{S}$" or $p \Vdash_R$ "$i \notin \underset{\sim}{S}$"$])$, and stationary (i.e., $\{i < \omega_1 : (\exists p \in G) p \Vdash$ "$i \in \underset{\sim}{S}$"$\}$ is stationary).

Before we start our iteration, we will define several forcing notions (which we will use later when we construct R, and also during the iteration), and we will explain some basic properties of these forcing notions.

Convention. Trees T will be such that members are sequences with the order being $\triangleleft$ (initial segments) and T closed under initial segments so $\ell g(\eta)$ is the level of η in T. But later we will use trees T whose members are sets of ordinal ordered by initial segments, so we can identify a name η if η is strictly increasing sequence of ordinals, $a = \mathrm{Rang}(\eta)$.

2.2 Fact. Let T be a tree of height ω_1, $\kappa \geq \aleph_1$ with $\kappa = 2^{\aleph_2}$ if not said otherwise. Let $P = R_1 * \underset{\sim}{R}_2$, where R_1 is Cohen forcing and R_2 is $\mathrm{Levy}(\aleph_1, \kappa)$ (computed in V^{R_1}). *Then* every ω_1-branch of T in V^P is already in V.

Proof. Well-known and included essentially in the proof of III 6.1.

2.3 Definition. Let T be a tree of height ω_1 with $\aleph_1$ nodes and $\leq \aleph_1$ many ω_1-branches $\{B_i : i < i^* \leq \omega_1\}$ and let $\{y_i : i < \omega_1 \text{ and } [i < 2i^* \Rightarrow i \text{ odd}]\}$ list the members of T such that: $[y_j <_T y_i \Rightarrow j < i]$. Let B_i^* be: B_j if $i = 2j$, $j < i^*$ or $\{y_j\}$ if y_j is defined. Let $B'_j = B^*_j \setminus \bigcup_{i<j} B^*_i$, $x_j = \min(B'_j)$ if $B'_j \neq \emptyset$ so that the sets B'_j are disjoint nonempty end segments of some branch $B_{j'}$, or the singletons $\{y_j\}$ or $\emptyset$; let $B'_j \neq \emptyset \Leftrightarrow i \in w$ and so $\langle B'_j : j \in w\rangle$ form a partition of T. Let $A = \{x_i : i \in w\}$ (so A does not include any linearly ordered uncountable set). The forcing "sealing the branches of T" is defined as (see proof of 2.4(3)):

$$P_T = \{f : f \text{ a finite function from } A \text{ to } \omega, \text{ and}$$
$$\text{if } x < y \text{ are in } \mathrm{Dom}(f), \text{ then } f(x) \neq f(y)\}.$$

See its history in VII 3.23.

2.4 Lemma. For T, P_T as in Definition 2.3:

(1) P_T satisfies the c.c.c.

(2) Moreover: If $\langle p_i : i < \omega_1\rangle$ are conditions in P, then there are disjoint uncountable sets S_1, $S_2 \subseteq \omega_1$ such that: whenever $i < j$, $i \in S_1$, $j \in S_2$, *then* p_i and p_j are compatible.

(3) If $G \subseteq P_T$ is generic over V, $V[G] \subseteq V^*$, and $\aleph_1^{V^*} = \aleph_1^V$, then all ω_1-branches of T in V^* are already in V.

Proof. (1) Follows by (2).

(2) Recall that p and q are incompatible if:

either $p \cup q$ is not a function *or* there are $\eta \in \mathrm{Dom}(p)$, $\nu \in \mathrm{Dom}(q)$ such that $p(\eta) = q(\nu)$, and η and ν are distinct but comparable, i.e. $\eta <_T \nu$ or $\nu <_T \eta$.

Let $\langle p_i : i < \omega_1 \rangle$ be a sequence of conditions in P_T. By the usual Δ-system argument we may assume that for all $i, j < \omega_1$ $p_i \cup p_j$ is a function, and we may also assume that $|\mathrm{Dom}(p_i)| = n$ for all $i < \omega_1$. We will now get the desired result by applying the following subclaim n^2 times:

2.4A Subclaim. If $\langle \eta^1_\alpha : \alpha \in S_1 \rangle$, $\langle \eta^2_\alpha : \alpha \in S_2 \rangle$ are lists of members of A without repetitions, S_1, S_2 are uncountable, *then* there are uncountable sets $S'_1 \subseteq S_1$, $S'_2 \subseteq S_2$ such that: $\alpha \in S'_1$, $\beta \in S'_2 \Rightarrow \eta^1_\alpha, \eta^2_\beta$ are incomparable.

Proof of the subclaim. for $\ell = 1, 2$ and $\zeta < \omega_1$, let:

$$L_\ell(\zeta) = \{\eta^\ell_\alpha \restriction \zeta : \alpha < \omega_1,\ \ell g(\eta^\ell_\alpha) \geq \zeta\}.$$

Let $\zeta_\ell = \min\{\zeta : L_\ell(\zeta)$ is uncountable$\}$, and if all $L_\ell(\zeta)$ are countable, let $\zeta_\ell = \omega_1$.

We now distinguish 4 cases:

Case 1: $\zeta_1 < \zeta_2$: Since $L_2(\zeta_1)$ is countable, for some η the set $S'_1 \stackrel{\text{def}}{=} \{\alpha < \omega_1 : \ell g(\eta^2_\alpha) > \zeta_1$ and $\eta <_T \eta^2_\alpha\}$ is uncountable (as $\aleph_1 = \mathrm{cf}(\aleph_1) > \aleph_0$), and as $L_1(\zeta_1)$ is uncountable, $S'_2 \stackrel{\text{def}}{=} \{\alpha < \omega_1 : \ell g(\eta^1_\alpha) \geq \zeta$ and $\neg\eta <_T \eta^1_\alpha\}$ is uncountable. So S'_1, S'_2 as required. We are done.

Case 2: $\zeta_2 < \zeta_1$: Similar.

Case 3: $\zeta_1 = \zeta_2 < \omega_1$: By induction on $\gamma < \omega_1$ choose $\beta(1, \gamma)$ and $\beta(2, \gamma)$ such that:

$$\ell g(\eta^1_{\beta(1,\gamma)}) \geq \zeta_1 \text{ and } \eta^1_{\beta(1,\gamma)} \restriction \zeta_1 \notin \{\eta^\ell_{\beta(\ell,\gamma')} \restriction \zeta_1 : \gamma' < \gamma,\ \ell = 1, 2\}$$

$$\ell g(\eta^2_{\beta(2,\gamma)}) \geq \zeta_2 \text{ and } \eta^2_{\beta(2,\gamma)} \restriction \zeta_2 \notin \{\eta^\ell_{\beta(\ell,\gamma')} \restriction \zeta_2 : \gamma' < \gamma, \ell = 1, 2\} \cup \{\eta^1_{\beta(1,\gamma)} \restriction \zeta_1\}$$

and let $S'_\ell = \{\beta(\ell, \gamma) : \gamma < \omega_1\}$, $\ell = 1, 2$.

Case 4: $\zeta_1 = \zeta_2 = \omega_1$ and no earlier case. For $\ell = 1, 2$, $\zeta < \omega_1$ let $A^\ell_\zeta = \{\eta \in T : \ell g(\eta) = \zeta$ and there are $\aleph_1$ many α with $\eta^\ell_\alpha \restriction \zeta = \eta\}$, clearly $A^\ell_\zeta \neq \emptyset$.

So $T^\ell \stackrel{\text{def}}{=} \bigcup_{\zeta<\omega_1} A^\ell_\zeta$ is a downward closed subtree of T, possibly only a single branch.

Subcase 4a: For some ℓ and ζ, $|A^\ell_\zeta| > 1$. Without loss of generality $|A^1_\zeta| > 1$. Let $\nu_2 \in A^2_\zeta$, $\nu_1 \in A^1_\zeta \setminus \{\nu_2\}$, for $\ell = 1, 2$ we let $S_\ell = \{\alpha < \omega_1 : \nu_\ell <_T \eta^\ell_\alpha\}$.

Subcase 4b: For each $\ell = 1, 2$ the set $T^\ell = \bigcup_{\zeta<\omega_1} A^\ell_\zeta$ is a branch, say $B_{i(\ell)}$. If $i(1) \neq i(2)$ then we can again find ν_1 and ν_2 as in case 4a. So let $i = i(1) = i(2)$. It is impossible that uncountably many η^ℓ_α are on B_i (by the choice of A in Definition 2.3), so we may assume that no η^ℓ_α is on B_i. By induction we can find uncountable sets $S_1' \subseteq S_1$, $S_2' \subseteq S_1$ and sequences $\langle \nu^1_\alpha : \alpha \in S_1' \rangle$, $\langle \nu^2_\alpha : \alpha \in S_2' \rangle$ such that: $\nu^\ell_\alpha \in B_i$, $\nu^\ell_\alpha <_T \eta^\ell_\alpha$, $\eta^\ell_\alpha \restriction (\ell g(\nu^\ell_\alpha) + 1) \notin B_i$, and $\{\nu^1_\alpha : \alpha \in S_1'\} \cap \{\nu^2_\alpha : \alpha \in S_2'\} = \emptyset$. This shows that for $\alpha \in S_1'$, $\beta \in S_2'$ the nodes η^1_α and η^2_α are incomparable. So we have proved the subclaim and hence 2.4(2).

Proof of 2.4(3). Since $T = \bigcup_{j<\omega_1} B_j'$ is a partition of T, we can for each $y \in T$ find a unique $j = j(y)$ with $y \in B_j'$. Let $h(y) = \min B_{j(y)}' \in A$. In V^{P_T} we have a generic function $g : A \to \omega$, and we can extend it to a function $g : T \to \omega$ by demanding $g(y) = g(h(y))$. Now let B^* be an ω_1-branch of T in some $\aleph_1$-preserving extension of V^{P_T}. Clearly $g \restriction B^*$ takes some value uncountably many times, but $g(y_1) = g(y_2)$ & $y_1 <_T y_2$ implies $j(y_1) = j(y_2)$, so $B^* \subseteq B_j$ for some j. $\square_{2.4}$

2.5 Fact. There is a family $\langle \eta_\delta : \delta < \omega_1, \delta \text{ limit} \rangle$ such that:

(A) $\eta_\delta : \omega \to \delta$, and $\sup\{\eta_\delta(n) : n < \omega\} = \delta$

(B) For all limit $\delta_1, \delta_2 < \omega_1$ and $n_1, n_2 < \omega$ we have: if $\eta_{\delta_1}(n_1) = \eta_{\delta_2}(n_2)$, then $n_1 = n_2$ and $\eta_{\delta_1} \restriction n_1 = \eta_{\delta_2} \restriction n_2$.

(C) if $m < \ell < \omega$ and $\delta < \omega_1$ is limit, then $\eta_\delta(m) + \omega \leq \eta_\delta(\ell) + \omega$.

Proof. Easy. Let $H : {}^{\omega>}\omega_1 \to \omega_1$ be a 1-1 map such that for all $\eta \in {}^{\omega>}\omega_1$ we have $H(\eta) \in [\ \mathrm{maxRang}(\eta),\ \mathrm{maxRang}(\eta) + \omega)$ (and can add $\nu \triangleleft \eta \Rightarrow H(\nu) < H(\eta)$).

Now for any limit ordinal δ, let $\alpha_0 < \alpha_1 < \cdots$ be cofinal in δ, and define η_δ

inductively by

$$\eta_\delta(n) = H(\eta_\delta{\restriction}n\char`\^\langle\alpha_n\rangle).$$

$\square_{2.5}$

2.6 Definition. Assume that $\langle\eta_\delta : \delta < \omega_1, \delta \text{ limit}\rangle$ is as above.

(1) For $\eta \in {}^{\omega>}\omega_1$, let $E_\eta = \{\delta : \eta \trianglelefteq \eta_\delta\}$.

(2) Let $\mathbf{Z} = \{\eta \in {}^{\omega>}\omega_1 : E_\eta \text{ is stationary}\}$, $C_0 = \{\delta < \omega_1 : (\forall n < \omega)\, \eta_\delta{\restriction}n \in \mathbf{Z}\}$.

(3) Let $\mathbf{Z}^* = \{\eta \in \mathbf{Z} : (\exists^{\aleph_1} i < \omega_1)\, \eta\char`\^\langle i\rangle \in \mathbf{Z}\}$.

(4) Let $C^* = \{\delta \in C_0 : (\exists^\infty n)\, \eta_\delta{\restriction}n \in \mathbf{Z}^*\}$.

(5) Let $\mathbf{Z}_0 = \{\eta \in \mathbf{Z} : (\forall k < \ell g(\eta))\, \eta{\restriction}k \notin \mathbf{Z}^*\}$

2.6A Fact.

(1) $\mathbf{Z}$ is closed under initial segments, so $\mathbf{Z}$ is a tree (of height ω). $\mathbf{Z}^*$ is the set of those nodes of $\mathbf{Z}$ which have uncountably many successors.

(2) $\mathbf{Z}$ defines a natural topology on C_0, if we take the sets E_η as basic neighborhoods.

(3) C_0 and even C^* contains a club of ω_1.

(4) For every finite $u \subseteq \mathbf{Z} \setminus \mathbf{Z}_0$ there is $\rho \in \mathbf{Z}$ which is $\triangleleft$-incomparable with every $\eta \in u$ moreover $\rho \in \mathbf{Z} \setminus \mathbf{Z}_0$.

Proof. (1) and (2) should be clear.

For (3), let χ be some large enough regular cardinal. If $\omega_1 \setminus C^*$ as stationary, we could find a countable elementary submodel $N \prec (H(\chi), \in)$ such that $\delta \stackrel{\text{def}}{=} N \cap \omega_1 \notin C^*$ and $\langle\eta_\delta : \delta < \omega_1 \text{ limit}\rangle$ belongs to N (hence $\langle E_\eta : \eta \in {}^{\omega>}(\omega_1)\rangle$, $\mathbf{Z}$, C_0, $\mathbf{Z}^*$, C^*, $\mathbf{Z}_0$ belong to N). Assume that for some $n_0 < \omega$ for all $n \in (n_0, \omega)$ we have $\eta_\delta{\restriction}n \notin \mathbf{Z}^*$. So the set

$$Y \stackrel{\text{def}}{=} \{\nu \in \mathbf{Z} : \nu \trianglelefteq \eta_\delta{\restriction}n_0 \text{ } \mathit{or}\text{: } \eta_\delta{\restriction}n_0 \trianglelefteq \nu \text{ and } (\forall k \in (n_0, \ell g(\nu)))\, \nu{\restriction}k \notin \mathbf{Z}^*\}$$

is a subtree of $\mathbf{Z}$ with countable splitting, hence is countable. Let $\delta' = \sup\{\nu(k) : \nu \in Y, k \in \mathrm{Dom}(\nu)\}$. Since $Y \in N$, also $\delta' \in N$, but $(\forall k)\, [\eta_\delta{\restriction}k \in Y]$, so $\eta_\delta(k) \leq \delta' < \delta$, contradicting $\delta = \sup\{\eta_\delta(k) : k < \omega\}$.

(4) So if $u \subseteq \mathbf{Z} \setminus \mathbf{Z}_0$ is finite, let $\eta \in u$ be of minimal length and as $\eta \notin \mathbf{Z}_0$ there is $\nu \lhd \eta$, such that $\nu \in \mathbf{Z}^*$, so for some $i < \omega_1$, $\rho \overset{\text{def}}{=} \nu\hat{}\langle i \rangle \in \mathbf{Z}$ and ρ is $\lhd$-incomparable with every $\eta' \in u$ and $\rho \notin \mathbf{Z}_0$ as $\nu \lhd \rho$, $\nu \in \mathbf{Z}^*$. $\square_{2.6A}$

From $\mathbf{Z}$ we can now define the forcing notion R_4, to be used below:

2.6B Definition.

$$R_4 = \{(u, w) : w \text{ a finite set of limit ordinals } < \omega_1, u \text{ a finite subset of } \mathbf{Z} \setminus \mathbf{Z}_0, \text{ and } w \cap E_\eta = \emptyset \text{ for } \eta \in u\}.$$

with the natural order: $(u_1, w_1) \le (u_2, w_2)$ *iff* $u_1 \subseteq u_2$ & $w_1 \subseteq w_2$.

Note that $w \cap E_\eta = \emptyset$ just means that for all $\delta \in w$, $\eta \ntrianglelefteq \eta_\delta$. Actually $\eta = \eta_\delta$ never occurs as $[\eta \in w \Rightarrow \ell g(\eta) < \omega]$ and $[\delta \in u \Rightarrow \ell g(\eta_\delta) = \omega]$.

So we have that (u, w) and (u', w') are incompatible iff $(u \cup u', w \cup w')$ is not in R_4, i.e., either there is $\eta \in u$, $\delta \in w'$ such that $\eta \trianglelefteq \eta_\delta$, or there are such $\eta \in u'$, $\delta \in w$.

R_4 produces a generic set $\underset{\sim}{S}^4 = \bigcup\{w : (\exists u)[(u, w) \in \underset{\sim}{G}_{R_4}]\}$ (i.e. this is an R_4-name), which can easily be shown to be a stationary subset of ω_1(in V^{R_4}, see 2.6E(1))(actually $V[\underset{\sim}{S}^4] = V[\underset{\sim}{G}_{R_4}]$).

2.6C Fact. R_4 satisfies the $\aleph_1$-c.c.; in fact for every $\aleph_1$ conditions there are $\aleph_1$ pairwise compatible (and more).

Proof. Let $(u_i, w_i) \in R_4$ for $i < \omega_1$. Let $v_i \overset{\text{def}}{=} \bigcup\{\mathrm{Rang}(\eta) : \eta \in u_i\}$.

Thinning out to a Δ-system we may assume that there are $\alpha < \omega_1$, $w^* \subseteq \alpha$, $v^* \subseteq \alpha$, $u^* \subseteq {}^{\omega >}\alpha$ such that for all $i < \omega_1 \setminus \alpha$,

$$w_i \cap \alpha = w^*, \quad v_i \cap \alpha = v^*, \quad u_i \cap {}^{\omega >}\alpha = u^*$$

and for all $i \neq j$: $w_i \cap w_j = w^*$, $v_i \cap v_j = v^*$ and $u_i \cap u_j = u^*$. So $\eta \in u_j \setminus u^* \Rightarrow$ $\max\mathrm{Rang}(\eta) > \alpha$. We may also assume that none of the v_i or w_i is a subset of α, and thinning out further we may also assume that for all $i < j$ we have $\alpha < \max(w_i) < \min(v_j \setminus \alpha)$.

Now if $i < j$ and (u_i, w_i) and (u_j, w_j) are incompatible, then we must have one of the following:

(a) $(\exists \eta \in u_i \setminus u^*)\,(\exists \delta \in w_j)\, \eta \trianglelefteq \eta_\delta$

(b) $(\exists \eta \in u_j \setminus u^*)\,(\exists \delta \in w_i)\, \eta \trianglelefteq \eta_\delta$

Now if if clause (b) holds for $\eta \in u_j \setminus u^*$ and $\delta \in w_i$, this implies $\delta \leq \max(w_i) < \min(v_j \setminus \alpha) \leq \max(\mathrm{Rang}(\eta)) < \delta$. [Why? As $\delta \in w_i$; by assumption above; as $\eta \in u_j \setminus u^*$; as $\eta \trianglelefteq \eta_\delta$ and the choice of η_δ (see 2.5(1)) respectively.] A contradiction, so clause (a) must hold. Now we claim that: for each $j < \omega_1$ the set $s_j \stackrel{\text{def}}{=} \{i < j : p_i \text{ and } p_j \text{ are incompatible}\}$ is finite.

Why? Assume not; by the above for $i \in s_j$ necessarily there are $\eta^i \in u_i \setminus u^*$ and $\delta_i \in w_j$ such that $\eta^i \triangleleft \eta_{\delta_i}$. But for $i(0) < i(1)$, both in s_j, we get that $\eta^{i(0)}$ and $\eta^{i(1)}$ must be incomparable, since neither of $\mathrm{Rang}(\eta^{i(0)})$ and $\mathrm{Rang}(\eta^{i(1)})$ can be a subset of the other. Hence all the $\delta_i (i \in s_j)$ are distinct — a contradiction as w_j is finite. $\square_{2.6C}$

2.6D Fact.

(1) If $A \subseteq \omega_1$ is stationary, $n < \omega$, *then* there is $\delta \in A$ such that $E_{\eta_\delta \restriction n} \cap A$ is stationary.

(2) If $B \subseteq \omega_1$ is stationary, then also the set

$$B' \stackrel{\text{def}}{=} \{\delta \in B : (\forall n < \omega)\ [E_{\eta_\delta \restriction n} \cap B \text{ is stationary}]\}$$

is stationary, and in fact $B \setminus B'$ is nonstationary.

Proof. (1) Using Fodor's lemma we can find a stationary set $A' \subseteq A$ and a finite sequence η^* such that for all $\delta \in A'$ we have $\eta_\delta \restriction n = \eta^*$. So $A' \subseteq A \cap E_{\eta^*} = A \cap E_{\eta_\delta \restriction n}$ for all $\delta \in A'$.

(2) Let $A \stackrel{\text{def}}{=} B \setminus B'$, $A_n \stackrel{\text{def}}{=} \{\delta \in B : E_{\eta_\delta \restriction n} \cap B \text{ is nonstationary}\}$. By (1), each A_n must be nonstationary, so also $A = \bigcup_n A_n$ is nonstationary. $\square_{2.6D}$

2.6E Fact. Let $\underset{\sim}{S}^4$ be the R_4-name of a subset of ω_1 defined in 2.6B. *Then* we have

(1) $\underset{\sim}{S}^4$ is stationary in V^{R_4}.

(2) If $A \subseteq \omega_1$ is stationary in V, then in V^{R_4} there is $\eta \in \mathbf{Z}$ such that $A \cap E_\eta$ is stationary and $E_\eta \cap \underset{\sim}{S}^4 = \emptyset$.

(3) Every stationary subset of ω_1 from V has (in V^{R_4}) a stationary intersection with $\omega_1 \setminus \underset{\sim}{S}^4$.

Proof. (1) Easy; for each $p = (u, w) \in R_4$ and club $E \in V$ of ω_1, as $u \subseteq \mathbf{Z} \setminus \mathbf{Z}_0$ is finite there is $\eta \in \mathbf{Z} \setminus \mathbf{Z}_0$ which is $\lhd$-incomparable with every $\nu \in u$ (see 2.6A(4)) so E_η is stationary hence we can find $\delta \in E \cap E_\eta \setminus (\sup(w) + 1)$, so $q = (u, w \cup \{\delta\}) \in R_4$, $p \le q$ and $q \Vdash_{R_4}$ "$\underset{\sim}{S}^4 \cap E \neq \emptyset$". As R_4 satisfies the c.c.c. this suffice.

(2) Let A be stationary. By 2.6A(3) w.l.o.g. $A \subseteq C^*$ and by 2.6D(2) we may w.l.o.g. assume that $(\forall \delta \in A)\,(\forall n < \omega)[E_{\eta_\delta \restriction n} \cap A$ is stationary$]$. Fix a condition $(u, w) \in R_4$. Choose $\delta \in A \setminus w$, then for some large enough n, $E_{\eta_\delta \restriction n} \cap w = \emptyset$ and $\eta_\delta \restriction n \notin \mathbf{Z}_0$, so $(u \cup \{\eta_\delta \restriction n\}, w)$ is a condition in R_4 above $(u, v) \in R_4$ and it clearly forces $A \cap E_{\eta_\delta \restriction n} \cap \underset{\sim}{S}^4 = \emptyset$.

(3) Follows from (2). $\square_{2.6E}$

2.7 Definition of the iteration. We define by induction on $\zeta \le \kappa$ an RCS iteration (see X, §1) $\bar{Q}^\zeta = \langle P_i, \underset{\sim}{Q}_j : i \le \zeta, j < \zeta \rangle$, and if $\zeta < \kappa$, $\bar{Q}^\zeta \in H(\kappa)$, which is a semiproper iteration (i.e. for $i < j \le \zeta$, i non-limit P_j/P_i is semiproper but for a limit ordinal j the forcing notion $\underset{\sim}{Q}_j$ is not necessarily semiproper) and, if $\zeta = \delta$, δ a limit ordinal, also P_ζ-names, A_ζ, T_ζ (of a tree), and $P_{\zeta+1}$-name $W_\zeta = \langle H^\zeta_\alpha(a) : \alpha \in a \in A_\zeta \rangle$, as follows:

(a) Suppose ζ is non-limit, let $\kappa_\zeta < \kappa$ be the first supercompact $> |P_\zeta|$, so κ_ζ is a supercompact cardinal even in V^{P_ζ}, and let Q_ζ be a semiproper forcing notion of power κ_ζ collapsing κ_ζ to $\aleph_2$ such that $\Vdash_{P_\zeta * \underset{\sim}{Q}_\zeta}$ "any forcing notion not destroying stationary subsets of ω_1 is semiproper",
[it exists e.g. by Lemma 1.3 and X 2.8 but really $Q_\zeta = \text{Levy}(\aleph_1, <\kappa_\zeta)$ (in V^{P_ζ}) is okay, as

$$\Vdash_{P_\zeta * Q_\zeta} \text{“}Ax_{\omega_1}[\aleph_1 - \text{complete}]\text{”}$$

and even $\mathrm{Ax}_1[\aleph_1$-complete$]$ implies (by 1.1) the required statement.]

(b) Suppose ζ is limit, $\underset{\sim}{Q}_\zeta$ will be of the form $\underset{\sim}{Q}^a * \underset{\sim}{Q}^b * \underset{\sim}{Q}^c$. Remember that $f^* : \kappa \to H(\kappa)$ is a Laver Diamond (see Definition VII 2.8).

If $f^*(\zeta)$ is a P_ζ-name, $\Vdash_{P_\zeta}$ "$f^*(\zeta)$ is a semiproper forcing notion", *then* let $\underset{\sim}{Q}^a_\zeta = f^*(\zeta)$. If $f^*(\zeta)$ is not like that, let $\underset{\sim}{Q}^a_\zeta$ =the trivial forcing.

$\underset{\sim}{Q}^b_\zeta$ will satisfy the following property:

(∗) If $\xi < \zeta$, ξ is non-limit, $A \in V^{P_\xi}$, $A \subseteq \omega_1$, and A is stationary in V^{P_ξ} (equivalently in V^{P_ζ}) *then* A is stationary in $V^{P_\zeta * \underset{\sim}{Q}^a_\zeta * \underset{\sim}{Q}^b_\zeta}$.

(This property (∗) will follow from 2.6E, it will assure that the iteration remains semiproper)

If ζ is divisible by ω^2, we will let $\underset{\sim}{Q}^b_\zeta = \underset{\sim}{Q}^1_\zeta * \underset{\sim}{Q}^2_\zeta * \underset{\sim}{Q}^3_\zeta$. First in V^{P_ζ} choose (see 2.1, 2.3) $\underset{\sim}{Q}^1_\zeta = R_1 * \underset{\sim}{R}_2 * \underset{\sim}{P}_{T_\zeta}$, where $T_\zeta = \{b : b$ an initial segment of some $a \in \bigcup_{\xi<\zeta} A_\xi\}$ ordered by being initial segment (for the definition of A_ξ see the definition of W_ξ below). From the generic subset of Q^1_ζ (and $P_\zeta * \underset{\sim}{Q}^a_\zeta$) we can define, for each ω_1-branch B of T_ζ, a 2-coloring $H_\alpha(B)$ of $\omega_1 : H_\alpha(B) = \bigcup\{H^\zeta_\alpha(a) : \xi \in a \in B$ and $\zeta > \xi \geq \alpha$ and $H^\xi_\alpha(a)$ is well defined$\}$. (See the definition of W_ζ below, we can say that if $H_\alpha(B)$ is not a 2-coloring of ω_1 we use trivial forcing). Remember 2.4(3).

To define Q^2_ζ, we need the following concept:

We will say that a function $h : [\omega_1]^2 \to 2$ is almost homogeneous if there is a partition $\omega_1 = \bigcup_{n<\omega} A_n$ and an $\ell \in \{0,1\}$ such that for all n the function $h\restriction[A_n]^2$ is constantly $= \ell$. We may say h is almost homogeneous with value ℓ.

We choose $\underset{\sim}{Q}^2_\zeta \in H(\kappa)$ such that

⊗ *if* there is $\underset{\sim}{Q} \in H(\kappa)$ such that

(i) $\underset{\sim}{Q}$ is a $P_\zeta * \underset{\sim}{Q}^a_\zeta * \underset{\sim}{Q}^1_\zeta$-name of a forcing notion

(ii) For every $\xi < \zeta$ the forcing notion $(P_\zeta * \underset{\sim}{Q}^a_\zeta * \underset{\sim}{Q}^1_\zeta * \underset{\sim}{Q})/P_{\xi+1}$ is semi proper, (equivalently, preserves stationarity of subsets of $\omega_1\rangle$

(iii) if, in $V^{P_\zeta * \underset{\sim}{Q}^a_\zeta * \underset{\sim}{Q}^1_\zeta}$, B is a branch of T_ζ cofinal† in ζ, $\alpha < \omega_1$, then the coloring $H_\alpha(B)$ of ω_1, is almost homogeneous in $V^{P_\zeta * \underset{\sim}{Q}^a_\zeta * \underset{\sim}{Q}^1_\zeta * \underset{\sim}{Q}}$

then $\underset{\sim}{Q}^2_\zeta$ satisfies this.

Otherwise $\underset{\sim}{Q}^2_\zeta$ is trivial.

† Note: members of B are subsets of ζ with last element, so $\{\max(a) : a \in B\}$ is a subset of ζ.

In $V^{P_\zeta * \underset{\sim}{Q}^a_\zeta * \underset{\sim}{Q}^1_\zeta * \underset{\sim}{Q}^2_\zeta}$ we now define a set S_ζ, which is supposed to guess the set $\underset{\sim}{S}[G]$. More on $\underset{\sim}{S}$ will be said below (and see "overview").

We let $\alpha \in S_\zeta$ if for all the ω_1-branches B of T_ζ cofinal in ζ (i.e. such that $\bigcup\{a : a \in B, \mathrm{otp}(a)$ a successor ordinal$\}$ is unbounded in ζ) the function $H_\alpha(B)$ is almost homogeneous with value 1.

Now we let $\underset{\sim}{Q}^3_\zeta$ be the forcing notion which shots a club through the complement of S_ζ, unless S_ζ includes modulo $\mathcal{D}_{\aleph_1}$ some stationary set from $\bigcup_{\xi<\zeta} V^{P_\xi}$, in which case Q^3_ζ will be trivial. This completes the definition of Q^b_ζ when ζ is divisible by ω^2, otherwise Q^b_ζ is trivial.

We let $\underset{\sim}{Q}_\zeta = \underset{\sim}{Q}^a_\zeta * \underset{\sim}{Q}^b_\zeta * \underset{\sim}{Q}^c_\zeta$ where $\underset{\sim}{Q}^c_\zeta$ is the addition of $(\aleph_1 + 2^{\aleph_0})^{V^{P_\zeta}}$ Cohen reals with finite support. Clearly for $\xi < \zeta$, $(P_\zeta / P_{\xi+1}) * \underset{\sim}{Q}_\zeta$ preserves stationarity of subsets of ω_1, hence it is semiproper (see (a)), so Q_ζ is o.k. An alternative to (b): we can demand Q^a_ζ forces SPFA. If ζ is not divisible by ω^2 let $\underset{\sim}{Q}_\zeta$ be $\underset{\sim}{Q}^a_\zeta * \underset{\sim}{Q}^b_\zeta * \underset{\sim}{Q}^c_\zeta$, with $\underset{\sim}{Q}^a_\zeta, \underset{\sim}{Q}^b_\zeta$ trivial, $\underset{\sim}{Q}^c_\zeta$ as above.

(c) For ζ limit we also have to define W_ζ (in $V^{P_{\zeta+1}}$).

(i) W_ζ is a function whose domain is $A_\zeta = \{a : a \subseteq \zeta+1, \zeta \in a \in V^{P_\zeta}$, and a is a countable set of limit ordinals and $\xi \in a \Rightarrow a \cap (\xi+1) \in V^{P_\xi}\}$.

(ii) For $a \in A_\zeta$, $W_\zeta(a) = \langle H^\zeta_\alpha(a) : \alpha < \mathrm{otp}(a)\rangle$, where $H^\zeta_\alpha(a)$ is a function from $[\mathrm{otp}(a)]^2 = \{\{j_1, j_2\} : j_1 < j_2 < \mathrm{otp}(a)\}$ to $\{0,1\}$ (where $\mathrm{otp}(a)$ is the order type of a).

(iii) For every $\xi \in a \in A_\zeta$ (check definition of A_ζ), $a \cap (\xi+1) \in A_\xi$, and for $\alpha < \mathrm{otp}(a \cap (\xi+1))$, $H^\xi_\alpha(a \cap (\xi+1))$ is $H^\zeta_\alpha(a)$ restricted to $[\mathrm{otp}(A \cap (\xi+1))]^2$.

(iv) If $a \in A_\zeta$, we use the Cohen reals from $\underset{\sim}{Q}^c_\zeta$ to choose the values of $H^\zeta_\alpha(a)(\{j_1, j_2\})$ when $\alpha = \mathrm{otp}(a \cap \zeta)$ or $j_1 = \mathrm{otp}(a \cap \zeta)$ or $j_2 = \mathrm{otp}(a \cap \zeta)$ that is when not defined implicitly by condition (iii), i.e. by H^ξ_α (not using the same digit twice (digit from the Cohen reals from $\underset{\sim}{Q}^c_\zeta$)).

(v) $T_\zeta (\in V^{P_\zeta})$ is the tree $(\bigcup\{A_\delta : \delta < \zeta$ a limit ordinal$\}, <_{T_\zeta})$, ($<_{T_\zeta}$ is being an initial segment i.e. $a < b$ iff $a = b \cap (\max(a) + 1)$).

There is no problem to carry the inductive definition.

Note that we can separate according to whether the cofinality of ζ in V^{P_ζ} is $\aleph_0$ or $\geq \aleph_1$ (so for a club of $\zeta < \kappa$ we can ask this in V) and in each case some parts of the definition trivialize.

2.7A Toward the proof: Clearly P_κ is semiproper, satisfies the κ-c.c., and $|P_\kappa| = \kappa$. In $V_0 = V^{P_\kappa}$ let $T^* = \bigcup\{A_\delta : \delta < \kappa \text{ (limit)}\}$, and let $<_{T^*}$ be the order: being initial segment. Let $T = \{a : a \text{ an initial segment of some } b \in T^*\}$.
So T is a tree, and the $(\alpha+1)$'th level of T is $\{\alpha \in T : \mathrm{otp}(a) = \alpha + 1\}$. The height of T is ω_1 (since all elements of T are countable) and all elements of T have $\kappa = \aleph_2$ many successors and every member of T belongs to some ω_1-branch.

For every ω_1-branch B of T we get a family of ω_1 many coloring functions $H_\alpha(B) : [\omega_1]^2 \to 2$, by letting $H_\alpha(B)(\{j_1, j_2\}) = {H_\alpha}^{\max(a)}(a)(j_1, j_2)$ for any $a \in B$ with $\mathrm{otp}(a) > \max(j_1, j_2, \alpha)$ successor ordinal. Now we want to show that PFA$^+$ fails in V^{P_κ}. To this end, we will define a proper forcing notion R and R-name $\underset{\sim}{S}$ of a stationary set of ω_1. R will be obtained by composition. The components of R and of the proof are not new.

2.8 Definition of R. Let $V_0 = V^{P_\kappa}$. Let R_0 be Levy$(\aleph_1, \aleph_2)$ (in V_0). In $V_1 = V_0^{R_0}$, let R_1 be the Cohen forcing; in $V_2 \overset{\text{def}}{=} V_1^{R_1}$ let R_2 be Levy$(\aleph_1, 2^{\aleph_2})$. Let $V_3 = V_2^{R_2}$. Let $\langle B_i : i < i^* \rangle \in V_1$ list the ω_1-branches of T in V_1 and $i_0^* < i^*$ be such that $i < i_0^* \Leftrightarrow \kappa > \sup[\bigcup\{a : a \in B_i\}]$. Easily in V_1, T has ω_1-branches with supremum κ (just build by hand) so really $i_0^* < i^*$. Forcing with $R_1 * \underset{\sim}{R}_2$ over V_1 does not add ω_1-branches to T (by 2.2), hence in V_3 it has $\leq \aleph_1$ ω_1-branches, so let us essentially specialize it (see 2.4(3)), using the forcing notion $R_3 = P_T$ from 2.3. Let $V_4 = V_3^{R_3}$. Let R_4 be the forcing defined in 2.6B, and let $V_5 = V_4^{R_4}$. In V^5 we now define R_5: it is the product with finite support of $R^5_{\alpha,i} (\alpha < \omega_1, i_0^* \leq i < i^*)$, where the aim of $R^5_{\alpha,i}$ is making ω_1 the union of $\aleph_0$ sets, on each of which $H_\alpha^{[i]} \overset{\text{def}}{=} H_\alpha(B_i)$ is constantly 0 if $\alpha \in S^4$, constantly 1 *if* $\alpha \notin S^4$ (remember $H_\alpha(B_i)$ was defined just before 2.8 and S^4 was defined from G_{R_4}), see definition below. See definition 2.6B and Fact 2.6E. Let $V_6 = V_5^{R_5}$. So the decision does not depend on i.

Now $R^5_{\alpha,i}$ is just the set of finite functions h from ω_1 to ω so that on each $h^{-1}(\{n\})$ the coloring $H^{[i]}_\alpha$ is constantly 0 or constantly 1, as required above (so some case for all $n < \omega$).

Lastly, let $R = R_0 * \underset{\sim}{R}_1 * \underset{\sim}{R}_2 * \underset{\sim}{R}_3 * \underset{\sim}{R}_4 * \underset{\sim}{R}_5$. We define $\underset{\sim}{S}$ such that $\underset{\sim}{S}^4 \subseteq \underset{\sim}{S} \subseteq \underset{\sim}{S}^4 \cup \{\gamma + 1 : \gamma < \omega_1\}$ and, if $G \subseteq R$ is directed and $\underset{\sim}{S}[G]$ well defined, then all relevant information is decided; specifically: for the model N of cardinality $\aleph_1$ chosen below, for every R-name $\underset{\sim}{\alpha}$ of an ordinal which belongs to N we have $(\exists p \in G)$ [p forces a value to $\underset{\sim}{\alpha}$] (i.e., what is needed below including a well ordering of ω_1 of order type $\underset{\sim}{\zeta}_\alpha$ for $\alpha < \omega_2$).

2.9 Fact. The forcing R is proper (in V_0).

As properness is preserved by composition, we just have to check each R_i in V_i. The only nontrivial one (from earlier facts) is R_5. For this it suffices to show that the product of any finitely many $R^5_{\alpha,i}$ satisfies the $\aleph_1$-c.c. Let $m < \omega$, and let the pairs (α_l, i_l) for $l < m$ be distinct (so $\alpha_l < \omega_1, i^*_0 \leq i_l < i^*$). Note that each B_{i_ℓ} (an ω_1-branch of T) is from V_1. So for some $\beta^* < \omega_1$, $i_{\ell_1} \neq i_{\ell_2} \Rightarrow B_{i_{\ell_1}}, B_{i_{\ell_2}}$ have no common member of level $\geq \beta^*$. Now we claim that in V_5 (on $H^{[i]}_\alpha$ see in 2.8):

$(*)$ If for each $\ell < m, \langle w^\ell_\gamma : \gamma < \omega_1\rangle$ is a sequence of pairwise disjoint finite subsets of $\omega_1 \setminus \beta^*$, then for some $\gamma(1), \gamma(2) < \omega_1$, for each even $\ell < m$

$$[x \in w^\ell_{\gamma(1)} \;\&\; y \in w^\ell_{\gamma(2)} \Rightarrow H^{[i_\ell]}_{\alpha_\ell}(\{x, y\}) = 0]$$

and for each odd $\ell < m$

$$[x \in w^\ell_{\gamma(1)} \;\&\; y \in w^\ell_{\gamma(2)} \Rightarrow H^{[i_\ell]}_{\alpha_\ell}(\{x, y\}) = 1].$$

Why? First we show that this holds in V_1 (note: $R_5 \in V_1$!). Because R_0 is $\aleph_1$-complete, it adds no new ω-sequence of members of V_0, hence for some $\zeta < \kappa, \{\langle \ell, w^\ell_\gamma\rangle : \gamma < \omega, \ell < m\}$ belongs to V^{P_ζ} and to $H(\zeta)$. Note that for each $\ell < m$, the sequence $\langle w^\ell_\gamma : \ell < m, \gamma < \omega\rangle$ is a sequence of pairwise disjoint subsets of $\omega_1 \setminus \beta^*$ and remember the way we use the Cohen reals to define the $H_i^\xi(a)$'s. We can show that for any possible candidate $\langle w^\ell : \ell < m\rangle$ for $\langle w_\varepsilon^\ell : \ell < m\rangle$ or even just for a sequence $\langle w^\ell : \ell < m\rangle$, $w^\ell \subseteq w_\varepsilon^\ell$ (for any $\varepsilon < \omega_1$ large enough) for infinitely many $\gamma < \omega$, the conclusion of $(*)$ holds for $(\gamma(1), \gamma(2)) = (\gamma, \varepsilon)$.

Clearly $(*)$ implies that any finite product of $R^5_{\alpha,i}$ satisfies the $\aleph_1$-c.c if it holds in the right universe (V_5). So for proving the fact we need to show that the subsequent forcing by R_1, R_2, R_3, R_4 preserves the satisfaction of $(*)$.

The least trivial is why R_3 preserves it (as $\underset{\sim}{R}_2$ is $\aleph_1$-complete and as R_1 and R_4 satisfy: among $\aleph_1$ conditions $\aleph_1$ are pairwise compatible (see 2.6(C)).

Recall from 2.4 that for any sequence $\langle p_i : i < \omega_1\rangle$ of conditions we can find disjoint uncountable sets S_1, S_2 such that for $i \in S_1$, $j \in S_2$ the conditions p_i and p_j are compatible. (This is also true for R_1 and R_4). We will work in V_3. So assume that $\langle \underset{\sim}{w}_\gamma^\ell : \gamma < \omega_1, \ell < m\rangle$ is an R_3-name of a sequence contradicting property $(*)$ in $V_3^{R_3}$. For $\gamma < \omega_1$ let p_γ be a condition deciding $\langle w_\gamma^\ell : \ell < m\rangle$, say $p_\gamma \Vdash \underset{\sim}{w}_\gamma^\ell = {}^*w_\gamma^\ell$. Let S_1, S_2 be as above, $S_k = \{\gamma_\alpha^k : \alpha < \omega_1\}$. Let $u_\alpha^\ell = {}^*w^\ell_{\gamma^1_\alpha} \cup {}^*w^\ell_{\gamma^2_\alpha}$ for $\ell < m$. By thinnings out we may without loss of generality assume that the sets $\bigcup_{\ell<m} w_\alpha^\ell$ for $\alpha < \omega_1$ are pairwise disjoint, so we can apply $(*)$ in V_3. This gives us $\alpha(1)$, $\alpha(2)$ such that for all even ℓ, $x \in u^\ell_{\alpha(1)}$, $y \in u^\ell_{\alpha(2)} \Rightarrow H^{[i_\ell]}_{\alpha_\ell}(\{x,y\}) = 0$ and similarly for odd ℓ we have $x \in u^\ell_{\alpha(1)}$ & $y \in u^\ell_{\alpha(2)} \Rightarrow H^{[i_\ell]}_{\alpha_\ell}(\{x,y\}) = 1$. Let q be a condition extending $p_{\gamma^1_{\alpha(1)}}$ and $p_{\gamma^2_{\alpha(2)}}$, then $q \Vdash$ "$\gamma^1_{\alpha(1)}$ and $\gamma^2_{\alpha(2)}$ are as required". $\square_{2.9}$

So R is proper in V_0; as in V_5, S^4 is stationary and R_5 satisfies the $\aleph_1$-c.c, clearly S^4 is a stationary subset of ω_1 in V_6 too; hence, by the choice of $\underset{\sim}{S}$ (just before 2.9) we have $\Vdash_R$ "$\underset{\sim}{S} \subseteq \omega_1$ is stationary".

2.9A Claim. In V^{P_κ}, PFA$^+$ fail as exemplified by R, $\underset{\sim}{S}$.

Proof. In V^{P_κ}, let χ be e.g. $\beth_3(\kappa)^+$ and let $N \prec (H(\chi), \in, <^*_\chi)$ be a model of cardinality $\aleph_1$ containing all necessary information. i.e. the following belongs to N: i (if $i \le \omega_1$), $\langle R_0, \underset{\sim}{R}_1, \underset{\sim}{R}_2, \underset{\sim}{R}_3, \underset{\sim}{R}_4, \underset{\sim}{R}_5\rangle$, $\langle P_i, \underset{\sim}{Q}_j : i \le \kappa, j < \kappa\rangle$, G_{P_κ}, $\underset{\sim}{S}^4$ (but not $\underset{\sim}{S}$!), $\underset{\sim}{f}$ (see below), $\langle \underset{\sim}{B}_i : i < i^*\rangle, i_0^*$. Suppose that $G \in V^{P_\kappa}$, $G \subseteq R$ is directed and meets all dense sets of R which are in N. It suffices to show that $\underset{\sim}{S}[G]$ is not stationary. Note that N is a model of ZFC$^-$ etc.

Let $\underset{\sim}{f} \in N$ be the R_0-name of the function from ω_1 onto κ, then easily $\underset{\sim}{f}[G]$ is a function from ω_1 onto some $\delta < \kappa$, $\mathrm{cf}(\delta) = \aleph_1$, in V^{P_κ}. Note that $\underset{\sim}{T}[G] \in N[G]$ is just T_δ, and if $N[G] \models$ "$\underset{\sim}{B}[G]$ is an ω_1-branch of T cofinal in κ", then $\underset{\sim}{B}[G]$ is as ω_1-branch of T_δ cofinal in δ, and similarly with the coloring. We will now show how we could have predicted this situation in V^{P_δ}: Let $\underset{\sim}{h} : \omega_1 \times \omega_1 \to T$ be an R-name (belonging to N) which enumerates all ω_1-branches of T (we use the essential specialization by R_3) i.e.

$$\Vdash_R \text{“}\{\{\underset{\sim}{h}(i,j) : j < \omega_1\} : i < \omega_1\} = \{\underset{\sim}{B}_i : i < i^*\}\text{”}.$$

Then each set $\{\underset{\sim}{h}(i,j)[G] : j < \omega_1\}$ (for $i < \omega_1$) will be an ω_1-branch of T_δ (remember $T_\delta = \bigcup\{A_\zeta : \zeta < \delta \text{ limit}\}$), some of them cofinal in δ, and these ω_1-branches will be in $V^{P_\delta * \underset{\sim}{Q}^a_\delta}$, as $\underset{\sim}{Q}^b_\delta$ (more exactly $\underset{\sim}{Q}^1_\delta$, see 2.7) was chosen in such a way that no ω_1-branch can be added to T_δ without collapsing $\aleph_1$. Also all the ω_1-branches of $\underset{\sim}{T}[G] = T_\zeta$ will appear in this list.

Now we can recall how the set S_δ was defined: For each ω_1-branch B of T_δ (in $V^{P_\delta * \underset{\sim}{Q}^a_\delta * \underset{\sim}{Q}^1_\delta * \underset{\sim}{Q}^2_\delta}$ equivalently in $V^{P_\delta * \underset{\sim}{Q}^a_\delta}$) which is cofinal in δ, we have $\aleph_1$ many coloring functions $H_\alpha(B)$, and there are such ω_1-branches. We let $\alpha \in S_\delta$ if for all these ω_1-branches B the function $H_\alpha(B)$ is almost homogeneous with value 1.

Now note that the set G also interprets the names for the homogeneous sets for the colorings $H^{[i]}_\alpha$. These homogeneous sets exist in V^{P_κ} hence in V^{P_ξ} for $\xi < \kappa$ large enough, so in $V^{P_\delta * \underset{\sim}{Q}^a_\delta * \underset{\sim}{Q}^1_\delta}$ there is a forcing producing such sets, which, for every $\xi < \delta$ preserves stationarity of sets A, which are stationary subsets of ω_1 in $V^{P_{\xi+1}}$ (the forcing is $\underset{\sim}{Q}^3_\delta * \underset{\sim}{Q}^c_\delta * (P_\kappa / P_{\delta+1})$). Using the supercompactness of κ we can get such a forcing in $H(\kappa)$. But this implies that these sets are already almost homogeneous in $V^{P_\delta * \underset{\sim}{Q}^a_\delta * \underset{\sim}{Q}^1_\delta * \underset{\sim}{Q}^2_\delta}$ (see clause (b) in 2.7), so also $\underset{\sim}{S}[G]$ is in $V^{P_\delta * \underset{\sim}{Q}^a_\delta * \underset{\sim}{Q}^1_\delta * \underset{\sim}{Q}^2_\delta}$ (see the choice of R_5 in 2.8) and $\underset{\sim}{S}[G] = S_\delta$. But the forcing $\underset{\sim}{Q}^3_\delta$ ensures that S_δ is not stationary. $\square_{2.1}$

2.10 Lemma. We can reduce the assumption in 2.1 to "κ is supercompact"

Proof. We repeat the proof of 2.1 with some changes indicated below. We demand that every Q_δ is semiproper. We need some changes also in clause (b) of 2.7 (in the inductive definition of $\underset{\sim}{Q}_i$), we let $\underset{\sim}{Q}^a_\zeta = f^*(\zeta)$ only if: $f^*(\zeta)$ is a P_ζ-name, $\Vdash_{P_\zeta}$ "$f^*(\zeta)$ is semiproper" and let $\underset{\sim}{Q}^a_\zeta$ be trivial othervise. Let $\underset{\sim}{Q}^b_\zeta$ be trivial except when for some $\lambda_\zeta < \kappa, f^*(\zeta) \in H(\lambda_\zeta)$, and ζ is $\beth_8(\lambda_\zeta)$-supercompact. In this case we let (in $V^{P_\zeta * \underset{\sim}{Q}^a_\zeta}$), $\underset{\sim}{Q}^1_\zeta$ be defined as in the proof of 2.1 except that the $R^5_{\alpha,j}$ are now as defined below, $\underset{\sim}{Q}^2_\zeta$ is a forcing notion of cardinality $(2^{\aleph_1})^{V^{P_\zeta * \underset{\sim}{Q}^a_\zeta * \underset{\sim}{Q}^1_\zeta}}$ which forces MA. Now let $\underset{\sim}{S}_\delta \in V^{P_\zeta * \underset{\sim}{Q}^a_\zeta * \underset{\sim}{Q}^1_\zeta * \underset{\sim}{Q}^2_\zeta}$ be as described below, and $\underset{\sim}{Q}^3_\zeta$ is shooting a club through $\omega_1 \setminus \underset{\sim}{S}_\delta$ if $\underset{\sim}{Q}^a_\zeta * \underset{\sim}{Q}^1_\zeta * \underset{\sim}{Q}^2_\zeta * \underset{\sim}{Q}^3_\zeta$ is semiproper, and trivial otherwise. Now $\underset{\sim}{Q}^b_\zeta = \underset{\sim}{Q}^1_\zeta * \underset{\sim}{Q}^2_\zeta * \underset{\sim}{Q}^3_\zeta$. Lastly $\underset{\sim}{Q}^c_\zeta$ is as in the proof of 2.1 and $\underset{\sim}{Q}_\zeta = \underset{\sim}{Q}^a_\zeta * \underset{\sim}{Q}^b_\zeta * \underset{\sim}{Q}^c_\zeta$, now clearly $\underset{\sim}{Q}_\zeta \in H(\beth_8(\lambda_\zeta))$. This does not change the proof of 2.1. Now we let Q_κ = shooting a club called $\underset{\sim}{E}$ (of order type κ) through $\{i < \kappa : V \models \text{"cf}(i) = \aleph_0\text{"}$ or $V \models$ "i is strongly inaccessible in V, λ_ζ well defined and i is $\beth_8(\lambda_\zeta)$-supercompact"$\}$ (ordered by being an initial segment). Now it is easy and folklore that, for such Q_κ, we have $V^{P_\kappa * \underset{\sim}{Q}_\kappa} \models$SPFA, and show as before $V^{P_\kappa * \underset{\sim}{Q}_\kappa} \models \neg\text{PFA}^+$.

* * *

Why the need to change $\underset{\sim}{Q}^2_\zeta$? As the result of an iteration we ask "is there $\underset{\sim}{Q}$ such that (i), (ii), (iii) of $\otimes$", and this may well defeat our desire that $\underset{\sim}{Q}_\zeta$ hence $\underset{\sim}{Q}^1_\delta$ belongs to $H(\beth_8(\lambda_\zeta))$. We want to be able to "decipher" the possible "codings" fast, i.e., by a forcing notion of small cardinality, so we change $R^5_{\alpha,i}$'s inside the definition of R, in Definition 2.8).

We let $\gamma_{\alpha,j}$ be 0 if $\alpha \in S^4$ and 1 otherwise, and let $R^5_{\alpha,j}$ be defined by:

$$\begin{aligned} R^5_{\alpha,j} = \{(w,h) :& w \text{ is a finite subset of } \omega_1 \text{ and } h \text{ is a finite function} \\ & \text{from the family of nonempty subsets of } w \text{ to } \omega \text{ such that :} \\ & \textit{if } u_1, u_2 \in \operatorname{Dom}(h) \text{ and } h(u_1) = h(u_2) \\ & \textit{then } |u_1| = |u_2| \text{ and } [\zeta \in u_1 \setminus u_2 \ \&\ \xi \in u_2 \setminus u_1 \ \&\ \zeta < \xi \Rightarrow \\ & \qquad\qquad H^{[j]}_{\alpha}\{\{\zeta,\xi\}\} = \gamma_{\alpha,j}]\}. \end{aligned}$$

(actually coloring pairs suffice).

2.10A Definition. 1) A function $H : [\omega_1]^2 \to \{0,1\}$ is called *ℓ-colored* (where $[A]^\kappa = \{a \subseteq A : |a| = \kappa\}$) if $\ell \in \{0,1\}$ and there is a function $h : \mathcal{S}_{<\aleph_0}(\omega_1) \to \omega$ such that: if u_1, u_2 are finite subsets of ω_1 and $h(u_1) = h(u_2)$ then $|u_1| = |u_2|$ and $[\zeta \in u_1 \setminus u_2 \ \&\ \xi \in u_2 \setminus u_1 \ \&\ \zeta < \xi \Rightarrow H(\{\zeta,\xi\}) = \ell]$.
2) Called H (as above) explicitly non-ℓ-colored if there is a sequence $\langle u_\gamma : \gamma < \omega_1\rangle$ of pairwise disjoint finite subsets of ω_1 such that: for any $\alpha < \beta < \omega_1$ there are $\zeta \in u_\alpha$, $\xi \in u_\beta$ such that $H(\{\zeta,\xi\}) \neq \ell$.

2.10B Claim. 1) 1-colored, 0-colored are contradictory.
2) If H is explicitly non-ℓ-colored then it is not ℓ-colored.
3) If $MA + 2^{\aleph_0} > \aleph_1$, $\ell < 2$ and $H : [\omega_1]^2 \to \{0,1\}$ *then* H is ℓ-colored or explicitly non ℓ-colored.

Proof. 1) Clearly H cannot be both 0-colored and 1-colored.
2) Note also that if H is ℓ-colored, and u_ζ $(\zeta < \omega_1)$ are pairwise disjoint non empty finite subsets of ω_1 such that $\zeta < \xi \Rightarrow \sup(u_\zeta) < \min(u_\zeta)$ *then* for some $\zeta < \xi$, $H(u_\zeta) = H(u_\xi)$ hence $H{\restriction}\{\{\alpha,\beta\} : \alpha \in u_\zeta, \beta \in u_\xi\}\}$ is constantly ℓ.
3) Use R defined like $R^5_{\alpha,j}$ from above.

If it satisfies the c.c.c., from a generic enough subset of R_H we can define a "witness" h to H being ℓ-colored. If R_H is not c.c.c. a failure is exemplified say by $\langle u_\zeta : \zeta < \omega_1\rangle$; without loss of generality it is a Δ-system i.e. $\zeta < \xi < \omega_1 \Rightarrow u_\zeta \cap u_\xi = u^*$. Reflection shows that $\langle u_\zeta \setminus u^* : \zeta < \omega_1\rangle$ exemplifies "explicitly non-ℓ-colored". $\square_{2.10B}$

The needed forcing $\underset{\sim}{Q}^2_\zeta$ is not too large ($\leq \lambda_\zeta$), and by 2.10B it essentially determines the $\gamma_{\alpha,j}$ (i.e., we can find $\gamma^0_{\alpha,j}$ so that if we have an appropriate G, the values of the $\gamma_{\alpha,j}$ will be $\gamma^0_{\alpha,j}$). So we have at most one candidate for $\underset{\sim}{S}[G]$, namely S_δ, and if $\omega_1 \setminus S_\delta$ is not disjoint to any stationary subset of ω_1 from V^{P_δ} modulo $\mathcal{D}_{\aleph_1}$, we end the finite iteration defining Q_δ by shooting a club through $\omega_1 \setminus S_\delta$.

Why is Q_δ still semiproper? Clearly $\underset{\sim}{Q}^a_\zeta$, $\underset{\sim}{Q}^1_\zeta$, $\underset{\sim}{Q}^2_\zeta$ are semiproper and so preserve stationarity of subsets of ω_1, and also $\underset{\sim}{Q}^3_\zeta$ do this and $\underset{\sim}{Q}^c_\zeta$ satisfies the c.c.c. So it is enough to prove that. Now use Rss (see chapter XIII §1 but assume on δ (remember we should shoot a club through $\underset{\sim}{E}$) that we have enough supercompactness for δ) to show that we still have semiproper $\equiv$ not destroying the stationarity of subsets of ω_1 for the relevant forcing.

This finish the proof that we can define the iteration $\bar{Q}$ as required. Lastly in the proof of the parallel of 2.9A we use also $E \in N$ hence $\delta \in E$. $\square_{2.10}$

2.11 Claim. If $\alpha(0), \alpha(1) \leq \omega_1$ and $|\alpha(0)| < |\alpha(1)|$, *then* $Ax_{\alpha(0)}$ [semiproper] $\nvdash Ax_{\alpha(1)}$ [proper] (assuming the consistency of ZFC+$\exists$ a supercompact).

Proof: Similar. [Now the Laver Diamond is used to guess triples of the form $(\bar{Q}\restriction\delta, \underset{\sim}{Q}_\delta, \langle \underset{\sim}{S}_i : i < \alpha(1)\rangle)$, $\underset{\sim}{Q}_\delta$ is a P_δ-name of a semiproper forcing, $\Vdash_{P_\delta + Q_\delta}$ "$\underset{\sim}{S}_i$ is a stationary subset of ω_1". In (b) from the colourings corresponding to the branches we decode a sequence $\langle S^*_\alpha : \alpha < \alpha(2)\rangle$ of stationary sets and try to shoot a club through $\omega_1 \setminus \underset{\sim}{S}^*_\alpha$ for one of them such that $\underset{\sim}{S}^\delta_i \setminus \underset{\sim}{S}_\delta$ is stationary for every $i < \alpha(1)$ (in addition to the earlier demands.] $\square_{2.11}$

2.12 Observation. Properness is not productive, i.e. (provably in ZFC) there are two proper forcings whose product is not proper.

Proof: Let T be the tree $({}^{\omega_1>}(\omega_2), \triangleleft)$; now one forcing, P, adds a generic branch with supremum ω_2, e.g., $P = T$ (it is $\aleph_1$-complete). The second forcing, Q, guarantees that in any extension of V^Q, as long as $\aleph_1$ is not collapsed, T

will have no ω_1-branch with supremum ω_2^V. Use $Q = Q_1 * \underset{\sim}{Q_2} * \underset{\sim}{Q_3}$, where Q_1 is Cohen forcing, $Q_2 = \mathrm{Levy}(\aleph_1, 2^{\aleph_1})$ in V^{Q_1} (so it is well known that in $V^{Q_1 * Q_2}$, $\mathrm{cf}(\omega_2^V) = \omega_1$, and T has no branch with supremum ω_2 and has no new ω_1-branch so has $\leq \aleph_1$ ω_1-branchs), and Q_3 is the appropriate specialization of T (like R_3 in the proof of 2.1, see Definition 2.3). Since in $V^{P \times Q}$ there is a branch of T cofinal in ${\omega_2}^V$ not from V and $V^{P \times Q}$ is an extension of V^Q, $\aleph_1$ must have been collapsed (see 2.4(3)).

We could also have used the tree ${}^{\omega_1 >}2$, but then we should replace "no ω_1-branch with supremum ${\omega_2}^V$" by "no branch of T which is not in V". $\square_{2.12}$

2.13 Discussion. Beaudoin asks whether SPFA $\vdash Ax_1$[finite iteration of $\aleph_1$-complete and c.c.c. forcing notions]. So the proofs of 2.1 (and 2.2) show the implications fail (whereas it is well known that already Ax(c.c.c.)$\Rightarrow$Ax$_1$(c.c.c.)).

But $\aleph_1$-complete forcing would be a somewhat better counterexample. We have

2.14 Fact. SPFA $\vdash Ax_1[\aleph_1$-complete].

2.14A Reminder. We recall the following facts and definitions (see XIII):

(1) If P and Q are $\aleph_1$-complete, then $\Vdash_P$ "Q is $\aleph_1$-complete".

(2) For $\langle A_i : i < \omega_1 \rangle$ such that $A_i \subseteq \omega_1$ we define the diagonal union of these sets as $\nabla_{i<\omega_1} A_i = \{\delta < \omega_1 : (\exists i < \delta)(\delta \in A_i)\}$.
If $A_i \subseteq \omega_1$ is nonstationary for all $i < \omega_1$, then $\nabla_{i<\omega_1} A_i$ is nonstationary (and if A_i is stationary for some i, then $\nabla_{i<\omega_1} A_i \supseteq A_i \setminus (i+1)$ is stationary).

(3) If $S \subseteq \omega_1$ is stationary, then the forcing of "shooting a club through S" is defined as $\mathrm{club}(S) = \{h : h$ an increasing continuous function from some non-limit $\alpha < \omega_1$ into $S\}$. We have $\Vdash_{\mathrm{club}(S)}$ "$\omega_1 \setminus S$ is nonstationary", and for every stationary $A \subseteq S$ we have $\Vdash_{\mathrm{club}(S)}$ "A is stationary".

Proof of 2.14. Suppose $V \models$SPFA, and P is an $\aleph_1$-complete forcing, $\underset{\sim}{S}$ is a P-name, and $\Vdash_P$ "$\underset{\sim}{S} \subseteq \omega_1$ is stationary". For $i < \omega_1$ let $(P_i, \underset{\sim}{S}_i)$ be isomorphic to $(P, \underset{\sim}{S})$, and let P^* be the product of $P_i (i < \omega_1)$ with countable support; so

$P_i \lessdot P^*, P^*$ is $\aleph_1$-complete, and $\underset{\sim}{S}_i$ is a P^*-name and $\Vdash_{P_i}$ "P^*/P_i does not destroy stationarity of subsets of ω_1".

Let $\Xi = \{A \in V : A \subseteq \omega_1, A$ is stationary and $\Vdash_P$ "$\underset{\sim}{S} \cap A$ is not stationary"$\}$. Clearly if $A \in \Xi$ and $B \subseteq A$ is stationary then $B \in \Xi$. Let $\{A_i : i < i^*\} \subseteq \Xi$ be a maximal antichain $\subseteq \Xi$ (i.e., the intersection of any two elements is not stationary).

So, by 1.12 $|i^*| \leq \omega_1$, so without loss of generality $i^* \leq \omega_1$ and define $A_i = \emptyset$ for $i \in [i^*, \omega_1)$. Let $A = \nabla_{i<\omega_1} A_i$. Then also $\Vdash_P$ "$A = \nabla_{i<\omega_1} A_i$", so we have:

(i) $\Vdash_P$ "$\underset{\sim}{S} \cap A$ is not stationary", and

(ii) for every stationary $B \subseteq \omega_1 \setminus A$, for some $p \in P$, we have $p \Vdash_{P^*}$ "$\underset{\sim}{S} \cap B$ is stationary".

Let $\hat{S} \overset{\text{def}}{=} \omega_1 \setminus A$. So $\hat{S}$ is stationary (as $\Vdash_P$ "$\underset{\sim}{S}$ is stationary"). Also, clearly,

(iii) for each $i < \omega_1$, and stationary $B \subseteq \hat{S}$ for some $p \in P_i \lessdot P^*$, we have $p \Vdash_{P^*}$ "$\underset{\sim}{S}_i \cap B$ is stationary".

As P^* is the product of the P_i with countable support, P^*/P_i does not destroy stationarity of subsets of ω_1, so we have

(iv) for every stationary $B \subseteq \hat{S}, \Vdash_{P^*}$ "for some $i, \underset{\sim}{S}_i \cap B$ is stationary".

Let $\underset{\sim}{S}^*$ be the P^*-name: $\nabla_{i<\omega_1} \underset{\sim}{S}_i \overset{\text{def}}{=} \{\alpha < \omega_1 : (\exists i < \alpha) \alpha \in \underset{\sim}{S}_i\}$. So $\Vdash_{P^*}$ "for every stationary $B \subseteq \hat{S}$ (from V), we have $B \cap \underset{\sim}{S}^*$ is stationary".

In V^{P^*} let Q^* be shooting a club $\underset{\sim}{C}$ through $A \cup S^*$ (i.e., $Q^* = \{h : h$ an increasing continuous function from some non-limit $\alpha < \omega_1$ into $A \cup S\}$ ordered naturally). Now Q^* does not destroy any stationary subset of ω_1 from V (though it destroys some from V^{P^*}). So $P^* * \underset{\sim}{Q}^*$ does not destroy any stationary subsets of ω_1 from V; hence by Lemma 1.3 it is semiproper. Now if $G \subseteq P^* * \underset{\sim}{Q}^*$ is generic enough, for each $i < \omega_1, G \cap P_i$ is generic enough such that $\underset{\sim}{S}_i[G]$ is well-defined, and since $C^* = \underset{\sim}{C}[G]$ is a club set and $C^* \subseteq A \cup \nabla_{i<\omega_1} \underset{\sim}{S}_i[G]$, we have $\hat{S} \cap C^* \subseteq \nabla_{i<\omega_1} \underset{\sim}{S}_i[G]$. As $\hat{S}$ is stationary, for some $i, \underset{\sim}{S}_i[G]$ is stationary so the projection of G to $G_i \subseteq P_i$ is as required, and we have finished. $\square_{2.14}$

2.15 Remark. A similar proof works if $P = P^a * \underset{\sim}{P}^b$, where P^a satisfies the $\aleph_1$-c.c. and $\underset{\sim}{P}^b$ is $\aleph_1$-complete in V^{P_a}, if we use $P^* = \{f : f$ a function from ω_1 to $\underset{\sim}{P}, f(i) = (p_i, q_i) \in P^a * \underset{\sim}{P}^b, |\{i : p_i \neq \emptyset\}| < \aleph_0, |\{i : q_i \neq \emptyset\}| < \aleph_1\}$. Note that necessarily even any finite power of P^a satisfies the $\aleph_1$-c.c. In short, we need that some product of copies of P is semiproper, i.e:

2.16 Fact. [SPFA] Suppose Q is a semi proper forcing notion, and there is a forcing notion P and a family of complete embeddings f_i $(i < i^*)$ of P into Q such that:

(a) for any $p \in P$ and $q \in Q$ for some i, the conditions $f_i(p), q$ are compatible with Q.

(b) the forcing $Q/f_i(P)$ does not destroy the stationarity of subsets of ω_1.

Then for any dense subsets $\mathcal{I}_\alpha$ of P for $\alpha < \omega_1$, and $\underset{\sim}{S}$ a P-name of a subset of ω_1, $\Vdash_P$ "$\underset{\sim}{S} \subseteq \omega_1$ is stationary" *there is* a directed $G \subseteq P$, not disjoint to any $\mathcal{I}_\alpha$ (for $\alpha < \omega_1$) such that $\underset{\sim}{S}[G]$ is a well defined stationary subset of ω_1.

Proof. Like 2.14. We define $A \subseteq \omega_1$ satisfying for $\underset{\sim}{S}$ and P the following conditions (from the proof of 2.14): (i), (ii), hence (iii), (iv) (with $P_i = f_i(P)$ and $\underset{\sim}{S}_i = f_i(\underset{\sim}{S})$). $\square_{2.16}$

§3. Canonical Functions for ω_1

3.1 Definition. 1) We define by induction on α, when a function $f : \omega_1 \to$ ordinals is an α-th canonical function:

f is an α-th canonical function (sometimes abbreviated "f is an α-th function" *iff*

(a) for every $\beta < \alpha$ there is a β-th function, $f_\beta < f \bmod \mathcal{D}_{\omega_1}$

(b) f is a function from ω_1 to the ordinals, and for every $f^1 : \omega_1 \to \mathrm{Ord}$, if $A^1 = \{i < \omega_1 : f^1(i) < f(i)\}$ is stationary then for some $\beta < \alpha$ and β-th function $f^2 : \omega_1 \to \mathrm{Ord}$ the set $A^2 \stackrel{\text{def}}{=} \{i \in A^1 : f^2(i) = f^1(i)\}$ is stationary,

2) If we replace a "stationary subset of ω_1" by "$\neq \emptyset \bmod \mathcal{D}$" ($\mathcal{D}$ any filter on ω_1); we write "f is a $(\mathcal{D}, \alpha)$-th function". Of course we can replace ω_1 by higher cardinals.

Remember

3.2 Claim. 1) If $\alpha < \omega_2, \alpha = \bigcup_{i<\omega_1} a_i, \langle a_i : i < \omega_1 \rangle$ is increasing continuous, each a_i is countable, and $f_\alpha(i) \stackrel{\text{def}}{=} \text{otp}(a_i)$ *then* f_α is an α-th function.
2) If for every α there is an α-th function, *then* $\mathcal{D}_{\omega_1}$ is precipitous; really "for every $\alpha < (2^{\aleph_1})^+$ there is α-th function" suffices, in fact those three statements are equivalent.
3) If f is an α-th function; $Q = \mathcal{D}^+_{\omega_1} = \{A \subseteq \omega_1 : A \text{ is stationary}\}$ (ordered by inverse inclusion) then $\Vdash_Q$ "in $V^{\omega_1}/\underset{\sim}{G}_Q$, we have: $\{x : V^{\omega_1}/\underset{\sim}{G}_Q \models$ "x is an ordinal $< f_\alpha/\underset{\sim}{G}_Q$"$\}$ is well ordered of order type α" (remember $V^{\omega_1}/\underset{\sim}{G}_Q$ is the "generic ultrapower" with universe $\{f/\underset{\sim}{G}_Q : f \in V \text{ and } f : \omega_1 \to V\}$ and $\underset{\sim}{G}_Q$ is an ultrafilter on the Boolean algebra $\mathcal{P}(\omega_1)^V$).
4) Any two α-th functions are equal modulo $\mathcal{D}_{\omega_1}$.
5) Similarly for the other filters (we have to require them to be $\aleph_1$-complete, and for (1) - also normal).

Proof. Well known, see [J]. We will only show (1): Let $A^1 = \{i : f(i) < f_\alpha(i)\}$ be stationary. So there is a countable elementary model $N \prec H(\chi)$ (for some large χ) containing $\alpha, f, \langle a_i : i < \omega_1 \rangle$ such that $\delta \stackrel{\text{def}}{=} N \cap \omega_1 \in A^1$. We have $f(\delta) < f_\alpha(\delta) = \text{otp}(a_\delta)$, and $a_\delta = \bigcup_{i \in N} a_i \subseteq N$, so there is $\beta \in N$ such that $f(\delta) = \text{otp}(a_\delta \cap \beta)$. Let $A^2 = \{i \in A^1 : f(i) = \text{otp}(a_i \cap \beta)\}$. Since $A^2 \in N$, $f \in N$, $\beta \in N$, $\langle a_i : i < \omega_1 \rangle \in N$ and $\delta \in A^2$, we can deduce A^2 is stationary.

$\square_{3.2}$

The following answers a question of Velickovic:

3.3 Theorem. Let κ be a supercompact. For some κ-c.c. forcing notion P not collapsing $\aleph_1$ we have that V^P satisfies:

(a) there is $f \in {}^{\omega_1}\omega_1$ bigger (mod $\mathcal{D}_{\omega_1}$) than the first ω_2 function hence the Chang conjecture fails.

(b) PFA (so $\mathcal{D}_{\omega_1}$ is semiproper hence precipitous).

(c) not PFA^+

Outline of the proof: In 3.4 we define a statement $(*)_g$, which we may assume to hold in the ground model (3.5). We define a set $S^g_\chi \subseteq \mathcal{S}_{<\aleph_1}(\chi)$ and we show that if $(*)_g$ holds, then S^g_χ is stationary (3.8). In 3.9 we recall that the class of S^g_χ-proper forcing notions is closed under CS iterations, so assuming a supercompact cardinal we can, in the usual way, force $Ax[S^g_\chi\text{-proper}]$. Finally we find, for each $\alpha < \omega_2$, an S^g_χ-proper forcing notion R_α such that $Ax[R_\alpha] \Rightarrow f_\alpha <_{\mathcal{D}_{\omega_2}} g$.

3.3A Remark. Remember that the first clause of 3.3(a) implies that Chang's conjecture fails, so the negation of 3.3(a) is sometimes called the "weak Chang conjecture".

Proof of 3.3*A*. Let $M = (M, E, \omega_1, \ldots)$ be a model with universe ω_2 which codes enough set theory. Assume that there exists an elementary submodel $N \prec M$ with $\|N\| = \aleph_1$, $|\omega_1^N| = \aleph_0$. Let $\delta = \omega_1^N = \omega_1 \cap N$. In M we have the function f from 3.3(a) and also a family $\langle f_\alpha, E_\alpha : \alpha < \omega_2\rangle$, ($f_\alpha$ is an α-th canonical function, $E_\alpha \subseteq \omega_1$ is a club set, $f_\alpha{\restriction}E_\alpha < f{\restriction}E_\alpha$) as well as a family $\langle E_{\alpha,\beta} : \alpha < \beta < \omega_2\rangle$ of clubs of ω_1 satisfying $f_\alpha{\restriction}E_{\alpha\beta} < f_\beta{\restriction}E_{\alpha\beta}$. For $\alpha < \beta$, $\alpha, \beta \in N$ we have $\delta \in E_{\alpha,\beta} \cap E_\beta$, so

(A) $(\forall \alpha, \beta \in N)\ [\alpha < \beta \Rightarrow f_\alpha(\delta) < f_\beta(\delta)]$

(B) $(\forall \alpha \in N)$: $[f_\alpha(\delta) < f(\delta)]$

So the set $\{f_\alpha(\delta) : \alpha \in N\}$ is uncountable (by (A)) and bounded in ω_1 (by (B)), a contradiction. $\square_{3.3A}$

3.4 Definition. Let f_α be the α'th canonical function for every $\alpha < \omega_2$ (so without loss of generality the f_α are of the form described in 3.2(1)). Let

$g : \omega_1 \to \mathrm{Ord}$. We let $(*)_g$ be the statement:

$$(*)_g \qquad \text{for all } \alpha < \omega_2 \text{ we have} \quad \neg(g <_{\mathcal{D}_{\omega_1}} f_\alpha).$$

By 3.2(4) this definition does not depend on the choice of $\langle f_\alpha : \alpha < \omega_2 \rangle$.

3.5 Remark. It is easy to force a function $g : \omega_1 \to \omega_1$ for which $(*)_g$ holds: let $P = \{h : \text{for some } i < \omega_1,\ h : i \longrightarrow \omega_1\}$ ordered by inclusion. P is $\aleph_1$-complete and $(2^{\aleph_0})^+$-c.c., so assuming CH we get $\aleph_1^{V^P} = \aleph_1^V$ and $\aleph_2^{V^P} = \aleph_2^V$. Let $\langle f_\alpha : \alpha < \omega_2 \rangle$ be the first ω_2 canonical function in V, then they are still canonical in V^P, and it is easy to see that for any $f : \omega_1 \to \omega_1$ in V we have $V^P \models \neg(g <_{\mathcal{D}_{\omega_1}} f)$ where g is the generic function for P.

3.6 Definition. 1) We call $N \prec (H(\chi), \in, <^*_\lambda)$ g-small (in short $g - sm$ or more precisely (g, χ)-small) if N is countable and $\mathrm{otp}(N \cap \chi) < g(N \cap \omega_1)$.
2) We let $S^g_\chi \overset{\mathrm{def}}{=} \{a : a \in \mathcal{S}_{\leq\aleph_0}(\chi), a \cap \omega_1 \text{ is an ordinal and } \mathrm{otp}(a) < g(a \cap \omega_1)\}$

3.7 Definition. We call a forcing notion Q g-small proper *if*: for any large enough χ and $N \prec (H(\chi), \in, <^*_\chi)$, satisfying $\|N\| = \aleph_0$, $Q \in N$, $p \in N \cap Q$ such that N is g-small *there is* $q \geq p$ which is (N, Q)-generic. We write g-sm for g-small.

3.7A Observation. 1) Any proper forcing is g-sm proper.
2) Without loss of generality g is nondecreasing.

Proof. 1) Trivial.
2) Let $E = \{\alpha < \omega_1 : \alpha \text{ is a limit ordinal such that } \beta < \alpha \Rightarrow g(\beta) < \alpha \text{ and } (\forall\beta < \alpha)(\exists\gamma)(\beta < \gamma < \alpha \,\&\, g(\gamma) > \beta)\}$, and let

$$g'(\alpha) = \begin{cases} g(\alpha) & \text{if } \alpha \in E, g(\alpha) \geq \alpha \\ \sup\{g(\beta) : \beta < \alpha\} & \text{otherwise.} \end{cases}$$

Now, for our definition g', g are equivalent but g' is not decreasing. $\square_{3.7A}$

3.8 Claim. 1) $(*)_g$ holds
iff for every $\chi \geq \aleph_2$ the set S^g_χ is a stationary subset of $\mathcal{S}_{<\aleph_1}(\chi)$
iff $S^g_{\aleph_2}$ is a stationary subset of $\mathcal{S}_{<\aleph_1}(\aleph_2)$
iff for some $\chi \geq \aleph_2$, S^g_χ is a stationary subset of $\mathcal{S}_{<\aleph_1}(\chi)$.
2) For a forcing notion Q and $\chi > 2^{|Q|}$ we have: Q is g-sm proper *iff* Q is S^g_χ-proper (see V1.1(2)).
3) If $(*)_g$ holds and Q is g-sm proper *then*

$$\Vdash_Q \text{“}(*)_g\text{”}$$

Proof. 1) first implies second

Assume $(*)_g$ holds, $\chi \geq \aleph_2$ is given, and we shall prove that S^g_χ is a stationary subset of $\mathcal{S}_{<\aleph_1}(\chi)$. Let $x \in H(\chi_1)$ and $\chi_1 = \beth_3(\chi)^+$(e.g $x = S^g_\chi$).

We can choose by induction on $i < \omega_1, N_i \prec (H(\chi_1), \in, <^*_{\chi_1})$ increasing continuous, countable, $x \in N_i \in N_{i+1}$. Clearly for each i we have $\delta_i \stackrel{\text{def}}{=} N_i \cap \omega_1$ is a countable ordinal, and the sequence $\langle \delta_i : i < \omega_1 \rangle$ is strictly increasing continuous. Now letting $N = \bigcup_{i<\omega_1} N_i$, then $\omega_1 + 1 \subseteq N \prec (H(\chi_1), \in, <^*_{\chi_1})$ and N has cardinality $\aleph_1$, so $\text{otp}(N \cap \chi) = \alpha$ for some $\alpha < \omega_2$; let $h : N \cap \chi \to \alpha$ be order preserving from $N \cap \chi$ onto α.
Note: letting $a^1_i \stackrel{\text{def}}{=} N_i \cap \chi, a_i = \text{rang}(h{\restriction}a^1_i)$ we have: α is $\bigcup_{i<\omega_1} a_i$ where a_i is countable increasing continuous in i and $f_{\alpha+1}(i) \stackrel{\text{def}}{=} \text{otp}(a_i) + 1$ is an $(\alpha + 1)$-th function (see 3.2(1)). Also $C = \{i : \delta_i = i\}$ is a club of ω_1 so by $(*)_g$ we can find $i \in C$ such that $f_{\alpha+1}(i) \leq g(i)$, so $\text{otp}(N_i \cap \chi) = \text{otp}(a^1_i) = \text{otp}(a_i) < f_{\alpha+1}(i) \leq g(i) = g(\delta_i) = g(N_i \cap \omega_1)$. I.e. for this i, N_i is g-sm; easily $N_i \cap \chi \in S^g_\chi$ and it exemplifies that S^g_χ is stationary.

second implies fourth. Trivial

fourth implies third. Check. (note: for $\chi \geq \aleph_2$, $\text{otp}(\chi \cap N) \geq \text{otp}(\omega_2 \cap N)$).

third implies first. Let $\alpha < \omega_2, \alpha = \bigcup_{i<\omega_1} a_i$, where a_i are increasing continuous each a_i countable, so $f_\alpha(i) \stackrel{\text{def}}{=} \text{otp}(a_i)$ is an α-th function and let C be a club of ω_1. Let $\bar{a} = \langle a_i : i < \omega_1 \rangle$. Let χ be regular large enough (e.g.

$\geq \beth_3^+$). Clearly

$$\{N \cap \aleph_2 : N \text{ is countable, } N \prec (H(\chi), \in, <^*_\chi)\}$$

is a club of $\mathcal{S}_{\aleph_0}(\aleph_2)$. So by assumption for some countable $N \prec (H(\chi), \in, <^*_\chi)$ we have $C, \bar{a} \in N$ and

(i) $\mathrm{otp}(N \cap \aleph_2) < g(N \cap \omega_1)$.

But as $\bar{a} \in N$ also $f_\alpha \in N$ and we have $[j < N \cap \omega_1 \Rightarrow a_j \in N \Rightarrow a_j \subseteq N]$ hence $\bigcup\{a_j : j < N \cap \omega_1\} \subseteq N \cap \alpha$ but this union is equal to $a_{N\cap\omega_1}$($\bar{a}$ is increasing continuous:) so, as $\alpha \in N$,

(ii) $\mathrm{otp}(a_{N\cap\omega_1}) < \mathrm{otp}(a_{N\cap\omega_1} \cup \{\alpha\}) \leq \mathrm{otp}(N \cap \omega_2)$.

But

(iii) $f_\alpha(N \cap \omega_1) = \mathrm{otp}(a_{N\cap\omega_1})$.

By (i) + (ii) + (iii) we get $f_\alpha(N\cap\omega_1) < g(N\cap\omega_1)$ and trivially $N\cap\omega_1 \in C$, but C was any club of ω_1, hence $\{j < \omega_1 : f_\alpha(j) < g(j)\}$ is stationary. As α was any ordinal $< \omega_2$ we get the desired conclusion.

(2) This is almost trivial, the only point is that to check S^g_χ-properness it is enough to consider models $N \prec (H(\chi), \in, <^*_\chi)$, but for sm-g properness we should consider $N \prec (H(\chi_0), \in, <^*_{\chi_0})$ for all large enough χ_0. First assume Q is g-sm proper, and we shall prove that Q is S^g_χ-proper; and let χ_0 be large enough (say $> \beth_2(\chi)$). Let M be the Skolem Hull of $\{\alpha : \alpha \leq 2^{|Q|}\} \cup \{Q, \chi\}$ in $(H(\chi_0), \in, <^*_{\chi_0})$. Note $||M|| = 2^{|Q|} < \chi$ hence $\mathrm{otp}(M \cap \chi_0) < \chi$ and there is an order-preserving $h : M \cap \chi \to (2^{|Q|})^+ \leq \chi$ onto an ordinal belonging to N. Let N be a countable elementary submodel of $(H(\chi_0) \in, <^*_{\chi_0})$ to which $x = \langle Q, \chi, M, h\rangle$ belongs, and $(N \cap \chi) \in S^g_\chi$. Let $N' \overset{\mathrm{def}}{=} N \cap M$, so $N' \cap \omega_1 = N \cap \omega_1, N'$ is a countable elementary submodel of $(H(\chi_0), \in, <^*_{\chi_0})$ and

$$\begin{aligned}\mathrm{otp}(N' \cap \chi_0) &= \mathrm{otp}(h''(N' \cap \chi_0)) \leq \mathrm{otp}(N \cap \mathrm{Rang}(h)) \\ &\leq \mathrm{otp}(N \cap \chi) < g(N \cap \omega_1) = g(N' \cap \omega_1).\end{aligned}$$

[Why? as h is order presserving; as N is closed under h, h^{-1} and $N' \prec N$; as $\mathrm{rang}(h) \subseteq \chi$; as $N \cap \chi \in S^g_\chi$; as $N' = N \cap M$ respectively.]

Applying "Q is g-sm proper" to N', for every $p \in Q \cap N'$ there is q such that

$p \le q \in Q$ and q is (N', p)-generic. But $Q \cap N = Q \cap N'$ and [q is (N', p)-generic $\Leftrightarrow$ q is (N, p)-generic] as $N \cap 2^{|Q|} = N' \cap 2^{|Q|}$. As we can eliminate "$x \in N$" (as some such x for some $\chi', H(\chi') \in H(\chi_0)$ and χ' belongs to N) we have proved Q is S^g_χ-proper.

The other direction should be clear too.

3) Let $\chi = (2^{|Q|})^+$.

By part (2) we know Q is S^g_χ-proper; by V 1.3 - 1.4(2) as Q is S^g_χ-proper, we have that $\Vdash_Q$ "$(S^g_\chi)^V \subseteq \mathcal{S}_{<\aleph_1}(\chi)^{V^Q}$ is stationary". Clearly $\Vdash_Q$ "$(S^g_\chi)^V \subseteq (S^g_\chi)^{V^Q}$" hence $\Vdash_Q$ "$(S^g_\chi)^{V^Q}$ is a stationary subset of $\mathcal{S}_{<\aleph_1}(\chi)$". So by part (1) (fourth implies first), we have $\Vdash_Q$ "$(*)_g$". $\square_{3.8}$

3.9 Claim.

Assume $(*)_g$ (where $g \in {}^{\omega_1}\omega_1$). *Then* the property "(a forcing notion is) g-sm proper" is preserved by countable support iteration (and even strongly preserved).

Proof. Immediate by V 2.3 and by 3.8(2) above. $\square_{3.9}$

3.10 Claim. Suppose, $g \in {}^{\omega_1}\omega_1$, and $(*)_g$ holds, κ supercompact, $L^* : \kappa \to H(\kappa)$ is a Laver diamond (see VII 2.8) and we define $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \le \kappa, j < \kappa \rangle$ as follows:

(i) it is a countable support iteration

(ii) for each i, *if* $L^*(i)$ is a P_i-name of a g-sm proper forcing and i is limit then $\underset{\sim}{Q}_i = L^*(i)$, *otherwise* $\underset{\sim}{Q}_i = \text{Levy}(\aleph_1, 2^{\aleph_2})$, (in V^{P_i}, i.e. a P_i-name).

Then

(a) P_κ is g-sm proper, κ-c.c. forcing notion of cardinality κ, and $\aleph_2^{V[P_\kappa]} = \kappa$

(b) Ax_{ω_1}[g-sm proper] holds in V^{P_κ}

(c) PFA holds in V^{P_κ}

(d) in V^{P_κ} for every $\alpha < \kappa, g$ is above the α-th function (by $<_{\mathcal{D}_{\omega_1}}$).

Proof. $\bar{Q}$ is well defined by III 3.1B.

Clearly $\Vdash_{P_i}$ "Q_i is g-sm proper" - by choice or as $\mathrm{Levy}(\aleph_1, 2^{\aleph_2})^{V[P_i]}$ is $\aleph_1$-complete hence proper hence (by 3.7) g-sm proper. So by 3.9 the forcing P_κ is g-sm proper; P satisfies κ-c.c. by III 4.1 hence $\Vdash_{P_\kappa}$ "κ regular, $\aleph_1^V$ regular".

The use of Levy $(\aleph_1, 2^{\aleph_2})^{V[P_i]}$ for i non-limit will guarantee $\kappa = \aleph_2$ in V^{P_κ}. Also $|P_\kappa| = \kappa$ is trivial, so (a) holds.

The proof of (b) is like the consistency of $\Vdash_P$ "Ax_{ω_1}[proper]", in VII 2.8 hence (by 3.7A(1)) we have $\Vdash_{P_\kappa}$ "PFA" i.e. (c) hold.

So it remains to prove (d), so let $\alpha < \aleph_2^{V[P_\kappa]} = \kappa$. This will follow from 3.10A, 3.10B, 3.10C below together with (b) above. Let us define a forcing notion R_α:

3.10A Definition. $R_\alpha = \{\langle a_i : i \leq j\rangle : j$ is a countable ordinal, each a_i is a countable subset of α and $\langle a_i : i \leq j\rangle$ is increasing continuous, and for i a limit ordinal $\mathrm{otp}(a_i) < g(i)\}$. The order is: $p \leq q$ iff p is an initial segment of q.
We can assume g is nondecreasing (see 3.7A(2)).

3.10B Observation. R_α is g-sm proper.

Proof. Left to the reader.

3.10C Observation. If $G \subseteq R_\alpha$ is sufficiently generic, *then* G defines an increasing continuous sequence $\langle a_i : i < \omega_1\rangle$ with $\bigcup_{i<\omega_1} a_i = \alpha$ and hence defines an α-th canonical function below g. $\square_{3.10,3.3}$

* * *

Answering a question of Judah:

Question. Does Ax[Countably Complete $*$ c.c.c.] imply PFA?

3.11 Claim. The answer is no.

Proof. Countably complete forcings and c.c.c. forcings and also their composition are ω-proper. So we have

$PFA \Rightarrow$Ax[ω-proper]$\Rightarrow$Ax[countably complete $*$ c.c.c.].

We will show that the first implication cannot be reversed:

3.12 Definition. $\bar{c} = \langle c(i) : i < \omega_1 \rangle$ is a ω-club guessing for ω_1 means that $c(i)$ is an unbounded subset of i of order type ω for each limit ordinal i less than ω_1, such that every closed unbounded subset c of ω_1 includes $c(i)$ for some limit ordinal $i < \omega_1$.

3.13 Claim. (1) If $\bar{c}$ is a ω-club guessing for ω_1, and P is ω-proper, then $\Vdash_P$ "$\bar{c}$ is a ω-club guessing for ω_1".
(2) $\diamondsuit_{\omega_1}$ implies that there is a ω-club guessing for ω_1 (so a ω-club guessing can be obtained by a small forcing notion).

Proof. (1): Let $\underset{\sim}{C}$ be a name for a closed unbounded subset of ω_1, $p \in P$. We need to find a condition $q \geq p$ and some $i < \omega_1$ such that $q \Vdash_P$ "$c(i) \subseteq \underset{\sim}{C}$". Let $\langle N_i : i < \omega_1 \rangle$ be an increasing continuous sequence of countable models $N_i \prec (H(\chi), \in <^*_\chi)$, χ large enough, $\{p, \underset{\sim}{C}, P\} \in N_0$. Let $\delta_i = N_i \cap \omega_1$. Let $C^* = \{i < \omega_1 : \delta_i = i\}$. Now C^* is closed unbounded, so there is some i such that $c(i) \subseteq C^*$, say $c(i) = \{i_0, i_1, \ldots\}$, $i_0 < i_1 < \ldots$. Let $q \geq p$ be N_{i_ℓ}-generic for all $n < \omega$. So $q \Vdash$ "$i_\ell = N_{i_\ell}[G] \cap \omega_1 = N_{i_\ell} \cap \omega_1$", and clearly $\Vdash$ "$N_{i_\ell}[G] \cap \omega_1 \in \underset{\sim}{C}$", so $q \Vdash$ "$c(i) \subseteq \underset{\sim}{C}$".
(2) Should be clear. $\square_{3.13}$

3.14 Claim. Suppose $\bar{c} = \langle c_\delta : \delta < \omega_1 \rangle$ is such that: c_δ is a closed subset of δ of order type $\leq \alpha^*$. Let
$R_{\bar{c}} \overset{\text{def}}{=} \{(i, C) : i < \omega_1, C$ is a closed subset of $i + 1$, such that for every $\delta \leq i$, $\sup(c_\delta \cap C) < \delta\}$,
order is natural. Let
$\mathcal{I}_\gamma \overset{\text{def}}{=} \{(i, C) \in R_{\bar{c}} : \gamma < \max(C)\}$.

Then: $R_{\bar{c}}$ is proper, each $\mathcal{I}_\gamma$ is a dense subset of $R_{\bar{c}}$, and if $G \subseteq R_c$ is directed not disjoint to each $\mathcal{I}_\gamma$, then $C^* = \cup\{C : (i, C) \in G\}$ is a club of $\aleph_1$ such that: $\delta < \omega_1 \Rightarrow \sup(C \cap c_\delta) < \delta$.

Proof. Straight.

For proving "$R_{\bar{c}}$ is proper" denote $q = (i^q, C^q)$, $i^q = \mathrm{Dom}(q)$, let $N \prec (H(\chi), \in, <^*_\chi)$, N countable, $p \in N \cap R_{\bar{c}}$, and $\{\bar{c}, R_{\bar{b}}, \alpha\} \in N$. W.l.o.g. $\beth_7^+ < \chi$. Let $\delta = N \cap \omega_1$, and so we can find $\langle N_i : i < \delta \rangle$, an increasing continuous sequence of elementary submodels of $(H(\beth_7^+), \in)$, $N_i \subseteq N$, $N \cap H(\beth_7^+) = \bigcup_{i<\delta} N_i$ and $p \in N_0$. So we can find $i_0 < i_1 < \dots$, $\delta = \bigcup_{\ell<\omega} i_\ell$ such that $\omega_1 \cap N_{i_\ell+1} \setminus N_{i_\ell}$ is disjoint to c_δ. Let $\langle \underset{\sim}{\tau}_n : n < \omega \rangle$ list the $R_{\bar{c}}$-names of ordinals from N, and we can choose by induction on n a condition p_n, q_n such that: $p \leq p_0 \in N_{i_0+1}$, i^{p_0} is $N_{i_0} \cap \omega_1$, and $[i^p, N_{i_0+1} \cap \omega_1)$ is disjoint to C^{p_0}, $p_n \leq q_n \in R_{\bar{c}} \cap N_{i_n+1}$, q_n force a value to $\underset{\sim}{\tau}_\ell$ if $\ell \leq n$ & $\underset{\sim}{\tau}_\ell \in N_{i_n+1}$, and $q_n \leq p_{n+1}$, $i^{p_{n+1}} = N_{i_{n+1}} \cap \omega_1$, and $[i^{p_{n+1}}, N_{i_{n+1}} \cap \omega_1)$ is disjoint to $c^{p_{n+1}}$. Now $\langle p_n : n < \omega \rangle$ has a limit as required.

Another presentation is noting:

(*) for each $p^* = (i^*, C^*) \in R_{\bar{c}}$ and dense subset $\mathcal{I}$ of P, there is a club $E = E_{q,\mathcal{I}}$ of ω_1 such that:

for every $\alpha \in E$, $\alpha > i^*$, and there is $(i^\alpha, C^\alpha) \in R_{\bar{c}}$, $(i^\alpha, C^\alpha) \geq (\alpha, C^*) \geq (i^*, C^*)$, (i^α, C^α) is in $\mathcal{I}$ and $i^\alpha < \min(E \setminus (\alpha + 1))$.

(**) *if* $p \in N \prec (H(\chi), \in, <^*_\chi)$, N countable, $\{\bar{c}, R_{\bar{c}}, \alpha^*\} \in N$, and $\mathcal{I} \in N$ a dense subset of $R_{\bar{c}}$, *then* $E_{p,\mathcal{I}} \cap N$ has order type $N \cap \omega_1$ hence for unbounded many $\alpha \in N \cap E_{p,\mathcal{I}}$, the interval $[\alpha, \min(E \setminus (\alpha+1)))$ is disjoint to $c_{N \cap \omega_1}$. $\square_{3.14}$

3.14A Conclusion. PFA$\Rightarrow$ there is no ω-club guessing on ω_1. On the other hand "Ax[ω-proper]+ there is a ω-club guessing" is consistent, since starting from a supercompact we can force Ax[ω-proper] with an ω-proper iteration (see V3.5). $\square_{3.11}$

3.15 Remark. The generalization to higher properness should be clear: for α additively indecomposable, Ax[α-proper] is consistent with existence of $\langle c(i) : i < \omega_1$ and α divides $i \rangle$ as in 3.12 only the order type of $c(i)$ is α (for a club of i's), for it to be preserved we use $\bar{c} = \langle c(i) : i < \omega_1$, and α devides $i \rangle$ such that for every γ the set $\{c(i) \cap \gamma : i < \omega_1$ divisible by α and $\gamma \in C(i)\}$ is countable.

On the other hand Ax[α-proper] implies there is no $\langle c(i) : i < \omega_1,\ \alpha\omega$ divides $i\rangle$ such that: $c(i)$ is a club of i of order type $\alpha\omega$ and for every club C of ω_1 for some i, $c(i) \subseteq C$.

§4. A Largeness of $\mathcal{D}_{\omega_1}$ in Forcing Extensions of L and Canonical Functions

The existence of canonical functions is a "large cardinal property" of ω_1, or more precisely, of the filter $\mathcal{D}_{\omega_1}$. For example, the statement "the α-th canonical function exists for any α" will hold if $\mathcal{D}_{\omega_1}$ is $\aleph_2$-saturated, and it implies that the generic ultrapower V^{ω_1}/G_Q (see 3.2(3)) is well-founded. If we know only that ω_1 is a canonical function, we can conclude that the generic ultrapower is well-founded at least below ω_1^V.

It was shown by Jech and Powell [JePo] that the statement "ω_1 is a canonical function" implies the consistency of various mildly large cardinals. Jech and Shelah [JeSh:378] showed how to force the $\aleph_2$-th (or the θ^{th}, for any θ) canonical function to exist (this is weaker than "ω_1 is a cannonical function"). After this paper Jech reasked me a question from [JePo]: "if the function ω_1 is a canonical function, does $0^\#$ exist?" We give here a negative answer. Our proof which uses large cardinals whose existence is compatible with the axiom $V = L$, is in the general style of this book: quite flexible iterations, quite specific to preserving $\aleph_1$. We thank Menachem Magidor for many stimulating discussions on the subject. Subsequently Magidor and Woodin find an equiconsistency results with different method.

This section consists of two parts: First we define a large cardinal property $(*)^1_\lambda$ and show (in 4.3)

$$\mathrm{Con}\Big(\ (\exists G)\Big[V = L[G] + G \subseteq \omega_1 \text{ is generic for a forcing in } L + (\exists\lambda)(*)^1_\lambda\Big]\Big),$$

assuming the existence of $0^\#$ or some suitable strong partition relation. Then we show (in 4.6, 4.7) that $(*)^1_\lambda$ implies that there is a generic extension of the universe in which ω_1 is a λ-function, and make some remarks about possible cardinal arithmetic in this extension.

We think that the proof of 4.6 is also interesting for its own sake, as it gives a method for proving large cardinal properties of $\mathcal{D}_{\omega_1}$ from consistency assumptions below $0^{\#}$.

4.1 Definition. $\lambda \rightarrow^{+} (\kappa)^{<\omega}_{\mu}$ means that for every club C of λ and function $F : [\lambda]^{<\omega} \rightarrow \mu$ there is $X \subseteq C, \mathrm{otp}(X) = \kappa$ such that: $u_1, u_2 \subseteq X \cup \min(X), |u_1| = |u_2| < \aleph_0, u_1 \cap \min(X) = u_2 \cap \min(X)$ implies $F(u_1) = F(u_2)$. Let $\lambda \rightarrow (\kappa)^{<\omega}_{<\lambda}$ mean: if $F : [\lambda]^{<\omega} \rightarrow \lambda$, $F(u) < \min(u \cup \{\lambda\})$, then for some $X \subseteq \lambda$, $\mathrm{otp}(X) = \kappa$ and $F{\restriction}[X]^n$ constant for each n.

By the known analysis

4.2 Remark. 1) If λ is minimal such that $\lambda \rightarrow (\kappa)^{<\omega}_{\mu}$ then $\lambda \rightarrow (\kappa)^{<\omega}_{<\lambda}$ and λ is regular and $2^{\theta} < \lambda$ for $\theta < \lambda$, from which it is easy to see $\lambda \rightarrow^{+} (\kappa)^{<\omega}_{\mu}$. Such λ's are Erdős cardinals, which for $\kappa \geq \omega_1$ implies the existence of $0^{\#}$ so implies $V \neq L$. But of course it has consequences in L.
2) Remember $A^{[n]} = \{b : b \subseteq A, |b| = n\}$.
3) Of course $\mu \geq 2$ is assumed.
4) $\lambda \rightarrow^{+} (\kappa)^{<\omega}_{\mu}$ implies λ is regular, $\mu < \lambda$, and $\lambda \rightarrow^{+} (\kappa)^{<\omega}_{\mu_1}$ for any $\mu_1 < \lambda$.

4.3 Claim. If in V: $\lambda \rightarrow^{+} (\kappa)^{<\omega}_{\kappa}$ and κ is regular uncountable, (hence $\lambda > 2^{\kappa}$) *then* in $V^{\mathrm{Levy}(\aleph_0, <\kappa)}$ and even in $L^{\mathrm{Levy}(\aleph_0, <\kappa)}$ (the constructible universe after we force with the Levy collapse) $(*)^1_{\lambda}$ is satisfied, where:

4.4 Definition. For λ an ordinal, $(*)^1_{\lambda}$ is the following postulate:
for any $\chi > 2^{\lambda}$, and $x \in H(\chi)$, there are N_0, N_1 such that:
(a) N_0, N_1 are countable elementary submodels of $(H(\chi), \in, <^*_{\chi})$
(b) $x \in N_0 \prec N_1$
(c) $\mathrm{otp}(N_0 \cap \lambda) = \mathrm{otp}(N_1 \cap \omega_1)$
(d) in N_1 there is a subset of $\mathrm{Levy}(\aleph_0, N_0 \cap \omega_1)$ generic over N_0.

(e) The collapsing map $f : N_0 \cap \lambda \to \omega_1$ defined by $f(\alpha) = \mathrm{otp}(N_0 \cap \alpha)$ satisfies:

whenever $u \in N_0$, $u \subseteq \lambda$, $|u| \leq \aleph_1$, then $f{\restriction}u \in N_1$ (note $f{\restriction}u$ is $f{\restriction}(u \cap N_0)$).

Proof of 4.3. Straightforward: let $G \subseteq \mathrm{Levy}(\aleph_0, {<}\kappa)$ be generic over V hence it is also generic over L (note: $\mathrm{Levy}(\aleph_0, {<}\kappa)^V = \mathrm{Levy}(\aleph_0, {<}\kappa)^L$). It is also easy to check that $V[G] \models$ "$\lambda \to^+ (\kappa)^{<\omega}_\kappa$ and even $\lambda \to^+ (\kappa)^{<\omega}_{(2^\kappa)}$" because $|\mathrm{Levy}(\aleph_0, < \kappa)| < \lambda$, see 4.2.

Let $\chi > 2^\lambda$, in $L[G]$ and we shall find N_0, N_1, f as required for $L[G], x \in H(\chi)^{L[G]}$ (because $L[G]$ is the case we shall use, $V[G]$ we leave to the reader). In V we can find a strictly increasing sequence $\langle \alpha_i : i < \kappa \rangle$ of ordinals $< \lambda$, indiscernible in $(H(\chi)^{L[G]}, \in, \lambda, G)$, each $\alpha_i \in C^* \stackrel{\mathrm{def}}{=} \{\alpha < \lambda : \alpha$ belongs to any club of λ definable in $(H(\chi)^{L[G]}, \in, \lambda, G)\}$ (so each α_i is a cardinal in $L[G]$). We define, by induction on $n, i_n, N_{0,n}, N_{1,n}$ such that

(α) $\omega \leq i_n < i_{n+1} < \omega_1, i_n$ is limit, $i_0 = \omega$

(β) $N_{0,n}$ is the Skolem Hull of $\{x\} \cup \{\alpha_i : i < i_n\}$ in $(H(\chi)^{L[G]}, \in, \lambda, G)$

(γ) $N_{1,n}$ is the Skolem Hull of $N_{0,n} \cup \bigcup\{\mathrm{otp}(N_{0,n} \cap \lambda) + 1\} \cup \{f_u : u \in N_{0,n}$ is a set of at most $\aleph_1$ of ordinals $< \lambda\}$ where $f_u : u \cap N_{0,n} \to \omega_2$ is defined by $f_u(\alpha) = \mathrm{otp}(N_{0,n} \cap \alpha)$ in the model $(H(\chi)^{L[G]}, \in, \lambda, G)$.

(δ) $i_{n+1} = \mathrm{otp}(N_{1,n} \cap \omega_1)$.

There is no problem to do this. Let $i_\infty \stackrel{\mathrm{def}}{=} \sup\{i_n : n < \omega\}$.

Finally let $N_0 = \bigcup_{n<\omega} N_{0,n}$ and $N_1 = \bigcup_{n<\omega} N_{1,n}$. Now N_0, N_1, f are not necessarily in $L[G]$ but we now proceed to show that they satisfy requirements (a)–(e) from $(*)^1_\lambda$. Clauses (a) and (b) are clear, since the models N_0 and N_1 are unions of elementary chains and $N^0_n \prec N^1_n$ and $x \in N_{0,n}$.

Clearly $N_{1,n} \cap \kappa$ is an initial segment of κ (as $V[G] \models \kappa = \aleph_1$), so $N_{1,n} \cap \kappa$ is an initial segment of $N_{1,n+1} \cap \kappa$. Hence $\mathrm{otp}(N_1 \cap \kappa) = \sup\{\mathrm{otp}(N_{1,n} \cap \kappa) : n < \omega\} = \sup\{i_n : n < \omega\} = i_\infty$. Since $\{\alpha_i : i < i_\infty\} \subseteq N_0$ and the α_i are strictly increasing, we have $\mathrm{otp}(N_0 \cap \lambda) \geq \mathrm{otp}\{i_\alpha : \alpha < \bigcup_{n<\omega} i_n\} = i_\infty$. So $\mathrm{otp}(N_0 \cap \lambda) \geq \mathrm{otp}(N_1 \cap \kappa)$.

For the converse inequality, note that $N_{0,n} \cap \lambda$ is an initial segment of $N_{0,n+1} \cap \lambda$ (as the α_i are indiscernible and in C^* and see Definition 4.1) so $\mathrm{otp}(N_0 \cap \lambda) =$

$\sup\{\mathrm{otp}(N_{0,n} \cap \lambda) : n < \omega\} \leq \sup\{\mathrm{otp}(N_{1,n+1} \cap \omega_1) : n < \omega_1\} \leq \mathrm{otp}(N_1 \cap \omega_1)$. So (c) holds.

Next we have to check (d). Note that N_0 is the Skolem Hull of $\{\alpha_i : i < i_\infty\}$. Let $\delta = N_0 \cap \kappa$; by the previous sentence also $\delta = N_{0,n} \cap \kappa$, and even $N_0 \cap L_\kappa = N_{0,n} \cap L_\kappa$. Let $G = \langle G_\alpha : \alpha < \kappa \rangle$, so $\bigcup G_\alpha$ is a function from ω onto α. Define $Q = \mathrm{Levy}(\aleph_0, \aleph_1)^{N_0}$, $\mathcal{P} = \{\mathcal{I} \cap Q : N_0 \models$ "$\mathcal{I}$ is a dense subset of $Q\}$". Now in $V[G]$, we see that Q is $\mathrm{Levy}(\aleph_0, \delta)$ and $\mathcal{P}$ is a countable family of subsets of Q. Hence for some $\alpha < \kappa$, Q and $\mathcal{P}$ belongs to $V[\langle G_\beta : \beta < \alpha \rangle]$. Without loss of generality $\alpha > \delta$, and α is divisible by $\delta \times \delta$ and without loss of generality $\alpha \in N_{1,1}$ (this is a minor change in the choice of the $N_{0,n}, N_{1,n}$'s). Define $f : \alpha \to \delta$ by $f(\delta i + j) = j$ when $j < \delta$, now $f \circ (\bigcup G_\alpha)$ is a function from ω onto δ, is generic over $V[\langle G_\beta : \beta < \alpha \rangle]$ (for $\mathrm{Levy}(\aleph_0, \alpha)$) hence is generic over N_0 and it belongs to N_1, so demand (d) holds (alternatively we can demand $\langle \alpha_i : i < \kappa \rangle \in V$ and proceed from this.)

Finally clause (e) follows as $N_{0,n} \cap \lambda$ is an initial segment of $N_0 \cap \lambda$ hence defining $f : N_0 \cap \lambda \to \kappa$ by $f(\alpha) \stackrel{\mathrm{def}}{=} \mathrm{otp}(N_0 \cap \alpha)$, used in clause (e) we have: for $u \in N_{0,n}$, $|u| \leq \aleph_1$, $u \subseteq \lambda$, we have $u \cap N_{0,n} = u \cap N_{0,n+1} = u \cap N_0$ (by the choice of the α_i's) and f_u (defined is clause (γ) above) is $f \restriction u$ (i.e. $f \restriction (u \cap N_0)$) which we have put in $N_{1,n+1}$.

So N_0, N_1, f are as required except possibly not being in $L[G]$. But the statement that such models N_0, N_1 exist is absolute between $L[G]$ and $V[G]$.

$\square_{4.3}$

4.5 Claim. $0^\#$ implies that if $\aleph_0 < \kappa < \lambda$ (in V) *then* $L^{\mathrm{Levy}[\aleph_0, <\kappa]}$ satisfies $(*)^1_\lambda$.

Proof. Left to the reader as it is similar to the proof of 4.3. $\square_{4.5}$

4.6 Main Lemma. If $(*)^1_\lambda$, $\lambda = \mathrm{cf}(\lambda) > \aleph_1$, and $2^{\aleph_0} = \aleph_1$ *then* for some forcing notion P:

(i) P satisfies the $\aleph_2$-c.c and has cardinality $(\lambda^{\aleph_1})^+$.

(ii) P does not add new ω-sequences of ordinals.

(iii) $\Vdash_P$ "ω_1 (i.e. the function $\langle \omega_1 : \alpha < \omega_1 \rangle$) is a λ-function".

(iv) $\Vdash_P$ "$2^{\aleph_1} = |P| = [(\lambda^{\aleph_1})^+]^V$" (so for $\mu \geq \aleph_1$ we have $(2^\mu)^{[V^P]} = (2^\mu)^V + \lambda^{\aleph_1}$).

(v) in V^P, for large enough χ and $x \in H(\chi)$ and stationary $S \subseteq \omega_1$ there is a countable $N \prec (H(\chi), \in)$, $x \in N$ such that $N \cap \omega_1 \in S$ and $(\forall f \in N)\big[f \in N \& f \in {}^{\omega_1}\omega_1 \Rightarrow (\exists \alpha \in \lambda \cap N)[N \cap \omega_1 \in \mathrm{eq}(f_\alpha, f)]\big]$, where $\mathrm{eq}(f_\alpha, f) \stackrel{\mathrm{def}}{=} \{i < \omega_1 : f_\alpha(i) = f(i)\}$, and f_α is an α-th function (and $\langle f_\alpha : \alpha < \lambda \rangle \in N$).

4.6A Remark. (a) Let us call a model $N \prec (H(\chi), \in, <^*_\chi)$ "good" if $(\forall f \in N \cap {}^{\omega_1}\omega_1)(\exists \alpha \in \lambda \cap N)\,[N \cap \omega_1 \in \mathrm{eq}(f_\alpha, f)]$ (where $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ is as above); note that this implies $\mathrm{eq}(f_\alpha, f) \subseteq \omega_1$ is stationary.

Let, for $x \in H(\chi)$,

$$\mathcal{M}_x \stackrel{\mathrm{def}}{=} \{N \cap 2^{\aleph_1} : N \text{ is good and, } x \in N\}$$

Note $\mathcal{M}_x \cap \mathcal{M}_y = \mathcal{M}_{\{x,y\}}$. So (v) can be rephrased as:

(v)′ The family $\langle \mathcal{M}_x : x \in H(\chi) \rangle$ is a base for a nontrivial filter on $\mathcal{S}_{<\aleph_1}(2^{\aleph_1})$ (i.e. on the Boolean algebra $(\mathcal{S}_{<\aleph_1}(2^{\aleph_1}))$.)

(b) Note that 4.6(ii) implies $\Vdash_P$CH, and (i) and (ii) together imply that P does not change any cofinalities.

(c) 4.6(v) implies almost 4.6(iii): for some $\beta \leq \lambda$, $\langle \omega_1 : \alpha < \omega_1 \rangle$ is a β-th function.

Proof of (c). Let $f : \omega_1 \to \mathrm{Ord}$, $S \stackrel{\mathrm{def}}{=} \{i : f(i) < \omega_1\}$ is stationary, and assume that for all $\alpha < \lambda$ and α-th function f_α the set $\mathrm{eq}(f, f_\alpha) \cap S$ is nonstationary (if there is such a f_α) say disjoint to the club set C_α. Let N be a model as in (v) containing all relevant information. Let $\delta = N \cap \omega_1$ so $\delta \in S$. Then for some $\alpha \in N$ we have $\delta \in \mathrm{eq}(f, f_\alpha) \cap S$ where $f_\alpha \in N$ is an α-th function. But as $\alpha \in N$ we also have $\delta \in C_\alpha$, a contradiction.

4.7 Conclusion. 1) If in V we have $\lambda \to^+ (\kappa)^{<\omega}_\kappa$ (or just $0^\# \in V, \aleph_0 < \kappa < \lambda$ are cardinals in V or just $V = L^{\mathrm{Levy}(\aleph_0, <\kappa)}$ and $V \models (*)^1_\lambda$), *then* in some generic

extension V^P of L, $2^{\aleph_0} = \kappa = \aleph_1^V$ and $2^\mu = \lambda^+$ when $\kappa \leq \mu \leq \lambda, 2^\mu = \mu^+$ when $\mu > \lambda$ and ω_1 is a λ-th function (and (v) of 4.6).
2) We can, in the proof of 4.6 below, have $\alpha^* = \gamma$ if $\mathrm{cf}(\gamma) > \lambda$, γ divisible by $|\gamma|$ and $|\gamma| = |\gamma|^{\aleph_1}$ (just more care in bookkeeping) so $\Vdash_P$ "$2^{\aleph_1} = |\gamma|$" is also possible.
3) If e.g. (1) above, and we let $Q = \mathrm{Levy}(\aleph_2, \lambda^+)^{V^P}$ *then* in V^{P*Q} we have $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$ (and conditions (iii)+(v) from 4.6 hold but λ is no longer a cardinal) and V^P, V^{P*Q} has the same functions from ω_1 to the ordinals.
4) We can have in 4.6(1), that V^P satisfies $2^\mu = \lambda$ for $\mu \in [\kappa, \lambda)$ and $2^{\aleph_1} = \lambda$ (and $2^\mu = \mu^+$ when $\mu \geq \lambda$ and ω_1 is a λ-th function).

We shall prove 4.7 later.

Proof of Lemma 4.6. We use a countable support iteration $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \leq \alpha^*, \beta < \alpha^* \rangle$, such that:

(1) $\alpha^* = (\lambda^{\aleph_1})^+$

(2) if $\beta < \lambda$, then $\underset{\sim}{Q}_\beta$ is adding a function $f^*_\beta : \omega_1 \rightarrow \omega_1$:

$$Q_\beta = \{f : \text{for some non-limit countable ordinal } i < \omega_1, \\ f \text{ is a function from } i \text{ to } \omega_1\},$$

order: inclusion.

(3) if $\beta = \lambda + \lambda\beta_1 + \beta_2$ where $\beta_1 < \beta_2 < \lambda$ then $\underset{\sim}{Q}_\beta$ is shooting a club to ω_1 on which $f^*_{\beta_1}$ is smaller than $f^*_{\beta_2}$:

$$\underset{\sim}{Q}_\beta = \big\{a : \text{ for some } i < \omega_1, a \text{ is a function from } \{j : j \leq i\} \text{ to } \{0,1\} \\ \text{such that: } \{j \leq i : a(j) = 1\} \text{ is a closed subset of } \mathrm{sm}(\underset{\sim}{f}^*_{\beta_1}, \underset{\sim}{f}^*_{\beta_2})\big\}$$

where $\mathrm{sm}(f,g) \stackrel{\mathrm{def}}{=} \{i < \omega_1 : f(i) < g(i)\}$,

order: inclusion.

(4) *if* $\beta < (\lambda^{\aleph_1})^+, \beta \geq \lambda^2$ and for some $\underset{\sim}{g}, \underset{\sim}{A}$ and $\gamma \leq \beta$ and p we have

$\otimes^{\beta}_{\underset{\sim}{g},\underset{\sim}{A},\gamma,p}$ $\underset{\sim}{g}$ is a P_γ-name of a function from ω_1 to ω_1, $\underset{\sim}{A}$ is a P_γ-name of a subset of ω_1 and $p \in P_\beta$:

$$p \Vdash_{P_\beta} \text{“}\underset{\sim}{A} \text{ is a stationary subset of } \omega_1, \text{ but for no } \alpha < \lambda,$$
$$\text{is } \mathrm{eq}[\underset{\sim}{g}, \underset{\sim}{f}_\alpha] \cap \underset{\sim}{A} \text{ stationary”}$$

then for some such $(\underset{\sim}{g}^*_\beta, \underset{\sim}{A}^*_\beta, \gamma^*_\beta, p^*_\beta)$, with minimal γ_β, the forcing notion $\underset{\sim}{Q}_\beta$ is killing the stationarity of $\underset{\sim}{A}^*_\beta$, that is: $\underset{\sim}{Q}_\beta = \{a : \text{for some } i < \omega_1, a$ is a function from $\{j : j \le i\}$ to $\{0,1\}$ and $\{j : j \le i$ and $a(j) = 1\}$ is closed and if $p^*_\beta \in \underset{\sim}{G}_{P_\beta}$ then a is disjoint to $\underset{\sim}{A}^*_\beta\}$
order: inclusion

(5) if no previous case applies let $\underset{\sim}{A}_\beta = \emptyset, \gamma_\beta = 0, \underset{\sim}{g}_\beta = 0_{\omega_1}$, and define Q_β as in (4).
There are no problems in defining $\bar{Q}$. Let $P = P_{(\lambda^{\aleph_1})^+}$.

Explanation. We start by forcing the f_α's, which are the witnesses for the desired conclusion and then forcing the easy condition: $f_\alpha < f_\beta \bmod \mathcal{D}_{\omega_1}$ for $\alpha < \beta < \lambda$. Then we start killing undesirable stationary sets. Note that given $f \in V^{P_i}$, maybe in V^{P_i} we have $S = \{\alpha < \lambda : \mathrm{eq}[f, f_\alpha]$ is stationary in $V^{P_i}\}$ has cardinality λ, and increasing i it decreases slowly until it becomes empty, so it is natural to use iteration of length of cofinality $> \lambda$ e.g. $\lambda^{\aleph_1} \times \lambda^+$ (ordinal multiplication) is O.K. The problem is proving e.g. that $\aleph_1$ is not collapsed.

Continuation of the proof of 4.6.

The main point is to prove by simultaneous induction that for $\alpha \le (\lambda^{\aleph_1})^+$ the conditions $(a)_\alpha - (e)_\alpha$ listed below hold:

$(a)_\alpha$ forcing with P_α adds no new ω-sequences of ordinals.

$(b)_\alpha$ P_α satisfies $\aleph_2$-c.c.

$(c)_\alpha$ the set P'_α of $p \in P_\alpha$ such that each $p(\beta)$ is an actual function (not just a P_β-name) is dense.

Before we proceed to define $(d)_\alpha$, note that for each $\beta < \alpha$ (using the induction hypothesis),

$$\Vdash_{P_\beta} \text{“}CH \text{ and } |\underset{\sim}{Q}_\beta| = \aleph_1 \text{ and } \underset{\sim}{Q}_\beta \text{ is a subset of}$$

$$H \overset{\text{def}}{=} \{h : h \in V \text{ is a function from some } i < \omega_1 \text{ to } \omega_1\} \in V$$

$$\text{ordered by inclusion”}.$$

So (as P_β satisfies the $\aleph_2$-c.c.), the name $\underset{\sim}{Q}_\beta$ can be represented by $\aleph_1$ maximal antichains of P_β : $\langle\langle p^\beta_{\zeta,h} : \zeta < \omega_1\rangle : h \in H\rangle$, i.e. for each $\zeta < \omega_1, p^\beta_{\zeta,h}$ forces $h \in \underset{\sim}{Q}_\beta$ or forces $h \notin \underset{\sim}{Q}_\beta$. So, $u^*_\beta \overset{\text{def}}{=} \bigcup_{\zeta,\ell} \mathrm{Dom}(p^\beta_{\zeta,\ell})$ is a subset of β of cardinality $\leq \aleph_1$ (all done in V). We may increase u^*_β as long as it is a subset of β of cardinality $\leq \aleph_1$. W.l.o.g. $p^\beta_{\zeta,h} \in P'_\beta$.

Call $u \subseteq \alpha$ closed (more exactly $\bar{Q}$-closed) if $\beta \in u$ implies: $u^*_\beta \subseteq u$ and $\underset{\sim}{g}^*_\beta, \underset{\sim}{A}^*_\beta$ are names represented by $\aleph_1$ maximal antichains $\subseteq P'_\beta$ with union of domains $\subseteq u^*_\beta$ and $\mathrm{Dom}(p^*_\beta) \subseteq u^*_\beta$. W.l.o.g. each u^*_β is closed. For a closed $u \subseteq \alpha$ we define P_u by induction on $\sup(u)$: let $P_u = \{p \in P_\alpha$: $\mathrm{Dom}(p) \subseteq u$ and for each $\beta \in \mathrm{Dom}(p), p(\beta)$ is a $P_{u\cap\beta}$-name$\}$. Let $P'_u = P_u \cap P'_\alpha$. Lastly let

$(d)_\alpha$ $P_u \lessdot P_\alpha$ for every closed $u \subseteq \alpha$; moreover

$(e)_\alpha$ if $u \subseteq \alpha$ is closed, $p \in P'_\alpha$ then:

(1) $p{\restriction}u \in P'_u \subseteq P'_\alpha$ and

(2) $p{\restriction}u \leq q \in P'_u$ implies $q \cup [p{\restriction} (\mathrm{Dom}(p) \setminus u)]$ is a least upper bound of p, q (in P'_α).

Of course the induction is divided to cases (but $(a)_\alpha$ is proved separately). Note that $(e)_\alpha \Rightarrow (d)_\alpha$.

<u>Case A</u>: $\alpha = 0$ Trivial

<u>Case B</u>: $\alpha = \beta + 1$, proof of $(b)_\alpha, (c)_\alpha, (d)_\alpha, (e)_\alpha$.

So we know that $(a)_\beta - (e)_\beta$ holds. By $(a)_\beta$ (as noted above), Q_β has power $\aleph_1$. So we know P_β satisfies $\aleph_2$-c.c., and $\Vdash_{P_\beta}$ “Q_β satisfies the $\aleph_2$-c.c.” hence P_α satisfies the $\aleph_2$-c.c., i.e. $(b)_\alpha$ holds.

If $p \in P_\alpha$, then $p(\beta)$ is a countable subset of $\omega_1 \times \omega_1$ from V^{P_β}, hence by $(a)_\beta$ for some $f \in V$ and q we have $p{\restriction}\beta \leq q \in P_\beta$ and $q \Vdash_{P_\beta}$ “$p(\beta) = f$”. By

$(c)_\beta$ w.l.o.g. q is in P'_β. So $q \cup \{\langle \beta, f\rangle\}$ is in P_α, is $\geq p$ and is in P'_α; so $(c)_\alpha$ holds.

As for $(d)_\alpha$ and $(e)_\alpha$, if $p \in P'_\alpha$, we can observe $(e)_\alpha(1)$ which says: "$p{\restriction}u \in P_u \subseteq P_\alpha$". [Why? If $\beta \notin u$, it is easy, so assume $\beta \in u$; now just note that $p{\restriction}(\beta \cap u) \in P_{\beta\cap u} \lessdot P_\alpha$ by the induction hypothesis, now $p{\restriction}\beta \Vdash_{P_\beta}$ "$p(\beta) \in \underset{\sim}{Q}_\beta$", but $\underset{\sim}{Q}_\beta$ is a $P_{\beta\cap u}$-name, $P_{\beta\cap u} \lessdot P_\beta$ (as u is closed and the induction hypothesis), so by $(d)_\beta$ we have $(p{\restriction}u){\restriction}\beta \Vdash_{P_{u\cap\beta}}$ "$p(\beta) \in Q_\beta$"; so $p{\restriction}u \in P_\alpha$ and as $\mathrm{Dom}(p{\restriction}u) \subseteq u$ we have $p{\restriction}u \in P_u$.]

Next $(e)_\alpha(2)$ follows (check) and then $(d)_\alpha$, $(e)_\alpha$ follows.

<u>Case C</u>: α limit $\mathrm{cf}(\alpha) > \aleph_0$, proof of $(b)_\alpha, (c)_\alpha, (d)_\alpha, (e)_\alpha$.

Clearly $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$ (as the iteration is with countable support), hence $(c)_\alpha$ follows immediately; from $(c)_\alpha$ clearly $(b)_\alpha$ is very easy [use a Δ-system argument, and CH], and clause $(\mathrm{e})_\alpha$ also follows hence $(\mathrm{d})_\alpha$.

<u>Case D</u>: α is limit $\mathrm{cf}(\alpha) = \aleph_0$, proof of $(b)_\alpha, (c)_\alpha, (d)_\alpha, (e)_\alpha$.

As in Case (C), it is enough to prove $(c)_\alpha$. So let $p \in P_\alpha$. Let χ be regular large enough; $N_0 \prec N_1$ be a pair of countable elementary submodels of $(H(\chi), \in, <^*_\chi)$ to which $\bar{Q}, \alpha, \lambda, p$ belongs, satisfying (a)–(e) of $(*)^1_\lambda$ in Def 4.4.

We can find an ω-sequence $\langle u_m : m < \omega\rangle$ such that:

- (i) each u_m is a member of N_0, and is a bounded subset of α of power $\leq \aleph_1$ which is closed for $\bar{Q}{\restriction}\alpha$
- (ii) $u_m \subseteq u_{m+1}$
- (iii) if $u \in N_0$ is a bounded subset of α of power $\leq \aleph_1$ closed for $\bar{Q}{\restriction}\alpha$ *then* for some m we have $u \subseteq u_m$.

There is no problem to choose such a sequence as the family of such u's is directed and countable. Let $\langle \mathcal{I}_m : m < \omega\rangle$ be a list of the dense open subsets of P_α which belong to N_0.

Note that in general, neither $\langle u_m : m < \omega\rangle$ nor $\langle \mathcal{I}_m : m < \omega\rangle$ are in N_1.

Let $\delta \overset{\mathrm{def}}{=} N_0 \cap \omega_1$ and note that $\delta \in N_1$. Let R be $\mathrm{Levy}(\aleph_0, \delta)^\omega$, the ω-th power of $\mathrm{Levy}(\aleph_0, \delta)$ with finite support, so R is isomorphic to $\mathrm{Levy}(\aleph_0, \delta)$ and it (and such isomorphisms) belongs to N_1 so there is $G^* \in N_1$, a (directed) subset of R, generic over N_0. Note that from the point of view of N_0, $\mathrm{Levy}(\aleph_0, \delta)$ is $\mathrm{Levy}(\aleph_0, \aleph_1)$ hence $((\mathrm{Levy}(\aleph_0, \aleph_1))^\omega)^{N_0} = (\mathrm{Levy}(\aleph_0, \delta))^\omega$, so G^* is an N_0-

generic subset of $(\mathrm{Levy}(\aleph_0, \aleph_1)^\omega)^{N_0}$. Let $G^* = \langle G^*_\ell : \ell < \omega \rangle$. Note that $N_0[G^*] \models ZFC^-$ and $N_0[G^*] \subseteq N_1$.

By the induction hypothesis $P_{u_m} \lessdot P_{u_{m+1}} \lessdot P_{(\sup u_{m+1})+1} \lessdot P_\alpha$ for every m. Now we choose by induction on $m < \omega, p_m$ and $G_m \subseteq P_\alpha \cap N_0$ such that:

$$p \le p_m \le p_{m+1},$$

$$p_{m+1} \in \mathcal{I}_m \cap N_0$$

$$p_m \restriction u_m \in G_m$$

$$G_m \subseteq N_0 \cap P'_{u_m} \text{ is generic over } N_0$$

$$\bigcup_{\ell < m} G_\ell \subseteq G_m,$$

$$G_m \in N_1, \text{ moreover } G_m \in N_0[\langle G^*_\ell : \ell \le m \rangle].$$

Why is this possible? Arriving to $m(> 0)$ we have $P'_{u_{m-1}} \lessdot P_\alpha$, $G_{m-1} \subseteq P'_{u_{m-1}} \cap N_0$ is generic for N_0, we can choose p_m as required ($p_m \in \mathcal{I}_m \cap N_0$ and $p_{m-1} \le p_m$ and $p_m \restriction u_{m-1} \in G_{m-1}$). Also $P'_{u_m} = P_{u_m} \cap P'_\alpha$ belongs to N_0, (as $\bar{Q}, P'_\alpha$, and u_m belongs), now it has cardinality $\aleph_1$ (and of course all its members are in V as well as itself), so some list $\langle r^{u_m}_\zeta : \zeta < \omega_1 \rangle$ of the members of P'_{u_m} of length ω_1 belongs to N_0. So as $\delta = N_0 \cap \omega_1 \in N_1$, clearly $P'_{u_m} \cap N_0 = \{r^{u_m}_\zeta : \zeta < \delta\}$ belongs to N_1 and N_1 "know" that it is countable.

As G^*_m is a subset of $\mathrm{Levy}(\aleph_0, \aleph_1)^{N_0} = \mathrm{Levy}(\aleph_0, \aleph_1^{N_0})^{N_0[\langle G^*_\ell : \ell < m \rangle]}$, generic over $N_0[\langle G^*_\ell : \ell < m \rangle]$ there is in $N[\langle G^*_\ell : \ell \le m \rangle]$ a subset of $P'_{u_m} \cap N_0$ generic for $\{\mathcal{I} : \mathcal{I} \in N_0[G_{m-1}]$ and $\mathcal{I} \subseteq P'_{u_m}$ and $\mathcal{I}$ is dense in $P_u\}$ extending G_{m-1}. So in N_1 and even $N_0[\langle G^*_\ell : \ell \le m \rangle]$ we can find $G_m \subseteq P_{u_m} \cap N_0$ generic over N_0 with $p_m \restriction u_m \in G_m$ and $G_{m-1} \subseteq G_m$.

Note: as $P_{u_m} \lessdot P_{u_{m+1}}$ we succeeded to take care of "$G_m \subseteq G_{m+1}$". Let $G = \bigcup_m G_m$, $\delta = N_0 \cap \omega_1$. We define $q = q_G$, a function with domain $\alpha \cap N_0$: for $\beta \in u_m \cap N_0$ let

$$q'_G(\beta) = \bigcup\{r(\beta) : \text{for some } m < \omega \text{ we have } r \in G_m \text{ and } r(\beta) \text{ is an actual (function not just a } P_\beta\text{-name)}\}$$

$q_G(\beta)$ is: $q'_G(\beta) \cup \{\langle \delta, \mathrm{otp}(N_0 \cap \beta) \rangle\}$ if $\beta < \lambda$, and $q'_G(\beta) \cup \{\langle \delta, 1 \rangle\}$ if $\beta \ge \lambda$.

Clearly q is a function with domain $\alpha \cap N_0$, each $q(\beta)$ a function from $\delta + 1$ to ω_1. (Here we use the induction hypothesis $(\mathrm{c})_\beta$.)

If $q \in P_\alpha$ then we will have $q \in P'_\alpha$ and q is a least upper bound of $\bigcup_{m<\omega} G_m$ and of $\{p_m : m < \omega\}$. Hence in particular $q \geq p$ thus finishing the proof of $(c)_\alpha$, hence (as said above) of the present case (Case D). Now we shall show:

$\otimes$ $q{\restriction}u_m \in N_1$ for each $m < \omega$

Clearly $q'_G{\restriction}u_m \in N_1$ as $G_m \in N_1$ (and $P'_{u_m} \in N_1$), hence to prove $\otimes$ we have to show that $\{\langle\beta, (q_G(\beta))(\delta)\rangle : \beta \in u_m\}$ belongs to N_1. Now $\{\langle\beta, (q(\beta))(\delta)\rangle : \beta \in u_m \cap N_0 \backslash \lambda\}$ is $\{\langle\beta, 1\rangle : \beta \in u_m \cap N_0 \backslash \lambda\} = (u_m \cap N_0 \backslash \lambda) \times \{1\}$ belongs to N_1 as $u_m \in N_0 \prec N_1$ and as said earlier, as $N_0 \cap \omega_1 \in N_1$, $N_0 \models |u_0| \leq \aleph_1$ we have $u_m \cap N_0 \in N_1$ and $\lambda \in N_0 \prec N_1$. Next the set $\{\langle\beta, q(\beta)(\delta)\rangle : \beta \in u_m \cap N_0 \cap \lambda\}$ is exactly $f{\restriction}u_m$, where f is the function from 4.4(e).

So by Claim 4.8 below we finish.

<u>Case E</u>: α nonzero, proof of $(a)_\alpha$.

So by cases $(B), (C), (D)$ we know that $(b)_\alpha, (c)_\alpha, (d)_\alpha, (e)_\alpha$ holds.

Now we imitate the proof of Case (D) except that in (i) and (iii) we omit the "bounded in α". So now $P_{u_m} \lessdot P_\alpha$" is justified not by "$(c)_\beta$ for $\beta < \alpha$" but by $(c)_\alpha + (d)_\alpha$. We can finish now, by using again 4.8.

4.8 Claim. If

(a) $N_0 \prec N_1 \prec (H(\chi), \in, <^*_\chi)$ are countable, $\bar{Q}$ is as in the proof of 4.6, $\bar{Q} \in N_0, \alpha = \ell g(\bar{Q}) \in N_0$, $\delta = N_0 \cap \omega_1$, $\mathrm{otp}(\lambda \cap N_0) = \mathrm{otp}(N_1 \cap \omega_1)$, and part (d) of $(*)^1_\lambda$ of Definition 4.4 holds.

(b) $G \subseteq P_\alpha \cap N_0, G$ is directed,

(c) there is a family U such that:

 (α) if $u \in U$ then $u \in N_0, u \subseteq \alpha$ is closed (for $\bar{Q}$ i.e. $\alpha \in u \Rightarrow u^*_\alpha \subseteq u$) of power $\leq \aleph_1$,

 (β) $\bigcup\{u : u \in U\} = N_0 \cap \alpha$, U is directed (by $\subseteq$) and if $u \in N_0$ is closed (for $\bar{Q}$) bounded subset of α of cardinality $\leq \aleph_1$ then $u \in U$.

 (γ) if $u \in U$ then $G \cap P_u$ is generic over N_0

 (δ) if $u \in U$ then $G \cap P_u \in N_1$

(d) $q = q_G$ is defined as in case D of the proof of 4.6 above, i.e. $\mathrm{Dom}(q) = \alpha \cap N_0$ and

$q'(\beta) = \bigcup\{r(\beta)$ for some $u \in U$, $r \in G_m$, $r(\beta)$ an actual function$\}$.

$q(\beta)$ is: $q'(\beta) \cup \{\langle \delta, \mathrm{otp}(N_0 \cap \beta)\rangle\}$ if $\beta < \lambda$, $q'(\beta) \cup \{\langle \delta, 1\rangle\}$ otherwise.

Then

(i) q is in P_α (and even in P'_α)

(ii) $q \in P'_\alpha$ is a least upper bound of G.

Proof. We prove by induction on $\beta \in N_0 \cap \alpha$ that $q \restriction \beta \in P_\alpha$ (hence $\in P'_\alpha$).

This easily suffices.

Note. if $u \in N_0$ is closed and $\subseteq u' \in U$ then we can add it to U.

Case 1: $\beta = 0$, or β is limit. Trivial.

Case 2: $\beta = \gamma + 1, \gamma < \lambda$. Check.

Case 3: $\beta = \gamma + 1, \beta \geq \lambda$.

We should prove $q \restriction \gamma \Vdash_{P_\gamma}$ "$q(\gamma) \in \underset{\sim}{Q}_\gamma$". Recall that u^*_γ is the subset of γ (of size $\aleph_1$) which was needed for the antichains defining Q_γ, and $\delta = N_0 \cap \omega_1$. Clearly u^*_γ and $u^*_\gamma \cup \{\gamma\}$ belongs to U (being closed bounded and in N_0). As $G \cap P_{u^*_\gamma \cup \{\gamma\}}$ is generic over N_0, clearly

$$q \restriction \gamma \Vdash_{P_\gamma} \text{“}q(\gamma) \text{ is a function from } \delta + 1 \text{ to } \omega_1, \text{ such that for every non limit } \zeta < \delta \text{ we have } q(\gamma) \restriction \zeta \in \underset{\sim}{Q}_\gamma\text{”}.$$

Noting $(q(\gamma)) \restriction \zeta$, where $\zeta \leq \delta$, is of the right form; and $\gamma \geq \lambda \Rightarrow (q(\gamma))^{-1}(\{1\})$ is closed and by the choice of $q(\gamma)(\delta)$, clearly it is enough to prove that:

$\otimes_a$ if $\lambda \leq \beta < \lambda^2$ and $\beta = \lambda + \lambda\beta_1 + \beta_2, \beta_1 < \beta_2 < \lambda$ *then* $q \restriction \beta \Vdash_{P_\beta}$ "$f^*_{\beta_1}(\delta) < f^*_{\beta_2}(\delta)$"

$\otimes_b$ if $\lambda^2 \leq \beta < \ell g(\bar{Q})$ *then* $q \restriction \beta \Vdash$ "$\underset{\sim}{p}^*_\beta \in \underset{\sim}{G}_{Q_\beta} \Rightarrow \delta \notin \underset{\sim}{A}^*_\beta$".

Now $\otimes_a$ holds as $q \Vdash_{P_\alpha}$ "$\langle f^*_\gamma(\delta) : \gamma \in N_0 \cap \alpha\rangle$ is strictly increasing" (just see how we have defined $q_G(\gamma)$ in clause (d) of 4.8 above).

So let us prove $\otimes_b$; remember $\underset{\sim}{Q}_\beta$ is a $P_{u^*_\beta}$-name and (u^*_β being closed) $\underset{\sim}{A}_\beta, g^*_\beta$ are $P_{u^*_\beta}$-names, $p^*_\beta \in N_0 \cap P'_{u_\beta}$. If $q \restriction u^*_\beta \Vdash$ "$\delta \notin \underset{\sim}{A}_\beta$ or $p^*_\beta \notin G_{P_{u^*_\beta}}$" we finish. Otherwise there is $r, q{\restriction}u^*_\beta \le r \in P_{u^*_\beta}$ and $r \Vdash$ "$\delta \in \underset{\sim}{A}_\beta$ & $p^*_\beta \in \underset{\sim}{G}_{P_\beta}$"; w.l.o.g. $r \in P'_{u^*_\beta}$. As $G{\restriction}P_{u^*_\beta} \in N_1$ by the proof of $\otimes$ in 4.6, case D (near the end), also $q{\restriction}u^*_\beta \in N_1$, and remembering $\beta \in N_0 \Rightarrow P_\beta \in N_0$ and $\delta \in N_1$, and $P_{u^*_\beta}$, $P'_{u^*_\beta} \in N_1$ and $\underset{\sim}{A}_\beta, p^*_\beta \in N_1$, clearly w.l.o.g. $r \in N_1$. As $\beta \in N_0, \underset{\sim}{g}^*_\beta \in N_0 \subseteq N_1$ is a $P_{u^*_\beta}$-name and $\delta \in N_1$, w.l.o.g. r forces a value to $\underset{\sim}{g}^*_\beta(\delta)$, say $\Vdash$ "$\underset{\sim}{g}^*_\beta(\delta) = \xi(*)$".

Now $\xi(*) \in N_1$ hence $\xi(*) < \mathrm{otp}(N_1 \cap \omega_1) \le \mathrm{otp}\ (N_0 \cap \lambda)$ (here we are finally using 4.4(c)), hence there is $\gamma \in \lambda \cap N_0$ such that $\xi(*) = \mathrm{otp}(N_0 \cap \gamma)$.

But now (see definition of $\underset{\sim}{Q}_\beta$) we have $r \Vdash_{P_\beta}$ "$eq[\underset{\sim}{g}^*_\beta, \underset{\sim}{f}^*_\gamma] \cap \underset{\sim}{A}_\beta$ is not stationary, so it is disjoint to some club $\underset{\sim}{C}^*_\beta$ of ω_1" where $\underset{\sim}{C}^*_\beta$ is a P_β-name and w.l.o.g. $\underset{\sim}{C}^*_\beta \in N_0$.

[Why? As $g^*_\beta, f^*_\gamma, \underset{\sim}{A}_\beta \in N_0$ there is a P_β-name $\underset{\sim}{C}^*_\beta$ such that $\Vdash_{P_\beta}$ " if $\mathrm{eq}[g^*_\xi, f^*_\gamma] \cap \underset{\sim}{A}_\beta$ is not a stationary subset of ω_1 then $\underset{\sim}{C}^*_\beta$ is a club of ω_1 disjoint to this intersection, otherwise $\underset{\sim}{C}^*_\beta = \omega_1$"].

So $\Vdash$ "$\underset{\sim}{C}^*_\beta$ is a club of ω_1". By the induction hypothesis for β (in particular $(\mathrm{b})_\beta$ from the proof of 4.6 which says that P_β satisfies the $\aleph_2$-c.c.), for some $\bar{Q}$-closed bounded $u \subseteq \beta, |u| \le \aleph_1, u \in N_0$ and $\underset{\sim}{C}^*_\beta$ is a P_u-name.

By the induction hypothesis $q{\restriction}\beta \in P'_\beta$; now by the construction of $q, q{\restriction}\beta \Vdash_{P_\beta}$ "$\underset{\sim}{C}^*_\beta \cap \delta$ is unbounded in δ" hence $(q{\restriction}\beta) \cup r$ [i.e. $r \cup (q{\restriction}(\beta \cap \mathrm{Dom}(q) \setminus u^*_\beta))$] is in P'_α, is an upper bound of $q{\restriction}\beta$ and r and it forces $\delta \in \underset{\sim}{C}^*_\beta$, hence $\delta \in eq[g^*_\beta, f^*_\gamma] \Rightarrow \delta \notin A^*_\beta$. But the antecedent holds by the choice of r, γ and $\xi(*)$. So we finish the proof. $\square_{4.8}$

Continuation of the proof of 4.6: So we have to check if conditions (i)-(v) of 4.6 hold for $P = P_{\alpha^*}$. Now (i) holds by $(b)_{\alpha^*} + (c)_{\alpha^*}$ (α^* is the length of the iteration- $(\lambda^{\aleph_1})^+$); condition (ii) holds by $(a)_{\alpha^*}$. Condition (iii) should be clear from the way $Q_\alpha(\lambda \le \alpha < \alpha^*)$ were defined (see the explanation after the definition of Q_α). Prove by induction on $\gamma < \lambda^+$ that

$(*)_\gamma$ if $\underset{\sim}{g}$ is a P_γ-name of a function from ω_1 to ω_1, $\underset{\sim}{A}$ is a P_γ-name of a subset of ω_1 and $p^* \in P_\gamma$ then:

if $p^* \Vdash$ " for every $\alpha < \lambda$ the set $\underset{\sim}{A} \cap eq(\underset{\sim}{g}, \underset{\sim}{f}_\alpha)$ is not stationary subset of ω_1"

then $p^* \Vdash$ "$\underset{\sim}{A} \subseteq \omega_1$ is not stationary".

Arriving to γ let $\langle(g_\zeta, \underset{\sim}{A}_\zeta, p^*_\zeta) : \zeta < \lambda\rangle$ list the set of such triples (their number is $\leq \lambda$ as $|P_\gamma| \leq \lambda = \lambda^{\aleph_1}$ and P_γ satisfies $\aleph_2$-c.c. and the list includes such triples for smaller γ's). For each ζ we can find a club E_ζ of λ^+ such that: if $\alpha < \beta \in E_\zeta$, then for some P_β-name $\underset{\sim}{C}_{\alpha,\underset{\sim}{A}_\zeta,\underset{\sim}{g}_\zeta}$ we have

$$\Vdash_{P_{\lambda^+}} \text{“}\textit{if } \underset{\sim}{A}_\zeta \cap eq(g_\zeta, \underset{\sim}{f}_\alpha) \text{ is not stationary}$$

$$\textit{then} \text{ it is disjoint to } \underset{\sim}{C}_{\alpha,\underset{\sim}{A}_\zeta,\underset{\sim}{g}_\zeta}\text{”}$$

$$\Vdash_{P_{\lambda^+}} \text{“}\underset{\sim}{C}_{\alpha,\underset{\sim}{A}_\zeta,\underset{\sim}{g}_\zeta} \text{ is a club of } \omega_1\text{”}.$$

For any $\delta \in \bigcap_{\zeta<\lambda} E_\zeta$ which has cofinality $> \aleph_1$, we ask whether when choosing $(g^*_\beta, \underset{\sim}{A}_\beta, \gamma_\beta, p^*_\beta)$ do we have a candidate $(\underset{\sim}{g}, \underset{\sim}{A}, \gamma', p)$ as in $\otimes^\delta_{\underset{\sim}{g},\underset{\sim}{A},\gamma'}$, $\gamma' \leq \gamma$.

If for every such δ the answer is no, we have proved $(*)$; if yes, we get easy contradiction.

For finishing the proof of condition (iii) note that we can let $f_\lambda(i) = \omega_1$, and prove by induction on $\alpha \leq \lambda$ that $\underset{\sim}{f}_\alpha$, is an α'th function as follows: $\beta < \alpha < \lambda \Rightarrow f_\beta <_{\mathcal{D}_{\omega_1}} f_\alpha$ (see $Q_{\lambda+\lambda\beta+\alpha}$'s definition) and if $S \subseteq \omega_1$, $f \in {}^{\omega_1}\omega_1$, $S \cap \mathrm{eq}(f, \underset{\sim}{f}_\alpha)$ not stationary for every $\alpha < \lambda$ we get S is not stationary by the definition of Q_β (for $\beta \in [\lambda^2, \alpha^*)$) so if $g <_{\mathcal{D}_{\omega_1}} f_\alpha$ then for every $\beta \in [\alpha, \lambda)$ the set $\mathrm{eq}[g, f_\beta]$ is not stationary and compare the definition of the α'th function and the definition of the forcing condition).

Lastly clause (iv) of 4.6 holds as $\alpha^* = (\lambda^{\aleph_1})^+$, each Q_α has cardinality $\aleph_1$, and $\dot{P}'_{\alpha^*}$ is a dense subset of P_{α^*}. Finally, condition (v) follows from 4.8.

$\square_{4.6}$

4.9. Proof of 4.7. 1)By 4.3, $(*)^1_\lambda$ holds in $L^{\mathrm{Levy}(\aleph_0,<\kappa)}$ and λ is regular hence $\lambda^{\aleph_1} = \lambda$. By 4.6 we can define a forcing notion P in $L^{\mathrm{Levy}(\aleph_1<\kappa)}, |P| = [\lambda^+]^{L[\mathrm{Levy}(\aleph_0,<\kappa)]} = \lambda^+$ as required.

2) Iterate as above for α^* with careful bookkeeping.

3) Left to the reader.

4) Lastly over V^P force with $\mathrm{Levy}(\lambda, \lambda^+)$ such that $2^{\aleph_1} = \lambda$. $\square_{4.7}$

4.10 Discussion. 1) Can we omit the Levy collapse of λ^+ in the proof of 4.7(4) and still get $2^{\aleph_1} = \lambda$ (and $\langle \omega_1 : i < \omega_1 \rangle$ is the λ-th function)? Yes, if we strengthen suitably $(*)^1_\lambda$. (e.g. saying a little more than there is a stationary set of such $\lambda' < \lambda, (*)^1_{\lambda'}$).

2) In 4.6 we can add e.g. that in V^P, Ax[proper of cardinality $\aleph_1$ not adding reals as in XVIII §2]. We have to combine the two proofs.

3) Suppose $V \models$ "$(*)^1_\lambda$", and for simplicity, $V \models$ "G.C.H., λ is regular $\neg(\exists \mu)[\lambda = \mu^+ \ \& \ \mu > \mathrm{cf}\mu \leq \aleph_1]$". (E.g. $L^{\mathrm{Levy}(\aleph_0, <\kappa)}$ when $0^\#$ exists, κ is a cardinal of V.) For some forcing notion P, $|P| = \lambda^+$, and in V^P we have: ω_1 is an ω_3-th function, $\Vdash_P$ "$\aleph_1 = \aleph_1^V, \aleph_2 = (\aleph_2)^V, \aleph_3 = \lambda, \aleph_4 = (\lambda^+)^V$ and CH and $2^{\aleph_1} = \aleph_4$", (so we can then force by Levy$(\aleph_3, \aleph_4)$ and get $2^{\aleph_1} = \aleph_3$).

Proof. 3) Let $R = \mathrm{Levy}(\aleph_2, < \lambda)$, R is $\aleph_2$-complete and satisfies the λ-c.c. and $|R| = \lambda$, so forcing by R adds no new ω_1-sequences of ordinals, make λ to $\aleph_3$. Let P'_{α^*} be the one from 4.6 (or 4.7(2)). As R is $\aleph_2$-complete, also in V^R we have: P'_{α^*} satisfies the $\aleph_2$-c.c., and P'_{α^*} has the same set of maximal antichains as in V. So the family of P'_{α^*}-name of a subset of ω_1 (or a function from ω_1 to ω_1) is the same in V and V^R. So clearly $P'_{\alpha^*} \times R$ is as required. $\square_{4.10}$

Problem. Is ZFC + "θ is an α-th function for some α (for $\mathcal{D}_{\omega_1}$)" + $\neg 0^\#$ consistent? For $\theta \in \{\aleph_1, \aleph_{\omega_1}\}$ or any preassumed θ? (Which will be $< 2^{\aleph_1}$.)