X. On Semi-Proper Forcing

§0. Introduction

We weaken the notion of proper to semiproper, so that some important properties (the most important is not collapsing $\aleph_1$, being preserved by some iterations) still hold for this weaker notion. But the class of semiproper forcing will also include some forcings which change the cofinality of a regular cardinal $> \aleph_1$ to $\aleph_0$. We will also describe how to iterate such forcings preserving semiproperness. So, using the right iterations, we can iterate such forcings without collapsing $\aleph_1$. As a result, we solve the following problems of Friedman, Magidor and Abraham respectively, by proving (modulo suitable large cardinals) the consistency of the following with G.C.H.:

(1) for every $S \subseteq \aleph_2, S$ or $\aleph_2 \setminus S$ contains a closed copy of ω_1,

(2) there is a normal precipitous filter D on $\aleph_2$, $\{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_0\} \in D$,

(3) for every $A \subseteq \aleph_2$, $\{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_0, \delta \text{ is regular in } L[\delta \cap A]\}$ is stationary.

However, the countable support iteration does not work, so we introduce the revised countable support. Though it is harder to define, it satisfies more of the properties we intuitively assume iterations satisfy and is applicable for the purpose of this chapter.

Notation.

Ord is the class of ordinals, Car the class of cardinals, ICar the class of infinite cardinals, UCar $=$ ICar $\setminus \{\aleph_0\}$ and RCar the class of infinite regular cardinals, SCar $=$ RCar $\cup \{2\}$, RUCar $=$ RCar $\cap$ UCar, and we let

$$S^\alpha_\beta = \{\delta < \aleph_\alpha : \mathrm{cf}\delta = \aleph_\beta\}.$$

§1. Iterated Forcing with RCS (Revised Countable Support)

Iterated forcing with countable support is widely used since Laver [L1]. One of its definitions is that at the limit stage with cofinality $\aleph_0$ we take the inverse limit, and at the limit stage with cofinality $> \aleph_0$ we take the direct limit. Another formulation is given in Definition III 3.1. However, the applications, as far as we remember, are for forcing notions which preserve the property "the cofinality of δ is uncountable", and in fact are E-proper, for some E which is a stationary subset of $\mathcal{S}_{\leq\aleph_0}(\cup E)$.

However, in our case we are interested just in forcing notions which do change some cofinality to $\aleph_0$. In these cases, we cannot break the iterated forcing into an initial segment and the rest (i.e., break $\langle P_i, \underset{\sim}{Q}_i : i < \alpha\rangle$ into $\langle P_i, \underset{\sim}{Q}_i : i < \beta\rangle)$ and $\langle P_i/P_\beta, \underset{\sim}{Q}_i : \beta \leq i < \alpha\rangle)$. The reason is that maybe the first forcing changes the cofinality of some $\delta, \beta < \delta < \alpha$ to $\aleph_0$; but then P_δ/P_β is not the inverse limit of $\langle P_i/P_\beta, \underset{\sim}{Q}_i : \beta \leq i < \delta\rangle$, and $\Vdash_{P_\beta}$ "$\langle P_i/P_\beta, \, , \underset{\sim}{Q}_i : \beta \leq i < \alpha\rangle$ is not a CS iteration". In fact, as every $p \in P_\delta$ has domain a bounded subset of δ, if $\Vdash_{P_\beta}$ "$\underset{\sim}{\alpha}_n \in (\beta, \delta), \underset{\sim}{\alpha}_n < \underset{\sim}{\alpha}_{n+1}, \delta = \bigcup\limits_{n<\omega} \underset{\sim}{\alpha}_n$, and $\langle \underset{\sim}{p}_{n,i} : i < \lambda\rangle$ is a sequence of pairwise incompatible conditions in $\underset{\sim}{Q}_{\underset{\sim}{\alpha}_n}$ or just in $P_{\underset{\sim}{\alpha}_{n+1}}/P_{\underset{\sim}{\alpha}_n}$ i.e $P_{\underset{\sim}{\alpha}_n}$-names of members of $P_{\underset{\sim}{\alpha}_{n+1}}/\underset{\sim}{G}_{P_{\underset{\sim}{\alpha}_n}}$" and we let $\underset{\sim}{\tau} : \omega \to \lambda$ be $\underset{\sim}{\tau}(n) = i$ *if* $\underset{\sim}{p}_{n,i}[\underset{\sim}{G}_{P_\delta} \cap P_\beta]$ belongs to $\underset{\sim}{G}_{P_\delta}$ or there is no such i and we let $\underset{\sim}{\tau}(n) = 0$, *then* $\Vdash_{P_\delta}$ "$\underset{\sim}{\tau}$ is a function from ω onto $\lambda + 1$". So if each $\underset{\sim}{Q}_i$ has two incompatible members and δ is divisible by ω^2, then P_δ will collapse $\aleph_1$ and even $(2^{\aleph_0})^{V^{P_\beta}}$ for $\beta < \delta$.

Hence we suggest another iteration, RCS (revised countable support), which seems to be the reasonable solution to this dilemma.

The essence of the solution is that a name of a condition is really a condition. More exactly, in countable support iteration a condition may be $\{(\beta, \underset{\sim}{q})\}$ such that $\underset{\sim}{q}$ is a P_β-name of a member of $\underset{\sim}{Q}_\beta$, so $\underset{\sim}{q}$ is a name but β is a "real" ordinal. But now we allow $\underset{\sim}{\beta}$ to be a name. But a name with respect to which forcing notion? We would like to use P_α-names, but then we get a vicious circle, defining what is a condition of P_α using P_α-names. So we can allow P_γ-names $\underset{\sim}{\beta}$ for some $\gamma < \alpha$, such that $\Vdash_{P_\gamma}$ "$\gamma \leq \underset{\sim}{\beta} < \alpha$", and then allow a P_γ-name of condition as above etc (this is the successor case in clause (B) of Definition 1.2(1), and shall use it freely in later sections). The exact definition appears below; though it has a somewhat cumbersome definition, it seems to conform better to our intuitive idea of iteration. A first version of it can found in [Sh:119]. For other realizations of this (and alternatives to §1 here) see [Sh:250], which is redone here in Chapter XIV. In XIV §1 we deal with κ-RS. There, all the induction on γ disappears as $\kappa > \aleph_1$ makes it unavailable. An alternative way is XIV 2.6=[Sh:250, 2.6] where we simplify matters by demanding, e.g., for $\bar{Q}$-named ordinal $\underset{\sim}{\zeta}$ that: $q \Vdash$"$\underset{\sim}{\zeta} = \xi$" $\Rightarrow q{\restriction}\xi \Vdash$ "$\underset{\sim}{\zeta} = \xi$", the price is the loss of the associativity law (see 1.1A(1)), this makes the treatment later less elegant, but does not cause real damage as far as we know: i.e. we cannot restrict inductive proofs to the cases the length δ of the iteration being 1, 2, ω, ω_1, κ inaccessible, but rather have 1, successor, for some $\alpha < \delta$, $p{\restriction}\alpha \Vdash \mathrm{cf}(\delta) = \aleph_0$ (where we are interested in the forcing above p) etc. As things are, we need to consider in e.g. 1.1(B), not only $r \in P_\xi$ but also $r \in P_{\xi+1}$ except when $\xi + 1 = \alpha$ (to avoid vicious circle), hence we have $\gamma = \beta + 1 < \alpha$ or $\gamma = \beta = \alpha - 1$ there. Compared to the previous (i.e. [Sh:b]) version, for smoothness we essentially complete the Q_i's and we also give (for completeness) the equivalent outside definition of $\bar{Q}$-named ordinals (and conditions (1.3(2))).

1.0 Remark.

(1) If $P_1 = P_0 * \underset{\sim}{Q}_0$, $\underset{\sim}{x}$ a P_1-name, $G_0 \subseteq P_0$ generic, then in $V[G_0]$, $\underset{\sim}{x}$ can be naturally interpreted as a Q_0-name, called $\underset{\sim}{x}/G_0$, which has a P_0-name $\underset{\sim}{x}/\underset{\sim}{G}_0$ or $\underset{\sim}{x}/P_0$; but usually we do not care to make those fine distinctions.

(2) Using $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$, P_α will mean Rlim $\bar{Q}$ (see Definition 1.1).

(3) If D is a filter on a set $J, D \in V$, $V \subseteq V^\dagger$ (e.g., $V^\dagger = V[G]$) then in an abuse of notation, D will denote also the filter it generates (on J) in $V^\dagger$.

(4) Formally, if $\Vdash_{P_0}$ "$\underset{\sim}{Q}_0$ is a forcing notion " then $P_0 * \underset{\sim}{Q}_0$ is a class, but this is for superficial reasons. We can demand that the set of members of $\underset{\sim}{Q}_0$ (in V^{P_0}) is a cardinal, and use only "canonical" P_0-names (as in 1.1 (B)), or restrict ourselves to members of some $H(\chi)$. In the iteration in this section (see 1.1), writing $|P|$, we mean $|P/\approx|$ (see I 5.5). We may use instead $d(P)$, the density character, which is defined as $\mathrm{Min}\{|P'| : P' \subseteq P, \forall p \in P\ \exists p' \in P'[p \leq p']\}$ or the essential density $d'(P) = \mathrm{Min}\{|P'| :$ for some P'', $P \lessdot P''$, P dense in P'' and $P' \subseteq P''$ and $(\forall p \in P)(\exists p' \in P')[p' \Vdash_{P''}$ "$p \in G_{P''}$"$]\}$ (we say P' is essentially dense in P; this means it is dense in the Boolean completion of P). The change does not make much difference.

(5) $\mathcal{D}_\kappa$ is the closed unbounded filter on κ.

(6) For a forcing notion Q, an almost member q of Q is $\{(p_i,\ q_i) : i < i^*\}$ such that $[p_i,\ q_i \in Q]$ &$[p_i, p_j$ compatible $\Rightarrow q_i = q_j]$, *and* for $r \in Q$, $q \leq r$ means $r \Vdash_Q$ "for every $i < i^*$ if $p_i \in \underset{\sim}{G}_Q$ then $q_i \in \underset{\sim}{G}_Q$"; if q', q'' are almost members of Q we define: $q' \leq q''$ iff $(\forall r \in Q)[q'' \leq r \Rightarrow q' \leq r]$. If, as we normally agree, $\emptyset_Q \in Q$ is minimal in Q then we can identify $r \in Q$ and the almost member $\{(\emptyset_Q,\ r)\}$. The set of almost members of Q will be denoted by $\hat{Q}$ (this is in fact just the completion of Q but if p, $q \in Q$ are equivalent (i.e. $\Vdash$ "$p \in \underset{\sim}{G}_Q \leftrightarrow q \in \underset{\sim}{G}_Q$" then in $\hat{Q}$, $p \leq q \leq p$ so they can be identified).

(7) Note that an almost member of $\hat{Q}$ is equivalent to a member of $\hat{Q}$, but is not a real almost member, but we usually ignore the distinction.

(8) See more on why the iteration is good in XI §1.

1.1 Definition. We define and prove the following (A), (B), (C), (D), Def. 1.2 and claims 1.3(1), 1.4, by *simultaneous induction on* α (also for generic extensions of V):

(A) $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ is an RCS iteration (RCS stands for revised countable support).

(B) a $\bar{Q}$-named ordinal (or $[j, \alpha)$-ordinal), (above a condition r).

(C) a $\bar{Q}$-named condition (or $[j, \alpha)$-condition), and we define $\underset{\sim}{q} \restriction \xi$, $\underset{\sim}{q} \restriction \{\xi\}$ for a $\bar{Q}$-named $[j, \alpha)$-condition $\underset{\sim}{q}$ and ordinal ξ and they are a member of P_ξ and a P_ξ-name of a member of $\hat{Q}_\xi$ respectively; of course $\xi \in [j, \alpha]$ (and $\xi \in [j, \alpha)$ respectively).

(D) the RCS-limit of $\bar{Q}$, $\operatorname{Rlim} \bar{Q}$ which satisfies $P_i \lessdot \operatorname{Rlim} \bar{Q}$ for every $i < \alpha$ and $p \restriction \xi$, $p \restriction \{\xi\}$ for $\xi < \alpha$, $p \in \operatorname{Rlim} \bar{Q}$.

(A) We define "$\bar{Q}$ is an RCS iteration"

$\alpha = 0$: no condition.

α is limit: $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ is an RCS iteration iff for every $\beta < \alpha$, $\bar{Q} \restriction \beta$ is one.

$\alpha = \beta + 1$: $\bar{Q}$ is an RCS iteration iff $\bar{Q} \restriction \beta$ is one, $P_\beta = \operatorname{Rlim}(\bar{Q} \restriction \beta)$ and $\underset{\sim}{Q}_\beta$ is a P_β-name of a forcing notion.

(B) We define "$\underset{\sim}{\zeta}$ is a $\bar{Q}$-named $[j, \alpha)$-ordinal of depth Υ above r" by induction on the ordinal Υ (and $\alpha = \ell g \bar{Q}$).

The intended meaning is an $(\operatorname{Rlim} \bar{Q})$-name of an ordinal of a special kind, however $\operatorname{Rlim} \bar{Q}$ is still not defined. So we use the part already known.

For $\Upsilon = 0$: "$\underset{\sim}{\zeta}$ is a $\bar{Q}$-named $[j, \alpha)$-ordinal of depth Υ above r" means $\underset{\sim}{\zeta}$ is a (plain) ordinal in $[j, \alpha)$, i.e., $j \le \underset{\sim}{\zeta} < \alpha, r \in P_{\zeta+1}$; but if $\underset{\sim}{\zeta} + 1 = \alpha$ then $r \in P_\zeta$.

For $\Upsilon > 0$: "$\underset{\sim}{\zeta}$ is a $\bar{Q}$-named $[j, \alpha)$-ordinal of depth Υ above r" means that for some $\beta < \alpha$, (letting $\gamma = \beta + 1$ if $\beta + 1 < \alpha$ and $\gamma = \beta$ otherwise) $r \in P_\gamma$, and for some antichain $\mathcal{I}$ of P_γ, pre-dense above r, $\mathcal{I} = \{p_i : i < i_0\} \subseteq P_\gamma$, $\{\Upsilon_i : i < i_0\}$ and $\{\underset{\sim}{\zeta}_i : i < i_0\}$, we have $P_\gamma \vDash$ "$(r \restriction \gamma) \le p_i$" (for simplicity), $\underset{\sim}{\zeta}_i$ is a $\bar{Q}$-named $[\max\{j, \beta\}, \alpha)$-ordinal of depth Υ_i above $p_i, \Upsilon_i < \Upsilon$, and $\underset{\sim}{\zeta}$ is $\underset{\sim}{\zeta}_i$ if

p_i and r (i.e., if p_i, r will be in the generic set then $\underset{\sim}{\zeta}$ will be $\underset{\sim}{\zeta}_i$; this is informal but clear, see formal version in 1.2(1)).

Without Υ : We say $\underset{\sim}{\zeta}$ is a $\bar{Q}$-named $[j, \alpha)$-ordinal above r, if it is such for some depth.

Without $r : r = \emptyset$.

Similarly, we omit "$[j, \alpha) -$ " when $j = 0$.

(C) We define "$\underset{\sim}{q}$ is a $\bar{Q}$-named $[j, \alpha)$-condition of depth Υ above r" and also $\underset{\sim}{q}\restriction\{\xi\}, \underset{\sim}{q}\restriction\xi$ and the $\bar{Q}$-named $[j, \alpha)$-ordinal $\zeta(\underset{\sim}{q})$ associated with $\underset{\sim}{q}$.

The definition is similar to (B).

For $\Upsilon = 0$: We say "$\underset{\sim}{q}$ is a $\bar{Q}$-named $[j, \alpha)$-condition of depth Υ above r" if for some ordinal $\zeta, j \leq \zeta < \alpha$ and $\underset{\sim}{q}$ is a P_ζ-name of a member of $\underset{\sim}{\hat{Q}}_\zeta$ (see 1.0(6)), $r \in P_{\zeta+1}$ but if $\zeta + 1 = \alpha$ then $r \in P_\zeta$ and for simplicity $\underset{\sim}{q}$ is above $r\restriction\{\zeta\}$ i.e. if $\zeta + 1 < \alpha$ then $r\restriction\zeta \Vdash_{P_\zeta}$ " in $\underset{\sim}{\hat{Q}}_\zeta$, $r\restriction\{\zeta\} \leq \underset{\sim}{q}$" (note: $r\restriction\zeta \in P_\zeta$, $r\restriction\{\zeta\}$ is a member of $\hat{Q}_\zeta$). We let

$$\underset{\sim}{q}\restriction\xi = \begin{cases} \underset{\sim}{q} & \text{if } \xi > \zeta + 1 \\ \underset{\sim}{q} & \text{if } \xi = \zeta + 1 \\ \emptyset_{P_\xi} & \text{if } \xi \leq \zeta \end{cases}$$

$$\underset{\sim}{q}\restriction\{\xi\} = \begin{cases} \underset{\sim}{q} & \text{if } \xi = \zeta, \\ \emptyset_{\underset{\sim}{Q}_\xi} & \text{if } \xi \neq \zeta. \end{cases}$$

notes: $\emptyset \in P_0$ and remember 1.0(7). Finally we let $\zeta(\underset{\sim}{q}) = \zeta$. [What if we wave "$\underset{\sim}{q}$ above $r\restriction\{\zeta\}$"? Then $\xi = \zeta + 1$ need special attention as in $\bar{Q}\restriction\xi$, r may not be in P_ζ so we have to transfer the information of $\underset{\sim}{q}$ to "allowable" form, so $\underset{\sim}{q}\restriction\xi$ depend also on r; so $\underset{\sim}{q}$ should also tell us who is r or require $r\restriction\zeta \Vdash [\hat{Q}_\zeta \models r\restriction\{\zeta\} \leq \underset{\sim}{q}$" or we should write $\underset{\sim}{q}\restriction_r\xi$, $\underset{\sim}{q}\restriction_r\{\xi\}$.]

For $\Upsilon > 0$: We say $\underset{\sim}{q}$ is a $\bar{Q}$-named $[j, \alpha)$-condition of depth Υ above r, if for some $\beta < \alpha$ (letting $\gamma = \beta + 1$ if $\beta + 1 < \alpha$ and $\gamma = \beta$ otherwise) for some $\bar{Q}$-named $[j, \alpha)$-ordinal of depth Υ above $r, \underset{\sim}{\zeta}$, defined by β, γ, $\{p_i : i < i_0\} \subseteq P_\gamma, \{\Upsilon_i : i < i_0\}$, $\{\underset{\sim}{\zeta}_i : i < i_0\}$, we have for each $i < i_0$ a $\bar{Q}$-named $[\max\{\beta, j\}, \alpha)$-condition $\underset{\sim}{q}_i$ of depth Υ_i above $r \bigcup p_i$ (see clause (c) in (D) below), so informally $\zeta(\underset{\sim}{q}_i) = \underset{\sim}{\zeta}_i$, and $\underset{\sim}{q}$ is $\underset{\sim}{q}_i$ if p_i and r are in the generic set of P_γ).

We then let $\zeta(\underset{\sim}{q}) = \underset{\sim}{\zeta}$.

Now we define $\underset{\sim}{q}\restriction\xi$ and $\underset{\sim}{q}\restriction\{\xi\}$; [really, we can just replace $\underset{\sim}{q}_i$ by $\underset{\sim}{q}_i\restriction\xi$, $\underset{\sim}{q}_i\restriction\{\xi\}$ respectively. In order to be pedantic, we need the following]. We define $\underset{\sim}{q}\restriction\xi$ as follows (below we ask $r \in \bigcup_{\varepsilon<\xi} P_{\varepsilon+1}$, because if ξ is a successor, $r \in P_\xi$ is a reasonable situation, if ξ a limit ordinal - not). *If* $r \in \bigcup_{\varepsilon<\xi} P_{\varepsilon+1}$ and $\beta+1 < \xi$, then $\underset{\sim}{q}\restriction\xi$ is defined like $\underset{\sim}{q}$ replacing $\underset{\sim}{q}_i$ by $\underset{\sim}{q}_i\restriction\xi$. If $r \in \bigcup_{\varepsilon<\xi} P_{\varepsilon+1}$, $\beta+1 = \xi = \alpha$, then $\underset{\sim}{q}\restriction\xi$ is $\underset{\sim}{q}$. If $r \in \bigcup_{\varepsilon<\xi} P_{\varepsilon+1}$, $\beta+1 = \xi < \alpha$ then $\underset{\sim}{q}\restriction\xi$ is the following P_β-name of a member of $\hat{\underset{\sim}{Q}}_\beta$:

if $r\restriction\beta \in \underset{\sim}{G}_{P_\beta}$ then $\underset{\sim}{q}\restriction\xi$ is $\{(p_i\restriction\{\beta\}, \underset{\sim}{q}_i) : p_i\restriction\beta \in \underset{\sim}{G}_{P_\beta}, i < i_0\} \in \hat{\underset{\sim}{Q}}$.

If $r \in \bigcup_{\varepsilon<\xi} P_{\varepsilon+1}$, $\beta+1 > \xi$ or $r \notin \bigcup_{\varepsilon<\xi} P_{\varepsilon+1}$ *then*: $\underset{\sim}{q}\restriction\xi$ is $\emptyset$ (or not defined).

Similarly for $\underset{\sim}{q}\restriction\{\xi\}$. If $r \in P_{\xi+1}$ (or $r \in P_\xi$), $\gamma \le \xi$ then $\underset{\sim}{q}\restriction\{\xi\}$ is defined like $\underset{\sim}{q}$ replacing $\underset{\sim}{q}_i$ by $\underset{\sim}{q}_i\restriction\{\xi\}$. If $r \in P_{\xi+1}$, $\beta < \gamma = \xi+1$ (hence $\beta = \xi < \alpha$) then $\underset{\sim}{q}\restriction\{\xi\}$ is the following P_β-name of a member of $\hat{\underset{\sim}{Q}}_\beta$: $\{(r\restriction\{\beta\} \cup \underset{\sim}{p}_i\restriction\{\beta\}, \underset{\sim}{q}_i\restriction\{\beta\}) : p_i\restriction\beta \in \underset{\sim}{G}_{P_\beta}$ and $r\restriction\beta \in P_\beta$ and $i < i_0\}$. If $r \in P_{\xi+1}$, $\beta = \gamma = \xi+1$ (actually is ruled out) or $\gamma > \xi+1$ then $\underset{\sim}{q}\restriction\{\xi\}$ is $\emptyset$. If $r \notin P_{\xi+1}$, then $\underset{\sim}{q}\restriction\{\xi\}$ is $\emptyset$ (or not defined).

[The definitions of $\underset{\sim}{\zeta}(q\restriction\xi), \underset{\sim}{\zeta}(q\restriction\{\xi\})$ are left to the reader].

We omit Υ and/or "$[j,\alpha)-$" if this holds for some ordinal Υ and/or $j = 0$. We omit r when $r = \emptyset (= \emptyset_{P_0})$. We leave the definition of $\underset{\sim}{q}\restriction[\zeta,\xi)$ to the reader.

(D) We define Rlim$\bar{Q}$ as follows:

if $\alpha = 0$: Rlim$\bar{Q}$ is trivial forcing with just one condition: $\emptyset = \emptyset_{P_0}$;

if $\alpha > 0$: we call $\underset{\sim}{q}$ an atomic condition of *R*lim$\bar{Q}$, if it is a $\bar{Q}$-named condition.

The set of conditions in $P_\alpha = $ Rlim$\bar{Q}$ is

$\{p : p$ a countable set of atomic conditions; and for every $\beta < \alpha$, $p\restriction\beta \overset{\text{def}}{=} \{r\restriction\beta : r \in p\} \in P_\beta$, and $p\restriction\beta \Vdash_{P_\beta}$ "$p\restriction\{\beta\} \overset{\text{def}}{=} \{r\restriction\{\beta\} : r \in p\}$ has an upper bound in $\hat{\underset{\sim}{Q}}_\beta$"$\}$.

The order is inclusion, (but in later sections we sometimes ignore the difference between $p \le q$ and $p \Vdash$ "$q \in \underset{\sim}{G}$")

Now we have to show:

(a) $P_\beta \lessdot \mathrm{Rlim}\bar{Q}$ (for $\beta < \alpha$). [By 1.4(1) below.]

(b) For $\beta < \alpha$, any $(\bar{Q}\restriction\beta)$-named $[j,\beta)$-ordinal (or condition) above r is a $\bar{Q}$-named $[j,\alpha)$-ordinal (or condition) above r. [Why? Obvious.]

(c) If $\xi < \alpha$, $\underset{\sim}{q}$ is a $\bar{Q}$-named (atomic) condition above r, $r \in \bigcup_{\varepsilon<\xi} P_\varepsilon$, then $\underset{\sim}{q}\restriction\xi$ is a $(\bar{Q}\restriction\xi)$-named (atomic) condition above r. [Why? Obvious.]

(d) If $\beta_1 < \beta_2 < \alpha$, $p \in P_{\beta_2} \setminus P_{\beta_1}$, $p \le q$ in P_{β_2} then $q \notin P_{\beta_1}$ (though it may be equivalent to one).

(e) If $\xi < \alpha$, $\underset{\sim}{q}$ a $\bar{Q}$-named atomic condition above r, $r \in \bigcup\limits_{\varepsilon<\xi} P_\xi$ *then* $\Vdash_P$ "$\underset{\sim}{q}\restriction\{\xi\}$ is a member of $\underset{\sim}{\hat{Q}}_\xi$ ".

1.1A Explanation. 1)What will occur if we simplify by letting in 1.1(B), for $\Upsilon > 0$, $\gamma = \beta$ always? Nothing happens, except that 1.5(3) is no longer true; though this is used later, we can manage without it too, though less esthetically; for variety, XIV 2.6 = [Sh:250, 2.6] is developed in this way (for a generalization called κ-RS, our case is $\kappa = \aleph_1$). For the case which interests us the two definitions are equivalent - by the proof of 2.6 (here).

2) So why in 1.1(B), for $\Upsilon > 0$, we do not let $\gamma = \beta + 1$ always? If $\beta + 1 = \alpha$, we fall into a vicious circle; defining $P_{\beta+1}$ using conditions in $P_{\beta+1}$; alternatively see XIV §1.

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1.1B Remark. We can obviously define $\bar{Q}$-named sets; but for conditions (and ordinals for them) we want to avoid the vicious circle of using names which are interpreted only after forcing with them below.

1.2 Definition.

(1) Suppose $\underset{\sim}{\zeta}$ is a $\bar{Q}$-named $[j,\alpha)$-ordinal above r, $r \in G \subseteq \bigcup_{i<\alpha} P_i$ and $G \cap P_i$ generic over V (whenever $i < \alpha$) (say G is in some generic extension of V).

We define $\underset{\sim}{\zeta}[G]$ by induction on the depth: if the depth of $\underset{\sim}{\zeta}$ is 0, it is $\underset{\sim}{\zeta}$, if the depth of $\underset{\sim}{\zeta}$ is > 0, and it is defined by β, γ, $\{p_i : i < i_0\}$, $\{\underset{\sim}{\zeta}_i : i < i_0\}$, $\{\boldsymbol{\Upsilon}_i : i < i_0\}$ as in Definition 1.1(B) *then* for a unique $i < i_0, p_i \in G$ and we let $\underset{\sim}{\zeta}[G] = \underset{\sim}{\zeta}_i[G]$ (remember $\boldsymbol{\Upsilon}_i < \boldsymbol{\Upsilon}$). (If there is no such i, it is not defined but as we demand $\{p_i : i < i_0\}$ is a predense above $r\restriction\gamma$ in P_γ above r and $\gamma < \alpha$ and $r \in G$, it will be defined).

If $r \notin G$ then $\underset{\sim}{\zeta}[G]$ is undefined, or we can give it a default value, like ∞.

For a $\bar{Q}$-named $[j, \alpha)$-condition $\underset{\sim}{q}$ above r, we define $\underset{\sim}{q}[G]$ similarly (with default value $\emptyset$).

(2) For $\underset{\sim}{\zeta}$ a $\bar{Q}$-named $[j, \alpha)$-ordinal above r, and $q \in \bigcup_{i<\alpha} P_i$ let $q \Vdash_{\bar{Q}}$ "$\underset{\sim}{\zeta} = \xi$" if for every $G \subseteq \bigcup_{i<\alpha} P_i$, such that each $G \cap P_i$ $(i < \alpha)$ is generic over V, $q \in G \Rightarrow \underset{\sim}{\zeta}[G] = \xi$, (similarly $q \Vdash_{\bar{Q}}$ "$\underset{\sim}{\zeta}$ undefined".)

1.3 Claim.

(1) Suppose $\underset{\sim}{\zeta}$ is a $\bar{Q}$-named ordinal [above r], ($\bar{Q}$ an RCS iteration, $\alpha = \ell g(\bar{Q})$). If $G \subseteq \bigcup_{i<\alpha} P_i$ [and $r \in G$] and each $G \cap P_i$ (where $i < \alpha$) is a generic subset of P_i over V, *then* for some ξ, $\underset{\sim}{\zeta}[G] = \xi, j \leq \xi < \alpha$. Moreover for some $q \in P_{\xi+1} \cap G$ we have $q \Vdash_{\bar{Q}}$ "$\underset{\sim}{\zeta} = \xi$" and $[\xi + 1 = \alpha \Rightarrow q \in P_\xi]$.

(2) Suppose $\bar{Q}$ is an RCS iteration of length α, $j < \alpha$, $\varphi(x,\ y)$ a definition with parameters in V and $r \in \bigcup_{i<\alpha} P_i$ such that:

(i) *If* G^* is generic over V for some forcing notion, in $V[G^*]$ we have $G \subseteq \bigcup_{i<\alpha} P_i$ is directed, for each $i < \alpha$ the set $G \cap P_i$ is generic over V and $r \in G$ *then* $V[G] \models (\exists! x)\varphi(x,\ G)$ and we call this unique x, $x_\varphi[G]$.

Suppose further that for such G^*, G we have $x_\varphi[G]$ is an ordinal $\zeta_\varphi = \zeta_\varphi[G] \in [j, \alpha)$ (or it is a pair $(\zeta_x, q_x) = (\zeta_\varphi[G], q_\varphi[G])$, with $\zeta_\varphi[G]$ an ordinal $\in [j, \alpha)$, $q_\varphi[G] \in \underset{\sim}{\hat{Q}}_{\zeta_\varphi[G]}[G \cap P_{\zeta_\varphi[G]}]$) and $r \in P_{\zeta_\varphi[G]+1}$.

(ii) If G^*, $G, x = x_\varphi[G]$ are as in (i), *then* for some $q \in G \cap P_{\zeta_{\varphi[G]}+1} \cap (\bigcup_{i<\alpha} P_i)$ we have:

$(*)^q_x$ *if* G^{**}, $G' \in V[G^{**}]$ satisfy the requirements on G^*, G and $q \in G'$ *then* $x_\varphi[G'] = x (= x_\varphi[G])$; note $\zeta_x = \alpha - 1 \Rightarrow q \in P_{\alpha-1}$ follows,

(iii) if $\delta < \alpha$ is limit, $r \in P_\beta$, $\beta < \delta$, and G^* generic over V and $G \in V[G^*]$ and $r \in G \subseteq \bigcup_{\varepsilon<\delta} P_\varepsilon$ and $G \cap P_\varepsilon$ generic over V for $\varepsilon < \delta$, *then*

<u>either</u> for some $q \in G$, and x, $(*)^q_x$ above holds, (so $\zeta_x < \delta$)

<u>or</u> for some $\beta_1 \in (\beta, \delta)$ and r^*, $r \leq r^* \in P_{\beta_1} \cap G$ we have:

for any $\beta' \in (\beta_1, \delta)$ for any r', x':

$$r^* \leq r' \in P_{\beta'} \& (*)^{r'}_{x'} \Rightarrow \zeta_{x'} \geq \delta$$

Then there is a $\bar{Q}$-named $[j,\ \alpha)$-ordinal above r, $\underset{\sim}{\zeta}$ [or $\bar{Q}$-named $[j, \alpha)$-condition $\underset{\sim}{q}$] such that:

If G^* is generic over V for some forcing notion, in $V[G^*]$, $G \subseteq \bigcup_{i<\alpha} P_i$ directed, for each $i < \alpha$ the set $G \cap P_i$ is generic over V and $r \in G$ *then* $x_\varphi[G] = \underset{\sim}{\zeta}[G]$ [or $x_\varphi[G] = \underset{\sim}{q}[G]$ (i.e. equivalent members of $\hat{Q}_{\underset{\sim}{\zeta}[\underset{\sim}{q}][G]}[G \cap P_{\underset{\sim}{\zeta}[\underset{\sim}{q}][G]})$].

1.3A Remark. 1) Concerning 1.3(2), of course every $\bar{Q}$-named ordinal (or condition) [above r] satisfies these conditions.

Proof. (1) The proof is by induction on the depth of $\underset{\sim}{\zeta}$.

(2) The proof is straightforward. For notational simplicity we deal with the case of $\bar{Q}$-named $[j, \alpha)$-ordinals only; but for easing the induction we define in Definition 1.1 clause (B) also "extended $\bar{Q}$-named ordinals" by just allowing ζ also values $\geq \alpha$ (but still $j < \alpha$ and now in $(*)^q_x$ we have $\zeta_x \geq \alpha - 1 \Rightarrow q \in P_{\alpha-1}$ (and we stipulate for α not successor, $\alpha - 1 = \alpha$)), and so similarly in 1.3(2)(i); clearly it suffices to prove 1.3(2) for this extension. Let β^* be minimal such that $r \in P_{\beta^*}$; we know $\beta^* < \alpha$. Let $\mathcal{I}$ be the set of $r^* \in \bigcup_{i<\alpha} P_i$ such that:

$(*)[r^*]$ for some β, γ we have: $r \leq r^* \in P_\gamma, j \leq \beta < \alpha$, $\beta \leq \gamma < \alpha, \gamma \leq \beta + 1$ and there is an extended $\bar{Q}$-named $[\beta, \infty)$-ordinal $\underset{\sim}{\zeta}$ such that:

if G^* is generic over V for some forcing notion, $G \in V[G^*]$, $G \subseteq \bigcup_{i<\alpha} P_i$, $G \cap P_i$ is generic over V for $i < \alpha$ and $r, r^* \in G$
then $x_\varphi[G] = \underset{\sim}{\zeta}[G]$.

Let $\mathcal{J} = \{p \in \mathcal{I} :$ for some $\gamma < \alpha$ we have $p \in P_\gamma \setminus \bigcup_{\varepsilon<\gamma} P_\varepsilon$ and for no γ', $j \leq \gamma' < \gamma$ is there $p' \in \mathcal{I} \cap P_{\gamma'}$, p' compatible with p (say, in P_γ)$\}$. It is enough to prove $r \in \mathcal{I}$, so assume that this fails. Choose χ large enough such that $\bar{Q} \in H(\chi)$, G^* be such that in $V[G^*]$ the cardinal 2^χ becomes a countable ordinal.

Now

$(*)_0$ If β, γ, r^* are as in $(*)[r^*]$ and $r^* \leq r^{**} \in P_\gamma$ then $r^{**} \in \mathcal{I}$ [this is trivial].

$(*)_1$ If $r \leq r^* \in P_\beta$, $\beta^* \leq \beta < \alpha$, $\mathcal{I} \cap P_\beta \setminus \bigcup_{\gamma<\beta} P_\gamma$ is pre-dense above r^* in P_β then $r^* \in \mathcal{I}$.

[Why? Straightforward by the inductive step in (B) of Definition 1.1].

For $\beta' < \alpha$, $r \in G \subseteq P_{\beta'}$, G generic over V, we define $\mathcal{I}^{[G]} = \{p \in \bigcup_{i<\alpha} P_i : p \in \bigcup_{\beta' \leq \varepsilon < \alpha} P_\varepsilon / G$ and for some $r' \in G$ we have $p \cup r' \in \mathcal{I}\}$.

$(*)_2$ Assume $r \in G \subseteq P_{\beta'}$, G is generic over V, $p \in \bigcup_{\beta' \leq \varepsilon < \alpha} P_\varepsilon / G$ and for some extended $\bar{Q}$-named $[j, \infty)$-ordinal $\underset{\sim}{\zeta}'$ above p we have: $G \subseteq G' \subseteq \bigcup_{\varepsilon<\alpha} P_\varepsilon \,\&\, p \in G' \,\&$ [for $\varepsilon < \alpha$, $G' \cap P_\varepsilon$ is generic over V] $\Rightarrow x_\varphi[G'] = \underset{\sim}{\zeta}'[G']$. *Then* $p \in \mathcal{I}^{[G]}$. [Why? Check, using the successor case in clause (B) of Definition 1.1.]

We shall prove by induction on $\beta \in [\beta^*, \bigcup_{\varepsilon<\alpha} \varepsilon]$ that

$\otimes$ if $\beta^* \leq \beta(0) < \beta$, $G_{\beta(0)} \subseteq P_{\beta(0)}$ is generic over V, $r \in G_{\beta(0)}$, $G_{\beta(0)} \cap \mathcal{I} = \emptyset$ *then* there is G_β such that [$\beta < \alpha \Rightarrow G_\beta \subseteq P_\beta$ is generic over V], [$\beta = \alpha \Rightarrow G_\beta \subseteq \bigcup_{i<\alpha} P_i \& \bigwedge_{i<\alpha} G_\beta \cap P_i$ is generic over V], $G_{\beta(0)} \subseteq G_\beta$ and $G_\beta \cap \mathcal{I} = \emptyset$.

It suffice to prove $\otimes$, as from $\otimes$ for $\beta = \bigcup_{\varepsilon<\alpha} \varepsilon$ we get 1.3(2); why? there is $G_{\beta^*} \subseteq P_{\beta^*}$ generic over V, such that $r \in G_{\beta^*}$ and $\mathcal{I} \cap G_{\beta^*} = \emptyset$ (otherwise by $(*)_1$ applied to $r^* = r$, $\beta^* = \beta$ we get $r \in \mathcal{I}$). Now use $\otimes$ with $\beta(0) = \beta^*$, $\beta = \bigcup_{\varepsilon<\alpha} \varepsilon$, and $G_{\beta(0)} = G_{\beta^*}$ and get G_β; contradiction to the assumption (ii) of 1.3(2), thus finishing the proof of 1.3(2).

Note that as $G_{\beta(0)} \cap \mathcal{I} = \emptyset$ also $G \cap \mathcal{I}^{[G]} = \emptyset$.

First case: $\underline{\beta = \beta^*}$. Empty.

Second case: $\underline{\beta = \beta_1 + 1 > \beta^*}$. So by the induction hypothesis without loss of generality $\beta(0) = \beta_1$. Clearly, $\beta < \alpha$ (otherwise we are done). As $G_{\beta(0)} \subseteq P_{\beta(0)}$ is generic over V (and $r \in G_{\beta(0)}$), there is $r^* \in G_{\beta(0)}$ such that $r \le r^*$ and $r^* \Vdash$ "$\mathcal{I} \cap \underset{\sim}{G}_{\beta(0)} = \emptyset$". So there is no r', $r^* \le r' \in P_{\beta(0)} \cap \mathcal{I}$. Is there $r' \in \underset{\sim}{Q}_{\beta(0)}[G_{\beta(0)}]$ incompatible with every $\{p{\restriction}\{\beta_1\} : p \in P_{\beta(0)+1} \cap \mathcal{I},\, p{\restriction}\beta_1 \in G_{(\beta(0))}\}$? (Note $(*)_0$ and remember $\beta_1 = \beta(0)$.) If so, no problem to find G_β as required; otherwise, without loss of generality, r^* forces this and by $(*)_1$, $r^* \in \mathcal{I}$, contradiction.

Third case: $\underline{\beta = \alpha \text{ is limit}}$. Without loss of generality in $V[G^*]$, P_α (and α) are countable. Let in $V[G^*]$, $\langle \beta_n : n < \omega \rangle$ be increasing with limit β, $\beta_0 = \beta(0)$. We define by induction on $m < \omega$, $G_{\beta_m} \subseteq P_{\beta_m}$ generic over V, increasing in n such that: $G_{\beta_m} \cap \mathcal{I} = \emptyset$. Let $n(0) = 0$, $G_{\beta_0} = G_{\beta(0)}$. For $m+1$, use the induction hypothesis. Now $\bigcup_{m<\omega} G_{\beta_m}$ is as required.

Fourth case: $\underline{\beta = \delta < \alpha \text{ is limit}}$. Let $\alpha^* > \alpha$ be an ordinal never of the form $\zeta_\varphi[G]$. We shall define $\varphi'(x,y)$ such that for $\bar{Q}' = \bar{Q}{\restriction}\delta$, $r' = r, j' = j$ the assumption of 1.3(2) holds: if $r \in G \subseteq \bigcup_{\varepsilon<\delta} P_\varepsilon$ and $G \cap P_\varepsilon$ is generic over V for $\varepsilon < \delta$ *then:*

(a) if for some $q \in G$, $q \ge r$ and x, the statement $(*)^q_x$ holds then $x_\varphi[G] = x$.

(b) otherwise, $x_\varphi[G] = \alpha^*$.

Now to see that assumption (i) of 1.3(2) holds we use assumption (iii) of 1.3(2) and also the other assumption holds. So by the induction hypothesis on α, an extended $\bar{Q}'$-named $[j, \infty)$-ordinal $\underset{\sim}{\zeta}'$ exists, say of depth Υ. Looking at 1.1(B) there is a set T of strictly decreasing finite sequences of ordinals closed under initial segments and $\langle \underset{\sim}{\zeta}_\eta, \Upsilon_\eta, p_\eta, \beta_\eta : \eta \in T \rangle$, where

(α) $\underset{\sim}{\zeta}_{\langle\rangle} = \underset{\sim}{\zeta}'$, $p_{\langle\rangle} = r$, $\Upsilon_{\langle\rangle}$ the depth of $\underset{\sim}{\zeta}_{\langle\rangle}$, $r \in P_{(\beta_{\langle\rangle}+1)}$

(β) if η is maximal in T then $\Upsilon_\eta = 0$, $\beta_\eta < \delta$, $\underset{\sim}{\zeta}_\eta$ an ordinal $\underset{\sim}{\zeta}_\eta \ge \beta_\eta$, $p_\eta \in P_{(\beta_\eta+1)}$

(γ) if $\eta \in T$ is not maximal in T then $\nu \in \mathrm{Suc}_T(\eta) \Rightarrow p_\eta \le p_\nu \in P_{(\beta_\eta+1)}$ & $\beta_\eta \le$

β_ν, $\langle p_\nu : \nu \in \mathrm{Suc}_T(\eta)\rangle$ is a maximal antichain in $P_{(\beta_\eta+1)}$ above p_η, $\underset{\sim}{\zeta}_\eta$ is the following extended $\bar{Q}$-named $[\beta_\eta, \infty)$-ordinal above p_η: if p_ν then it is $\underset{\sim}{\zeta}_\nu$.

Suppose first: $[\eta$ maximal in T & $p_\eta \restriction \beta(0) \in G_{\beta(0)}$ & $\zeta_\eta = \alpha^* \Rightarrow p_\eta \in \mathcal{I}^{[G_{\beta(0)}]}]$. Let $T' = \{\eta \in T : p_\eta \restriction \beta(0) \in G_{\beta(0)}\}$; we define $\underset{\sim}{\zeta}$. Just for every maximal $\eta \in T'$ such that $\zeta_\eta = \alpha^*$, "plant" a witness to $p_\eta \in \mathcal{I}^{[G_{\beta(0)}]}$. In details, we prove that for every $\eta \in T$, there is a $\bar{Q}$-named $[j, \alpha)$-ordinal $\underset{\sim}{\zeta}^*_\eta$ above p_η such that: if $G_{\beta(0)} \subseteq G' \subseteq \bigcup_{\varepsilon<\alpha} P_\varepsilon$ & $p_\eta \in G'$ & $(\forall \varepsilon < \alpha)$ ($P_\varepsilon \cap G'$ is generic over V) then $x_\varphi[G'] = \underset{\sim}{\zeta}^*_\eta[G']$. This is shown by $\triangleleft$-downward induction on $\eta \in T$. In the case η is maximal in T, then: if $p_\eta \restriction \beta(0) \notin G_{\beta(0)}$ the demand is quite vacuous, if $\zeta_\eta \neq \alpha^*$ we can use a $\bar{Q}$-name of depth 0 and in the remaining case we know that $p_\eta \in \mathcal{I}^{[G_{\beta(0)}]}$ and this give the required conclusion. The remaining (=second) case is $\eta \in T$ not $\triangleleft$-maximal, and so use the induction hypothesis (and as in $(*)_1$, the successor case of clause (B), Definition 1.1). So we have gotten a name of the right kind in $V[G_{\beta(0)}]$, so by $(*)_2$ we get a contradiction. So for some maximal $\eta \in T$, $p_\eta \restriction \beta(0) \in G_{\beta(0)}, \zeta_\eta = \alpha^*$ and $p_\eta \notin \mathcal{I}^{[G_{\beta(0)}]}$. If for any such η, $\{q \in P_\delta : p_\eta \leq q \in \mathcal{I}\}$ is pre-dense in $P_\delta / G_{\beta(0)}$ above p_η, we again can get a witness to $p_\eta \in \mathcal{I}^{[G_{\beta(0)}]}$ (reread clause (iii) of 1.3(2)), again contradiction. So some $q^* \in P_\delta$ is $\geq p_\eta$ and is incompatible with any $q \in \mathcal{I} \cap P_\delta$ in $P_\delta / G_{\beta(0)}$. Any $G_\delta \subseteq P_\delta$ generic over V which include $G_{\beta(0)} \cup \{q^*\}$ is as required. $\square_{1.3}$

1.4 Claim. Let $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha\rangle$ be an RCS iteration, $P_\alpha = \mathrm{Rlim}\bar{Q}$.

(1) If $\beta < \alpha$, then not only $P_\beta \lessdot P_\alpha$, but if $q \in P_\beta$, $p \in P_\alpha$, then q, p are compatible in P_α iff $q, p \restriction \beta$ are compatible in P_β. Moreover if $q \in P_\beta, p \in P_\alpha, P_\beta \models$ "$p \restriction \beta \leq q$" then $p \cup q$ is a common upper bound of p, q in P_α (even a lub, and in particular $P_\beta \vDash$ "$q \restriction \alpha \leq q$").

(2) If $\underset{\sim}{\beta}, \underset{\sim}{\gamma}$ are $\bar{Q}$-named $[j, \ell g(\bar{Q}))$-ordinals, then $\mathrm{Max}\{\underset{\sim}{\beta}, \underset{\sim}{\gamma}\}$ (defined naturally) is a $\bar{Q}$-named $[j, \ell g(\bar{Q}))$-ordinal.

(3) If $\alpha = \beta_0 + 1$, in Definition 1.1, part (D), in defining the set of elements of P_α we can restrict ourselves to $\beta = \beta_0$. Also in such a case, $P_\alpha = P_{\beta_0} * \underset{\sim}{Q}_{\beta_0}$ (essentially). More exactly, $\{p \bigcup \{\underset{\sim}{q}\} : p \in P_{\beta_0}$, $\underset{\sim}{q}$ a P_{β_0}-name of a member

of $\underset{\sim}{Q}_{\beta_0}\}$ is a dense subset of P_α, and the order $p_1 \bigcup\{\underset{\sim}{q}_1\} \leq_1 p_2 \bigcup\{\underset{\sim}{q}_2\}$ iff $[p_1 \leq p_2$ (in P_{β_0}) and $p_2 \Vdash_{P_{p0}}$ "$\underset{\sim}{q}_1 \leq \underset{\sim}{q}_1$ in $\underset{\sim}{Q}_{\beta_0}$"$]$ is equivalent to that of P_α, i.e., we get the same completion to a Boolean Algebra.

(4) The following set is dense in P_α : $P'_\alpha \overset{\text{def}}{=} \{p \in P_\alpha$: for every $\beta < \alpha$, if $r_1, r_2 \in p$, then $\Vdash_{P_\beta}$ "if $r_1 \restriction \{\beta\} \neq \emptyset$, $r_2 \restriction \{\beta\} \neq \emptyset$ then they are equal"$\}$.

(5) $|P_\alpha| \leq (\sum_{i<\alpha} 2^{|P_i|})^{|\alpha|}$, for limit α (i.e. we count conditions only up to equivalence).

(6) If $\Vdash_{P_i}$ "$|\underset{\sim}{Q}_i| \leq \kappa$", κ a cardinal, then $|P_{i+1}| \leq 2^{|P_i|} + \kappa$ (i.e. identifying equivalent names).

(7) If $\Vdash_{P_i}$ "$d(Q_i) \leq \kappa$" then $d(P_{i+1}) \leq d(P_i) + \kappa$ (where d is density).

(8) For α limit $d(P_\alpha) \leq 2^{\Sigma_{i<\alpha} d(P_i)}$.

Proof. Easy.

1.5 The Iteration Lemma.

(1) Suppose F is a function, *then* for every ordinal α there is RCS-iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha^\dagger \rangle$, such that:

 (a) for every $i, \underset{\sim}{Q}_i = F(\underset{\sim}{Q} \restriction i)$,

 (b) $\alpha^\dagger \leq \alpha$,

 (c) either $\alpha^\dagger = \alpha$ or $F(\bar{Q})$ is not an $(\operatorname{Rlim} \bar{Q})$-name of a forcing notion.

(2) Suppose $\beta < \alpha$, $G_\beta \subseteq P_\beta$ is generic over V, *then* in $V[G_\beta]$, $\bar{Q}/G_\beta = \langle P_i/G_\beta, \underset{\sim}{Q}_i : \beta \leq i < \alpha \rangle$ is an RCS-iteration and $\operatorname{Rlim}(\bar{Q}) = P_\beta * (\operatorname{Rlim} \bar{Q}/\underset{\sim}{G}_\beta)$ (essentially).

(3) The Associative Law.

If $\alpha_\xi (\xi \leq \xi(0))$ is increasing and continuous, $\alpha_0 = 0$, $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha_{\xi(0)} \rangle$ is an RCS-iteration, $P_{\xi(0)} = \operatorname{Rlim} \bar{Q}$, *then* so are

$$\langle P_{\alpha(\xi)}, P_{\alpha(\xi+1)}/P_{\alpha(\xi)} : \xi < \xi(0) \rangle \text{ and } \langle P_i/P_{\alpha(\xi)}, \underset{\sim}{Q}_i : \alpha(\xi) \leq i < \alpha(\xi+1) \rangle;$$

and vice versa.

(4) If $\bar{Q}$ is an RCS iteration, $p \in \mathrm{Rlim}\bar{Q}$, $P'_i = \{q \in P_i : q \geq p \restriction i\}, \underset{\sim}{Q}'_i = \{p \in \underset{\sim}{Q}_i : p \geq p \restriction \{i\}\}$ *then* $\bar{Q} = \langle P'_i, Q'_i : i < \ell g\bar{Q}\rangle$ is (essentially) an RCS iteration (and $\mathrm{Rlim}\bar{Q}'$ is $P'_{\ell g\bar{Q}}$).

Proof. (1) Easy.

(2) Pedantically, we should formalize the assertion as follows:

(∗) There is a function $F = F_0$ (= a definable class), such that for every RCS-iteration $\bar{Q}$, and $\ell g(\bar{Q}) = \alpha$, and $\beta < \alpha$, $F_0(\bar{Q}, \beta)$ is a P_β-name of $\underset{\sim}{\bar{Q}}^\dagger$ such that:

a) $\Vdash_{P_\beta}$ "$\underset{\sim}{\bar{Q}}^\dagger$ is a RCS-iteration of length $\alpha - \beta$".

b) $P_\beta * (\mathrm{Rlim}\underset{\sim}{\bar{Q}}^\dagger)$ is equivalent to $P_\alpha = \mathrm{Rlim}\bar{Q}$, by $F_1(\bar{Q}, \beta)$ (i.e., $F_1(\bar{Q}, \beta)$ is an isomorphism between the corresponding completions to Boolean algebras).

c) if $\beta \leq \gamma \leq \alpha \Vdash_{P_\beta}$ "$F_0(\bar{Q} \restriction \gamma, \beta) = F(\bar{Q}, \beta) \restriction (\gamma - \beta)$" and $F_1(\bar{Q}, \beta)$ extends $F_1(\bar{Q} \restriction \gamma, \beta)$ and $F_1(\bar{Q} \restriction \gamma, \beta)$ transfer the P_γ-name $\underset{\sim}{Q}_\gamma$ to a P_β-name of a $(\mathrm{Rlim}(\underset{\sim}{\bar{Q}}^\dagger \restriction (\gamma - \beta))$-name of $\underset{\sim}{Q}^\dagger_{\gamma-\beta}$ (when $\underset{\sim}{\bar{Q}}^\dagger = \langle \underset{\sim}{Q}^\dagger_i : i < \gamma - \beta\rangle$).

The proof is by induction on α, and there are no special problems.

(3) Again, pedantically the formulation is

(∗∗) For $\bar{Q}$ is an RCS-iteration, $\ell g(\bar{Q}) = \alpha_{\xi(0)}$, $\bar{\alpha} = \langle \alpha_\xi : \xi \leq \xi(0)\rangle$ increasing continuous, $F_3(\bar{Q}, \bar{\alpha})$ is an RCS-iteration $\bar{Q}^\dagger$ of length $\alpha_{\xi(0)}$ such that:

a) $F_4(\bar{Q}, \bar{\alpha})$ is an equivalence of the forcing notions $\mathrm{Rlim}\bar{Q}, \mathrm{Rlim}\bar{Q}^\dagger$.

b) $F_3(\bar{Q} \restriction \alpha_\zeta, \bar{\alpha} \restriction (\zeta + 1)) = F_3(\bar{Q}, \bar{\alpha}) \restriction \zeta$.

c) $\underset{\sim}{Q}^\dagger_\xi$ is the image by $F_4(\bar{Q} \restriction \alpha_\xi, \bar{\alpha} \restriction (\xi + 1))$ of the $P_{\alpha_\xi} = \mathrm{Rlim}(\bar{Q} \restriction \alpha_\xi)$-name $F_0(\bar{Q} \restriction \alpha_{\xi+1}, \alpha_\xi)$

The proof again poses no special problems.

(4) Left to the reader. $\square_{1.5}$

1.6 Claim. If κ is regular, and $|P_i| < \kappa$ (or just $d(P_i) < \kappa$) for every $i < \kappa$, and $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \kappa\rangle$ is an RCS-iteration, then:

(1) every $\bar{Q}$-named ordinal is in fact a $(\bar{Q} \restriction i)$-named ordinal for some $i < \alpha$,

(2) like (1) for $\bar{Q}$-named conditions,

(3) $P_\kappa = \bigcup_{i<\kappa} P_i$.

Proof. Easy.

1.7 Claim. Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \delta \rangle$ is an RCS-iteration, δ limit and $p \in P_\delta$, and $\underset{\sim}{\zeta}$ is a $\bar{Q}$-named ordinal. *Then* there are $i < \delta$, and $p^\dagger \in P_{i+1}, p\restriction(i+1) \leq p^\dagger$ such that $p^\dagger \Vdash_{\bar{Q}}$ "$\underset{\sim}{\zeta} = i$" (or $p^\dagger \Vdash$ "$\underset{\sim}{\zeta}[G]$ undefined " if we allow this). The same holds for $\bar{Q}$-named conditions (if $\underset{\sim}{Q}_i \subseteq V$).

Proof. Easy. By 1.3(1).

§2. Proper Forcing Revisited

2.1 Discussion. Properness is a property of forcing notions which implies that $\aleph_1$ is not collapsed by forcing with P, and is preserved by countable-support iteration (and also $\aleph_1$-free iteration, see IX.). This property was introduced in chapter III, and (see VII §3, 4) many examples of forcing not collapsing $\aleph_1$ were shown to be proper ($\aleph_1$-complete, c.c.c., Sacks forcing, Laver forcing and more). It was argued that proper forcing is essentially the most general property implying $\aleph_1$ is not collapsed and preserved under iteration. So the forcing of shooting a closed unbounded set through a stationary subset S of $\aleph_1$ (see Baumgartner, Harrington and Kleinberg [BHK], and III 4.4), though not collapsing $\aleph_1$, is excluded as if $\aleph_1 = \bigcup_{n<\omega} S_n, S_n$ pairwise disjoint stationary subsets of $\aleph_1$ and we shoot a closed unbounded subset through each $\omega_1 \setminus S_n$, in the limit $\aleph_1$ is collapsed. Of course we can "kill" stationary sets in a fixed normal ideal of $\aleph_1$ (see e.g. [JMMP]) and properness really demands somewhat more than not destroying stationary subsets of $\aleph_1$ (also stationary subsets of $\mathcal{S}_{\leq\aleph_0}(\lambda) = \{A \subseteq \lambda : |A| {\leq} \aleph_0$ should not be destroyed); but those seemed technical points.

However, in Chapters III–IX we were mainly interested in forcings of cardinality $\aleph_1$, so another restriction of properness was ignored: if P is proper, any countable set of ordinals in V^P is included in a countable set of V. So forcing changing the cofinality of some λ, $\mathrm{cf}\lambda > \aleph_1$, to $\aleph_0$, are not included. In fact, there are such forcings which do not collapse $\aleph_1$, and moreover, do not add reals: Prikry forcing [Pr] (which changes the cofinality of a measurable cardinal to $\aleph_0$) and Namba [Nm] which changes the cofinality of $\aleph_2$ to $\aleph_0$ (and do not add reals when CH holds).

We suggest here a property of forcing, called semiproperness, such that most theorems proved for proper forcing hold (when we use RCS-iteration) and it includes Prikry forcing. We did not know whether there is a forcing changing the cofinality of $\aleph_2$ to $\aleph_0$ which is semiproper (i.e., provably from ZFC), but we shall have an approximation to this, (but see XII §2).

So in this section we introduce the notion, and prove the preservation under RCS-iteration. In this we weaken a little the assumptions: for limit δ, Q_δ is not necessarily semiproper, only $P_{\delta+1}/P_{i+1} (i < \delta)$ is semiproper. This change does not influence the proof, but is useful, as we can exploit the fact that δ was a large cardinal in V. Note that the useful result is Corollary 2.8.

2.2 Definition. A forcing notion P is $\underset{\sim}{S}$-semiproper ($\underset{\sim}{S}$ a P-name of a class of uncountable cardinals of V) if for any large enough regular λ, and well-ordering $<^*$ of $H(\lambda)$, and countable $N \prec (H(\lambda), \in, <^*)$, such that $P \in N$, $\underset{\sim}{S} \in N$, and for every $p \in P \cap N$ there is q, $p \leq q \in P$ such that: for every cardinal $\kappa \in N$ and P-name $\underset{\sim}{\beta} \in N$ of an element of κ,

$q \Vdash_P$ "if $\kappa \in \underset{\sim}{S}$ then there is $A \in N, |A|^V < \kappa, \underset{\sim}{\beta} \in A$"

Equivalently, if $\underset{\sim}{S}$ consists of regular cardinals of V, $q \Vdash_P$ "if $\kappa \in \underset{\sim}{S}$ then $\mathrm{Sup}(N \cap \kappa) = \mathrm{Sup}(N[G] \cap \kappa)$"; or even $q \Vdash$ " if $\mathrm{cf}(\kappa)^V \in \underset{\sim}{S}$, then $\mathrm{Sup}(N \cap \kappa) = \mathrm{Sup}(N[G] \cap \kappa)$"; the case "$\underset{\sim}{S} = \{\aleph_1\}$ is the main case.

(Note that we write A and not $\underset{\sim}{A}$, i.e., A is in V; also when κ is regular in V, without loss of generality $A = \gamma$ for some $\gamma < \kappa$; this is the main case.)

We call q, under such circumstances, $\underset{\sim}{S}$-semi-(N, P)-generic. "Semiproper" means "$\{\aleph_1\}$-semiproper", and "semi-generic" means "$\{\aleph_1\}$-semi-generic" (we change the conventions of [Sh:b] where they mean $\underset{\sim}{\mathrm{U}}\mathrm{RCar}^{V^P}$- semiproper, $\underset{\sim}{\mathrm{U}}\mathrm{RCar}^{V^P}$-semi-generic respectively (see below)).

2.2A Remark. We could here change the definition to:

$q \Vdash_P$ " if $\kappa \in \underset{\sim}{S} \cap N[G]$ then, letting N' = the Skolem Hull of $N \cup \{\kappa\}$, we have $\mathrm{Sup}(N' \cap \kappa) = \mathrm{Sup}(N[G] \cap \kappa)$"

(in this case every $\kappa \in S$ is regular $> \aleph_0$). We have not looked into this variant.

2.2B Remark. When we write "P is $\mathrm{U}\underset{\sim}{\mathrm{C}}\mathrm{ar}$-semiproper" or "$P$ is UCar-semiproper", UCar means $\{\delta : \delta = \aleph_1^V \text{ or } \mathrm{cf}^{V^P}(\delta) > \aleph_0\}$ so it is a P-name. Similarly for SCar, RUCar instead of RCar (and also $\aleph_1$) etc. But e.g. RUCar^V-semiproper means the regular uncountable cardinals of V.

2.3 Claim.

(1) If P is UCar^V-semiproper, or even S-semiproper, $S = \{\lambda : \mathrm{cf}\lambda > \aleph_0$ and λ a cardinal, in $V\}$, or even RUCar^V-semiproper, *then* P is proper, and vice versa. Moreover, in this case, q in Definition 2.2 is (N, P)-generic which means: if $\underset{\sim}{\beta} \in N$ is a P-name of an ordinal then $q \Vdash_P$ "$\underset{\sim}{\beta} \in N$".

(2) P is $\underset{\sim}{S}$-semiproper iff the condition of Definition 2.2 holds for some $\lambda > 2^{|P|}$, and well-ordering $<^*$ *iff* it holds for $\lambda = (2^{|P|})^+$ (provided that $P \in H(\lambda)$). Also, the well ordering $<^*$ is convenient but not really necessary.

(3) P is $\underset{\sim}{S}$-semiproper iff $(B^P \setminus \{0\}, \geq)$ is, where B^P is the complete Boolean algebra corresponding to P.

(4) In Definition 2.2, for $\kappa > \aleph_0$, and $\kappa > |P|$, the condition is trivially satisfied by any q, so only $\underset{\sim}{S} \cap \{\kappa : \aleph_0 < \kappa \leq |P|\}$ is relevant.

(5) If $P \lessdot Q$, $\underset{\sim}{S}$ a P-name and Q is $\underset{\sim}{S}$-semiproper *then* P is $\underset{\sim}{S}$-semiproper.

(6) If P is $\underset{\sim}{S}$-semiproper, $\Vdash_P$ "$\kappa \in \underset{\sim}{S}$", $\mathrm{cf}(\kappa) > \aleph_0$, *then* $\Vdash_P$ "$\mathrm{cf}(\kappa) > \aleph_0$". In particular, if $\aleph_1^V \in S$ then $\aleph_1^V = \aleph_1^{V^P}$.

(7) If $\Vdash_P$ "$\underset{\sim}{S}^1 \subseteq \underset{\sim}{S}^2$", P is $\underset{\sim}{S}^2$-semiproper then P is $\underset{\sim}{S}^1$-semiproper (similarly for semi generic).

Proof. Easy. $\square_{2.3}$

2.4 Definition.

(1) A property is preserved by RCS-iteration, provided that for any RCS-iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$, if $\underset{\sim}{Q}_i$ has the property (in V^{P_i}) for each i, then Rlim $\bar{Q}$ has the property.

(2) A property is strongly preserved by RCS-iteration provided that, for $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ an RCS-iteration, we have

(a) *if* for every $\gamma \leq \beta < \alpha$ such that γ not a limit ordinal, $P_{\beta+1}/P_\gamma$ has the property *then* Rlim$\bar{Q}$ has the property and

(b) if $\alpha = \beta + 1 > \gamma$, P_β/P_γ and $\underset{\sim}{Q}_\beta$ have the property, then P_α/P_γ has the property.

(3) We can replace RCS-iteration by any other kind of iteration in this definition.

2.4A Remark. In VI 1.6, 1.7, many properties were shown to be preserved by CS iteration. In fact we have proved they are strongly preserved for CS iteration – see VI 0.1(B) and even RCS iterations.

2.5 Claim.

(1) In Definition 2.4(1), (2) it suffices to consider the two-step iteration and the case where α is a regular cardinal and: $\gamma < \beta < \alpha$ implies P_β/P_γ has the property (where for 2.4(2) γ is zero or a successor ordinal).

(2) If a property is strongly preserved by RCS-iteration *then* the property is preserved by RCS-iteration.

(3) In (1), for α regular, we can add: [$\beta < \alpha \Rightarrow \Vdash_{P_\beta}$ "α is a regular cardinal"] provided that: P_α has the property *iff* $\{p \in P_\alpha : P_\alpha \restriction \{q : q \geq p\}$ has the property $\}$ is dense.

Proof. Easy, by induction on α; for (1) use the associative law 1.5(3). For (3) use 1.5(4). $\square_{2.5}$

2.6 The Semi-Properness Iteration Lemma.

(1) "Q is $\underset{\sim}{S}^Q$-semiproper" is strongly preserved by RCS-iteration for

$$\underset{\sim}{S}^Q = \{\aleph_1^V\} \cup \{\kappa : \text{ in } V^Q \text{ we have "}\kappa = \mathrm{cf}\kappa > \aleph_0\text{"}\},$$

so it is a Q-name.

(2) Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ is an RCS-iteration, for successor $j \leq \alpha$ for arbitrarily large non limit $i < j$, P_j/P_i is $\underset{\sim}{S}_{i,j}$-semiproper (and $\underset{\sim}{S}_{i,j}$ is defined, $\underset{\sim}{S}_{i,j}$ is a P_j-name). Let ($\underset{\sim}{S}$ is a P_α-name):
$\underset{\sim}{S} = \{\lambda : \lambda$ an uncountable regular cardinal, and for every i non-limit we have: $\mathrm{cf}(\lambda)^{V^{P_i}} \in \underset{\sim}{S}_{i,j}$ for every $j \in [i, \alpha)$, for which $\underset{\sim}{S}_{i,j}$ is well defined $\}$.
Then $P_\alpha = \mathrm{Rlim}\bar{Q}$ is $\underset{\sim}{S}$-semiproper provided that:

(C1) for every limit $\delta \leq \alpha$ there is $\xi < \delta$, such that

$$\Vdash_{P_\xi} \text{"}[\mathrm{cf}(\delta) = \aleph_0 \textit{ or } \text{ for every } \xi \leq i < j < \delta : \text{ if } \underset{\sim}{S}_{i,j} \text{ is defined then } \Vdash_{P_j/P_i} \text{"}\mathrm{cf}(\delta)^{V^{P_i}} \in \underset{\sim}{S}_{i,j}]\text{"}.$$

(3) In (2) we can weaken (C1) by replacing ξ by a $(\bar{Q}\restriction\delta)$- named $[0, \delta)$-ordinal $\underset{\sim}{\xi}$ i.e. if $p \in P_{\xi+1}$, $p \Vdash$ "$\underset{\sim}{\xi} = \xi$" then, for $\xi \leq i < j \leq \delta$, i non-limit we have, $p\restriction\xi \Vdash_{P_\xi}$ " $[\mathrm{cf}\delta = \aleph_0$ *or* $p\restriction[\xi, j) \Vdash_{P_j/P_\xi}$ "$(\mathrm{cf}\delta)^{V^{P_i}} \in \underset{\sim}{S}_{i,j}$"]", and replace S by $\underset{\sim}{S} = \{\lambda$: for every non-limit $i < \alpha$ and $j \in [i, \alpha)$ (such that $\underset{\sim}{S}_{i,j}[\underset{\sim}{G}_{i,j}]$ well defined), the cofinality of λ as computed in V^{P_i} is $> \aleph_0$ and belongs to $\underset{\sim}{S}_{i,j}[\underset{\sim}{G}_{P_j}]\}$.

(4) In part (2) we can omit the condition (C1) and replace "for arbitrarily large non-limit $i < j$" by "for every $i_0 < j$ there is a $\bar{Q}$-named $[i_0, j)$-ordinal $\underset{\sim}{i}$ forced to satisfy the demand on i".

Remark.

(1) For $i < \alpha$ non-limit clearly $\underset{\sim}{S}_{i,i+1}$ is defined, so $\underset{\sim}{Q}_i$ is $\underset{\sim}{S}_{i,i+1}$-semiproper.

(2) In 2.6(2) and (3), in (C1) we can replace "for every" by "for arbitrarily large" assuming $\underset{\sim}{S}_{i,j}$ decreases with j.

(3) See XII§1 for an alternative proof, using games.

Proof. (1) Follows from (2).

(2) We prove the theorem by induction on α, for all $\bar{Q}$'s and even for $\bar{Q}$'s in forcing extensions of V.

Let $T = \{(i,j) : \underset{\sim}{S}_{i,j} \text{ is defined }\}$ (here $T \in V$).

Note that for any $\beta \leq \gamma \leq \alpha, \beta$ non-limit, $\bar{Q}\restriction[\beta,\gamma) = \langle P_i/P_\beta, \underset{\sim}{Q}_i : \beta \leq i < \gamma\rangle$ satisfies the hypothesis on $\bar{Q}$. Let λ be big enough, $<^*$ a well-ordering of $H(\lambda)$, $\bar{Q} \in H(\lambda)$, $N \prec (H(\lambda), \in, <^*)$, N countable, $\underset{\sim}{S} \in N$, $P_\alpha \in N$ hence w.l.o.g. $\bar{Q} \in N$ [because $(H(\lambda), \in, <^*) \models$ "there is $\bar{Q}$, an RCS-iteration as in 2.6(2) such that $P_\alpha = \mathrm{Rlim}\,\bar{Q}$", as $P_\alpha \in N \prec (H(\lambda), \in, <^*)$ there is such a $\bar{Q}$ in N]. Similarly w.l.o.g. $\langle \underset{\sim}{S}_{i,j} : (i,j) \in T\rangle$ belongs to N. Furthermore, let $p \in P_\alpha \cap N$.

Case A. α non-limit.

The cases $\alpha = 0$, $\alpha = 1$ are too trivial to consider. For $\alpha > 1$ by the induction hypothesis on α and 1.5(3) we can assume $\alpha = 2$.

So by 2.3(3)+1.4(3) w.l.o.g. $P_2 = Q_0 * \underset{\sim}{Q}_1$, and let $p = (p_0, \underset{\sim}{p}_1) \in P_1 \cap N$. As clearly $Q_0 \in N$, there is $q_0 \in Q_0$, $p_0 \leq q_0$, which is $\underset{\sim}{S}_{0,1}$-semi (N, P)-generic. To help us in understanding let $G_0 \subseteq Q_0$ be generic, $q_0 \in G_0$. As $<^*$ is a well-ordering of $H(\lambda)$, $(H(\lambda)[G_0],\ H(\lambda), \in, <^*)$ has definable Skolem functions, and a definable well-ordering (and note: $H(\lambda)[G_0]$ is $H(\lambda)$ of the universe $V[G_0]$ as we know that any member of $H(\lambda)[G_0]$ has a name in $H(\lambda)$).

Now $N[G_0]$ is the Skolem Hull of N in $(H(\lambda)[G_0], \in, <^*)$. So: as $\underset{\sim}{p}_1[G_0] \in N[G_0]$ (because $\underset{\sim}{p}_1, G_0 \in N[G_0]$), $Q_1 = P_1/G_0$ is $\underset{\sim}{S}_{1,2}$-semiproper (i.e. $\underset{\sim}{S}_{1,2}[G_0]$-semiproper), and Q_1, $\underset{\sim}{p}_1[G_0] \in N[G_0] \prec (H(\lambda)[G_0], \in, <^*)$, *there is* $q_1 \in Q_1$ which is $\underset{\sim}{S}_{1,2}$-semi $(N[G_0], Q_1)$-generic and $q_1 \geq \underset{\sim}{p}_1[G_0]$. Let $G_1 \subseteq Q_1$ be generic, $q_1 \in G_1$. Note that $\underset{\sim}{S} \subseteq \underset{\sim}{S}_{0,1} \cap \underset{\sim}{S}_{1,2}$.

So if $\kappa \in N$ and $\mathrm{cf}(\kappa) \in \underset{\sim}{S}_{0,1}[G_0]$ then as q_0 is $\underset{\sim}{S}_{0,1}$-semi (N, Q_0)-generic and $q_0 \in Q_0$ clearly $\mathrm{Sup}(N \cap \kappa) = \mathrm{Sup}(N[G_0] \cap \kappa)$; and similarly if $\kappa \in N$ and $\mathrm{cf}^{V[G_0]}(\kappa) \in \underset{\sim}{S}_{1,2}[G_0, G_1]$ then $\mathrm{Sup}(N[G_0] \cap \kappa) = \mathrm{Sup}(N[G_0, G_1] \cap \kappa)$. We have described q_1 knowing G_0, hence there is an appropriate Q_0-name $\underset{\sim}{q}_1$ such that $q_0 \Vdash_{Q_0}$ "$\underset{\sim}{q}_1$ is as described above".

As $\underset{\sim}{S} \subseteq \underset{\sim}{S}_{0,1} \cap \underset{\sim}{S}_{1,2}$ and as G_0, G_1 were arbitrary except that $q_0 \in G_0$, $\underset{\sim}{q}_1 \in G_1$, clearly $(q_0, \underset{\sim}{q}_1)$ is $\underset{\sim}{S}$-semi (N, P_2)-generic.

Case B. α a limit ordinal and there are $\beta < \alpha$ and $p^\dagger$ such that $p{\restriction}\beta \leq p^\dagger \in P_\beta$ and $p^\dagger \Vdash_{P_\beta}$ "$\mathrm{cf}(\alpha) = \aleph_0$".

As $N \prec \langle H(\lambda), \in \rangle$, $\bar{Q} \in N$ and $\beta \in N$, $p \in N$, we can assume β is a successor ordinal and $p^\dagger \in N$, hence by 1.4(1) without loss of generality $p{\restriction}\beta = p^\dagger$. Moreover by Case A it suffices to prove that $P_\alpha/P_\beta, P_\beta$ are $\underset{\sim}{S}$-semiproper (for P_β, more exactly $\{\kappa :$ for no $q \in \underset{\sim}{G}_{P_\beta}, q \Vdash_{P_\alpha}$ "$\kappa \notin \underset{\sim}{S}\}|$. By the induction hypothesis this holds for P_β; for P_α/P_β (we are working in $V[G_\beta]$, $G_\beta \subseteq P_\beta$ generic over V, $p^\dagger \in G_\beta$ by 1.5(1)) w.l.o.g. $\beta = 0$ so $\mathrm{cf}\alpha = \aleph_0$, and as $\bar{Q} \in N$, $\alpha \in N$, clearly there are $\alpha_n < \alpha, \alpha_n < \alpha_{n+1}, \alpha = \bigcup_{n<\omega} \alpha_n$, and w.l.o.g. each α_n is a successor ordinal or 0 and $\alpha_n \in N$, $\alpha_0 = \beta$ and $(\alpha_n, \alpha_{n+1}) \in T$.

Now let $\{(\underset{\sim}{\beta}_n, \kappa_n) : n < \omega\}$ be a list of the pairs $(\underset{\sim}{\beta}, \kappa)$, where $\kappa \in N$ and $\underset{\sim}{\beta}$ a P_α-name of an ordinal $< \kappa$, $\underset{\sim}{\beta} \in N$. We define by induction on $n < \omega$ p_n, q_n such that:

(1) p_n is a P_{α_n}-name of a member of $N \cap P_\alpha, p_0 = p^\dagger$.
(2) $q_n \in P_{\alpha_n}$, $q_{n+1}{\restriction}\alpha_n = q_n, q_n$ is $(\bigcap_{k<n} \underset{\sim}{S}_{\alpha_k,\alpha_{k+1}})$ -semi (N, P_{α_n})-generic,
(3) $p_n{\restriction}\alpha_n \leq q_n$, (i.e. this is forced)
(4) $p_{n+1} \Vdash_{P_\alpha}$ "$\underset{\sim}{\beta}_n < \underset{\sim}{\gamma}_n$ for some $\underset{\sim}{\gamma}_n$ a P_{α_n}-name of an ordinal $< \kappa_n$, $\underset{\sim}{\gamma}_n \in N$".
(5) $q_n \Vdash_{P_{\alpha_n}}$ "$p_n \leq p_{n+1}$ (in P_α)"

This is easy ($q_{n+1}{\restriction}[\alpha_n, \alpha_{n+1})$ can be constructed like $\underset{\sim}{q}_1$ in case A). Of course the point is that a P_{α_n}-name of a condition in $P_{\alpha_n+1}/P_{\alpha_n}$ is essentially a condition in $P_{\alpha_{n+1}}$. Now $\bigcup_{n<\omega} q_n$ is as required.

Case C. α a limit ordinal and for no $\beta < \alpha$, $p^\dagger \in P_\beta$, $p{\restriction}\beta \leq p^\dagger$ does $p^\dagger \Vdash_{P_\beta}$ "$\mathrm{cf}(\alpha) = \aleph_0$".

Let $\xi = \xi^*$ be as guaranteed by condition (C1) from the hypothesis. By case A without loss of generality $\xi = 0$. Let $\alpha_n \in N$, $\alpha_n < \alpha_{n+1}$, $\bigcup_{n<\omega} \alpha_n = \mathrm{Sup}(N \cap \alpha)$ (exists, as $\alpha \in N$), and $\alpha_0 = 0$, α_n non-limit; and repeat the previous proof getting $\langle q_n : n < \omega \rangle$, adding

(6) if $r \in p_n$ (so r is a $\bar{Q}$-named atomic condition) *then* for some m and P_{α_n}-name $\underset{\sim}{\xi}_m < \alpha$ we have

$$p_{m+1} \restriction (\underset{\sim}{\xi}_m + 1) \Vdash \text{“}\underset{\sim}{\zeta}(r) = \xi_m\text{”},$$

in other words for n, $k < \omega$ for some $m > n$ and P_{α_m}-name $\underset{\sim}{\xi}_m$ we have:

for every $G_{\alpha_m} \subseteq P_{\alpha_m}$ generic over V to which q_m belongs, letting r be the k-th member of p_n in the canonical well ordering of p_n of order type ω, we have:

either for $\xi = \underset{\sim}{\xi}_m[G_{\alpha_m}] < \alpha_n$, and some $p' \in G_{\alpha_m} \cap P_{\xi_m+1}$, $p' \Vdash_{P_{\xi+1}} \underset{\sim}{\zeta}(r) = \xi$” or for some $\xi \in [\alpha_n, \alpha)(\cap N[G_{\alpha_n}])$, we have $p_m[G_{\alpha_m}] \restriction (\xi + 1) \Vdash_{P_\alpha / G_{\alpha_n}}$ “$\underset{\sim}{\zeta}(r) = \xi$”.

By condition (C1) from the hypothesis and as $\xi^* = 0$, we have $q_n \Vdash_{P_{\alpha_n}}$ “$N \cap \alpha$ is unbounded in $N[\underset{\sim}{G}_{\alpha_n}] \cap \alpha$, i.e. $\{\alpha_n : m < \omega\}$ is an unbounded subset of $N[\underset{\sim}{G}_{\alpha_n}] \cap \alpha$”. Let $q \in P_{\sup(N \cap \alpha)}$, $q \restriction \alpha_n = q_n$. The new point is that condition (3) above does not immediately give $p_n \leq q$, only yields $(*)$ $p_n \restriction (\bigcup_{i<\omega} \alpha_i) \leq \bigcup_{i<\omega} q_i$. But if $q \in G \subseteq P_\alpha$, G generic over V, then $p'_n \overset{\text{def}}{=} \underset{\sim}{p}_n[G \cap P_{\alpha_n}]$ is a member of $N[G_{\alpha_n}] \cap P_\alpha$, and for every $\bar{Q}$-named condition $r \in p'_n$ we know by (6) above that for some m, $\underset{\sim}{\xi}_m$ is a P_{α_m}-name and letting $\xi = \underset{\sim}{\xi}_m[G \cap P_{\alpha_m}]$ we have $p'_{m+1} \restriction \xi \Vdash_{P_\xi}$ “$\underset{\sim}{\zeta}(r) = \xi$”. But $\underset{\sim}{\zeta}(r) \in N[G_{\alpha_n}] \subseteq N[G]$ and q_n is $\{\mathrm{cf}\alpha\}$-semi $(N[G_{\alpha_n}], P_\alpha / P_{\alpha_n})$-generic hence $\underset{\sim}{\xi}_n[G] < \sup(N \cap \alpha)$ hence by $(*)$ we know $\{r\} \in G$. This insures that: *if* $q \in G \subseteq P_{\sup(N \cap \alpha)}$, G generic over V and $r \in p_n[G_{\alpha_n}]$ $(n < \omega)$ (so r is a $\bar{Q}$-named condition) *then* $\underset{\sim}{\zeta}(r)[G] < \sup(N \cap \alpha)$.

As this holds for every $r \in p'_n$ we necessarily have $p'_n \in G$ [as some $q^* \in G$ forces $(\forall r)(r \in p'_n \Rightarrow r \in G)$. Why? As this hold; assume toward contradiction that $q^* \not\Vdash$ “$p'_n \in \underset{\sim}{G}$” so, w.l.o.g. it force the negation, but you can check that $p'_n \cup q^* \in P_\alpha / G_{\alpha_n}$, contradiction].

As this holds for every appropriate G, we have $q \Vdash_{P_\alpha}$ “$p_n \in \underset{\sim}{G}_{P_\alpha}$” which is enough.

(3) A similar proof (only we increase p to determine ξ).

(4) The proof is like the proof of part (2), but in the case α is a limit ordinal (i.e. cases B, C), we use $\underset{\sim}{\alpha}_n$ a $\bar{Q}$-named ordinal, so conditions (1)-(5) (see case B) should be revised accordingly and if $n < \omega$, $\underset{\sim}{\xi}$ is a $\bar{Q}$-named ordinal in the Skolem-hull of $N \cup \{p_n\}$ then for some m, $\Vdash_{P_\alpha}$ "$\underset{\sim}{\xi} \leq \underset{\sim}{\alpha}_m$". $\square_{2.6}$

As we do not actually need 2.6(4) we have not elaborate. In fact, essentially we have proved above also the following, which will be useful e.g. for chain conditions:

2.7 Lemma. *If* $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \delta\rangle$ is an RCS-iteration as in 2.6(2) or 2.6(3), δ a limit ordinal, and $\emptyset \Vdash_{P_i}$ "cf$\delta > \aleph_0$" (and of course $\Vdash$ "$\aleph_1 \in \underset{\sim}{S}$", really (C1) of 2.6(2) is needed for δ only), for every $i < \delta$, *then* $\emptyset \Vdash_{P_\delta}$ "cf$(\delta) > \aleph_0$". Moreover, in this case, $\bigcup_{i<\delta} P_i$ is a dense subset of P_δ more exactly essentially dense (i.e. for every $p \in P_\delta$ for some $q \in \bigcup_{i<\delta} P_i$ we have $q \Vdash$ "$p \in \underset{\sim}{G}_{P_\delta}$").

Proof. Let $p \in P_\delta$. Let χ be large enough, $<^*$ a well ordering of $H(\chi)$, $N \prec (H(\chi), \in, <^*)$ is countable, $\{\bar{Q}, \underset{\sim}{S}, p\} \subseteq N$. In the proof of 2.6, for $\alpha = \delta$, necessarily case C occurs. Now $q \in P_{\sup(N\cap\alpha)} \subseteq \bigcup_{\beta<\alpha} P_\beta$ is above p_0 which is p. Now in (C1) the second possibility always holds, so if $\underset{\sim}{\tau} : \omega \to \delta$ is a P_δ-name from N, then q forces each $\underset{\sim}{\tau}(n)$ to be equal to some $P_{\alpha_{k(n)}}$-name of an ordinal $< \delta$ from N, which q forces to be $< \sup(N \cap \delta)$. Together we finish. $\square_{2.7}$

Also note that the most useful case of 2.6 is

2.8 Corollary. Suppose $\langle P_i, \underset{\sim}{Q}_i : i < \delta\rangle$ is an RCS-iteration, and for every $j < \delta$ for arbitrarily large non-limit $i < j+1$, P_{j+1}/P_i is $\{\aleph_1\}$-semiproper, and for every $i < \delta$, $\Vdash_{P_{i+n}}$ " the power of P_i is $\aleph_1$" for some $n < \omega$. *Then* P_δ is $\{\aleph_1\}$-semiproper. *If* in addition $|P_i| < |\delta|$, for $i < \delta$ and δ is inaccessible *then* P_δ is $\underset{\sim}{S}$-semiproper, for $\underset{\sim}{S} = \{\aleph_1^V\} \cup \{\kappa :\Vdash_{P_\delta}$ "κ is a cardinal, $\kappa = \mathrm{cf}(\kappa) > \aleph_0$"$\}$. If in addition cf$(\delta) = \aleph_1$ then $\bigcup_{i<\delta} P_i$ is a dense subset of P_δ more exactly essentially dense (i.e. for every $p \in P_\delta$ for some $q \in \bigcup_{i<\delta} P_i$ we have $q \Vdash$ "$p \in \underset{\sim}{G}_{P_\delta}$").

2.9 Remark. For iteration of proper forcings, there is really no difference between CS and RCS-iterations (see III 1.16), i.e. for $\bar{Q}$ an *RCS* iteration of proper forcing, $\{p \in \mathrm{RLim}\,\bar{Q}$: the set $\{\alpha$: for some $r \in p$ and $q \in P_{\beta+1}$ we have $q \Vdash_{P_{\beta+1}}$ "$\underset{\sim}{\zeta}(r) = \beta$"$\}$ is countable$\}$ is a dense subset (in a weak sense) of $\mathrm{Rlim}\,\bar{Q}$. In fact E-properness (for some stationary $E \subseteq \mathcal{S}_{\leq \aleph_0}(\cup E)$) suffices.

2.10 Conclusion. Suppose κ is supercompact (without loss of generality, with Laver indestructibility). *Then* for some κ-c.c. semiproper forcing notion P of power κ, $\Vdash_P$ "SPFA" and even $\Vdash_P$ "SPFA$^\alpha$" for all $\alpha \leq \omega_1$, where SPFA $=$ SPFA^0 and SPFA$^\alpha$ is the assertion Ax_α [semiproper], i.e.:
If Q is a semiproper forcing notion, $\langle \underset{\sim}{\tau}_i : i < \omega_1 \rangle$ a sequence of Q-names of members of V, $\langle \underset{\sim}{S}_\beta : \beta < \alpha \rangle$ a sequence of Q-names of stationary subsets of ω_1, *then* for some directed $G \subseteq Q$:
(a) for every $i < \omega_1$, for some $q \in G$, q forces a value to $\underset{\sim}{\tau}_i$.
(b) for every $\beta < \alpha$, $\{\zeta < \omega_1 : \exists q \in G, q \Vdash$ "$\zeta \in \underset{\sim}{S}_\beta\}$ is a stationary subset of ω_1.

Proof. Same as PFA-see VII, 2.7(2) or VII 2.9. We use iteration as in 2.8, e.g. require $\underset{\sim}{Q}_{2i+1}$ is Levy$(\aleph_1, 2^{|P_{2i+1}|})$. $\square_{2.10}$

§3. Pseudo-Completeness

A widely used family (or property) of forcing is $\aleph_1$-completeness, i.e., if $p_n \leq p_{n+1} \in P$, then there is $p \in P$, $p_n \leq p$ for every n. This is the simplest family of forcing which does not add reals, nor new ω-sequences of ordinals. In our perspective we want a condition parallel to this, including, e.g., Prikry forcing.

3.1 Definition. For a forcing notion P, a P-name $\underset{\sim}{S}$ of a set of cardinals of V, an ordinal δ (always a limit ordinal) and condition p we define a game $\Game^\delta_{\underset{\sim}{S}}(p, P)$: in the i-th move, player I chooses a cardinal (in V) λ_i and a P-name $\underset{\sim}{\beta}_i$ of an ordinal $< \lambda_i$, and player II has to find a condition p_i, and a set $A_i \subseteq \lambda_i$, $|A_i| < \lambda_i$, $A_i \in V$ such that:
(A) $p_i \Vdash$ "$\underset{\sim}{\beta}_i \in A_i$ or $\lambda_i \notin \underset{\sim}{S}$"; and
(B) $p_i \geq p$, $p_i \geq p_j$ for $j < i$.

The play continues for δ moves.

In a specific play, player II wins iff $\{p\} \bigcup \{p_i : i < \delta\}$ has an upper bound (and loses otherwise). If a player has no legal move (this can occur to player II only) then he loses instantly.

We say that a player wins the game if he has a winning strategy.

3.2 Claim.

(1) At most one player can win the game $\partial^{\delta}_{\underset{\sim}{S}}(p, P)$.

(2) If for every $\lambda_i \in \underset{\sim}{S}$ and $\mu \in \mathrm{SCar}$, $\mu \leq \lambda_i \Rightarrow \mu \in \underset{\sim}{S}$, then in the definition of the game, it does not matter if we demand $|A_i| = 1$ (i.e., if one side has a winning strategy iff he has a winning strategy in the revised game).

(3) If μ_1 is regular, $\mu_1 < \mu_0$, δ divisible by μ_1 (and if $\mu_1 = \mu_2^+$ "δ divisible by $(\mu_2)^2$" suffice) and for every cardinal μ, $[\mu_1 \leq \mathrm{cf}\mu \leq \mu \leq \mu_0 \Rightarrow \mu \in \underset{\sim}{S}]$ *then* in the definition of the game, it does not matter if we demand, when $\lambda_i = \mu_0$, that $|A_i| < \mu_1$.

(4) Also we can replace λ_i by any set $B \in V$, $|B| = \lambda_i$. If λ_i is regular (even if only in V) we can demand $A_i \in \lambda_i$ (i.e., it is a proper initial segment).

(5) If for every regular μ satisfying $\aleph_0 \leq \mu \leq \lambda$ we have $\mu \in \underset{\sim}{S}$ and there is $n \in \underset{\sim}{S}$, $1 < n < \aleph_0$ and for every $p \in P$, player II does not lose in the game $\partial^{\delta}_{\underset{\sim}{S}}(p, P)$, *then* forcing by P does not introduce new δ-sequences from λ. (Usually $n = 2$; for $n > 2$ we have to work somewhat more in the proof.)

(6) If $n \in \underset{\sim}{S}$, $n < \omega$, adding $\{m : n < m \leq \aleph_0\}$ to $\underset{\sim}{S}$ does not change anything; also if $\mathrm{cf}(\lambda) \in \underset{\sim}{S}$ adding λ does not change anything.

(7) In Definition 3.1, if $\mathrm{cf}^V(\lambda) \in \underset{\sim}{S}$ we can add λ to $\underset{\sim}{S}$ with nothing being changed.

Proof. E.g.(3), player II can find a response in the revised game by playing $< \mu_1$ many moves in the original game, each time having a family $\mathcal{P}$ of $< \mu_1$ candidates, and for each $A \in \mathcal{P}$, if $\mathrm{cf}(|A|) \in [\mu_1, \mu_0]$ we replace it by a subset of smaller cardinal by one more, and if $\mathrm{cf}(|A|) < \mu_1$, we represent it as the union of $< \mu_1$ sets each of cardinality $< |A|$. In (6) (as well as in (2), (5)) just let player II use several moves to "answer" one question (if $m = \omega$ it is still finitely many though without an a priory bound). $\square_{3.2}$

3.3 Definition. The forcing P is $(\underset{\sim}{S}, \delta)$-complete if player II wins in the game $\partial^{\delta}_{\underset{\sim}{S}}(p, P)$ for every $p \in P$.

We define "P is $(\underset{\sim}{S}, < \beta)$-complete" similarly. P is pseudo κ-complete if it is $(\kappa^+ \cap \mathrm{SCar}^V, \mu)$-complete for every (cardinal) $\mu < \kappa$.

3.4 Lemma.

(1) If P is $|\delta|^+$-complete *then* it is (Car^V, δ)-complete.

(2) If P is $(\lambda^+ \cap \mathrm{SCar}^V, \delta)$-complete, $\delta \leq \lambda$, *then* forcing by P does not change the cofinality of any μ, $\aleph_0 < \mu \leq |\delta|$, and forcing by P does not add new δ-sequences from λ .

(3) In particular if P is $(\{2\}, \omega)$-complete (or even $(\{n\}, \omega)$-complete) *then* forcing by P does not add reals.

(4) If P is $(\underset{\sim}{S}, \omega)$-complete *then* P is $\underset{\sim}{S}$-semiproper.

(5) If P is (S_1, δ_1)-complete, *then* it is (S_2, δ_2)-complete provided that $(\forall \gamma \in S_2)(\exists \beta \in S_1)$ $[\mathrm{cf}(\gamma) = \beta$ or $\gamma = \beta]$ and $\delta_2 \leq \delta_1$.

(6) P is $(\underset{\sim}{S}, \delta)$-complete implies $(B^P \setminus \{0\}, \geq)$ is $(\underset{\sim}{S}, \delta)$-complete, ($B^P$ is the complete Boolean algebra corresponding to P). (See also 3.8.)

Proof. Easy. $\square_{3.4}$

3.5 Theorem.

(1) RCS-iteration strongly preserves $(\mathrm{S\underset{\sim}{C}ar}, \omega)$-completeness, and $(\mathrm{R\underset{\sim}{C}ar}, \omega)$--completeness and $(\mathrm{RU\underset{\sim}{C}ar}, \omega)$ -completeness. Moreover, if the assumption holds for the iteration $\bar{Q}$, $\bar{Q}$ has limit length, and λ is in the sets of cardinals mentioned above in each V^{P_i}, $i < \ell g\bar{Q}$, then it is so in V^{P_δ}.

(2) RCS-iteration strongly preserves (S, ω)-completeness for $S \subseteq \{2, \aleph_0, \aleph_1\}$, if we restrict ourselves to $\bar{Q}$'s satisfying $(\forall i < \ell g(\bar{Q}))[(\exists n) \Vdash_{P_{i+n}}$ "$|P_i| \leq \aleph_1$"$]$.

(3) The strong preservation in (2) holds even without the extra assumption.

3.5A Remark. Actually, we demanded in 3.1 that $\underset{\sim}{S}$ is a set of cardinals but, for example, $\mathrm{S\underset{\sim}{C}ar}$ is essentially $|P|^+ \cap \mathrm{S\underset{\sim}{C}ar}$.

3.5B Remark.

We can also imitate 2.6, and vice versa.

Proof. (1) We use Claim 2.5(1), so have to deal only with iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha\rangle$ where $\alpha = 2$ or $\alpha = \lambda$ a regular cardinal.

Let $\underset{\sim}{S}$ be any one of those three classes of cardinals, (remember, the meaning of our $\underset{\sim}{S}$ depends on which forcing it applies to say $\underset{\sim}{S} = \underset{\sim}{S}^P$, which we know by the game we are using) so $\partial^\omega_{\underset{\sim}{S}}(p, P)$ means $\partial^\omega_{\underset{\sim}{S}^P}(p, P)$.

Case A. $\alpha = 2$.

Let $p = (p_0, \underset{\sim}{p}_1) \in Q_0 * \underset{\sim}{Q}_1$, and let $F_0, \underset{\sim}{F}_1$ be the winning strategies of player II in $\partial^\omega_{\underset{\sim}{S}}(p_0, Q_0)$, $\partial^\omega_{\underset{\sim}{S}}(\underset{\sim}{p}_1, \underset{\sim}{Q}_1)$ respectively. By 3.2(4), we can assume $\underset{\sim}{F}_1$ gives us an ordinal or a member of $\{0, 1\}$ if the corresponding λ is regular or 2 respectively. The idea of the proof is that the output $\underset{\sim}{F}_1$ gives us, a Q_0-name for an ordinal, can be used as input for F_0.

Let in the i-th move player I choose λ_i and a P_2-name $\underset{\sim}{\beta}_i$ of an ordinal $< \lambda_i$, and player II choose $(p_{0,i}, \underset{\sim}{p}_{1,i}) \in P_2$, a P_1-name $\underset{\sim}{A}_{1,i}$, and a set $A_{0,i} \subseteq \lambda_i$ (Officially player II plays $(p_{0,1}, \underset{\sim}{p}_{1,i})$, $A_{0,i}$, and chooses $\underset{\sim}{A}_{1,i}$ for himself). Player II preserves the following property:

(∗)(a) $p_{0,i} \Vdash_{Q_0}$ "the following is an initial segment of the play of $\partial^\omega_{\underset{\sim}{S}}(\underset{\sim}{p}_1, \underset{\sim}{Q}_1)$ in which player II uses the strategy $\underset{\sim}{F}_1 : \langle \ldots, \langle \lambda_j, \underset{\sim}{\beta}_j\rangle, \langle \underset{\sim}{p}_{1,j}, \underset{\sim}{A}_{1,j}\rangle, \ldots\rangle_{j\le i}$".

(b) $p_{0,i} \Vdash_{Q_0}$ "$\underset{\sim}{A}_{1,i}$ is an ordinal $\underset{\sim}{\alpha}_i < \lambda_i$ if $\lambda_i \geq \aleph_0$ and a singleton $\{\underset{\sim}{\alpha}_i\} \subseteq \lambda_i$ if $\lambda_i = 2$ and $\underset{\sim}{A}_{1,i} \subseteq A_{0,i}$".

(c) $A_{0,i}$ is an ordinal $< \lambda_i$ if $\lambda_i \geq \aleph_0$ and a singleton $\subseteq 2$ if $\lambda_i = 2$.

(d) The following is an initial segment of a play of the game $\partial^\omega_{\underset{\sim}{S}}(p_0, Q_0)$ in which player II uses his winning strategy F_0: in the j-th move player I chooses

$\lambda_j, \underset{\sim}{\alpha}_j$ such that: $[\lambda_i \geq \aleph_0 \Rightarrow \underset{\sim}{A}_{1,i} = \alpha_i]$ and $[\lambda_i = 2 \Rightarrow A_{1,i} = \{\underset{\sim}{\alpha}_i\}]$ and player II chooses $p_{0,j}, A_{0,j}$.

It is easy to see that player II can do this and that it is a winning strategy.

Case B. $\alpha = \lambda$ a regular cardinal and $p \in P_\lambda$ and there are $\beta < \lambda, p^\dagger \in P_\beta$, $p{\restriction}\beta \leq p^\dagger$ such that $p^\dagger \Vdash_{P_\beta}$ "$\mathrm{cf}(\lambda) = \aleph_0$".

By the previous case, it suffices to prove that $P_\alpha/P_{\beta+1}$ is $(\underset{\sim}{S}, \omega)$-complete, so w.l.o.g. $\mathrm{cf}(\lambda) = \aleph_0$ and in fact $\lambda = \aleph_0$, and there are no problems. We leave the details as an exercise to the reader.

Case C. $\alpha = \lambda$ is regular and for every $\beta < \alpha$, $p{\restriction}\beta \Vdash_{P_\beta}$ "$\mathrm{cf}(\lambda) > \aleph_0$".

We will first give an informal sketch of II's strategy. We will also choose ξ_n, $0 = \xi_0 < \xi_1 < \ldots < \xi_n < \lambda$. After each move $(\lambda_n, \underset{\sim}{\alpha}_n)$ of player I, player II starts a new game $\eth_n = \eth^\omega_{\underset{\sim}{S}}\ [p_n{\restriction}[\xi_n, \xi_{n+1}), P_{\xi_{n+1}}/P_{\xi_n}]$, where $p_n{\restriction}[\xi_n, \alpha)$ is chosen such that it decides $\underset{\sim}{\alpha}_n$ up to a P_{ξ_n}-name $\underset{\sim}{\alpha}_{n,n}$. He then plays one step in each of the games $\eth_m$ $(m = n-1, \ldots, 0)$, simulating for I_m (i.e. first player in $\eth_m$) the move $\langle \lambda_n, \underset{\sim}{\alpha}_{n,m+1} \rangle$ and II_m answer $\langle p_n{\restriction}[\xi_m, \xi_{m+1}), \underset{\sim}{\alpha}_{n,m} \rangle$ (where $\underset{\sim}{\alpha}_{n,m}$ is a P_{ξ_m}-name) where we choose a constant winning strategy for II_m (it is a P_{ξ_m}-name) and player II answers in the true game in $\alpha_{n,0}$. The ξ_ℓ's must be big enough such that all the p_n's are eventually forced to be essentially in $\bigcup_{\ell<\omega} P_{\xi_\ell}$ (i.e. equivalent to a member). We only have to deal with countably many $\bar{Q}$-named ordinals, so we can take care of finitely many at each step n.

We now describe more formally the winning strategy of player II. By a hypothesis, for every non-limit $\beta < \gamma < \alpha$, and $r \in P_{\beta,\gamma} (= P_\gamma/P_\beta)$ player II has a winning strategy $\underset{\sim}{F}_{\beta,\gamma}(r)$ (a P_β-name) for winning the game $\eth^\omega_{\underset{\sim}{S}}(r, P_\gamma/P_\beta)$. We can change a little the rules of the game $\eth^\omega_{\underset{\sim}{S}}(r, P_\gamma/P_\beta)$, letting in stage n player I choose $k < \omega$ and a finite sequence $\langle \lambda^n_1, \underset{\sim}{\beta}^n_1, \ldots, \lambda^n_k, \underset{\sim}{\beta}^n_k \rangle$ ($\underset{\sim}{\beta}^n_\ell$ a P_γ/P_β-name of an ordinal $< \lambda^n_\ell$) and player II will choose $\alpha^n_1, \ldots, \alpha^n_k \in V^{P_\beta}$, and a condition $p_n \in P_\gamma/P_\beta$ satisfying $V^{P_\beta} \models$ "$\alpha^n_\ell < \lambda^n_\ell$", $p_n \Vdash_{P_\gamma/P_\beta}$ "*if* $\lambda^n_\ell \in \underset{\sim}{S}$ *then* $\underset{\sim}{\beta}^n_\ell < \alpha^n_\ell < \lambda^n_\ell$ when $\lambda^n_\ell \geq \omega$ and $\underset{\sim}{\beta}^n_\ell = \alpha^n_\ell$ when $\lambda^n_\ell = 2$" and $p_n \geq p, p_n \geq p_{n-1}$, (remember here $\underset{\sim}{S}$ is really a P_γ-name). Note: if player II wins the usual game he will win also the revised one.

Let for every $p \in P_\alpha$, $p = \{p^{[\ell]} : \ell < \omega\}, p^{[\ell]}$ a $\bar{Q}$-named condition.

Now player II's winning strategy uses some auxiliary games which he plays on the side. In stage n, player I chooses $\lambda_n, \underset{\sim}{\alpha}_n$ (a P_α-name of an ordinal $< \lambda_n$), but player II chooses not only p_n, A_n, but also a non-limit ordinal $\xi_n < \lambda$, and for $\ell \le n$ P_{ξ_ℓ}-names $\underset{\sim}{\alpha}_{n,\ell}$ of ordinals $< \lambda_n$ and for $k \le n$ also $\underset{\sim}{\beta}^n_k$ which is a P_{ξ_k}-name of ordinal $< \lambda$ such that:

a) $p \le p_n, p_{n-1} \le p_n$, $\xi_0 = 0$, $\xi_{n+1} = \mathrm{Max}\{\xi_n + 1, \underset{\sim}{\beta}^\ell_0 + 1 : \ell \le n\}$

b) $p_n\restriction[\xi_n, \alpha) \Vdash_{P_\alpha}$ "$\underset{\sim}{\alpha}_n = \underset{\sim}{\alpha}_{n,n}$, $\zeta(p^{[\ell]}_k)$ is $\le \underset{\sim}{\beta}^n_n < \lambda (= \alpha)$ (or undefined) for $\ell, k < n$ such that $\ell = n-1 \vee k = n-1$" where $\underset{\sim}{\alpha}_{n,n}$, $\underset{\sim}{\beta}^n_n$ are P_{ξ_n}-names.

c) for each $m < n$, the following is an initial segment of a play in the game $\underset{\sim}{G}^\omega_{\underset{\sim}{S}}(p_m\restriction[\xi_m, \xi_{m+1}), P_{\xi_{m+1}}/P_{\xi_m})$ in which player II uses his winning strategy $\underset{\sim}{F}_{\xi_m,\xi_{m+1}}$ $(p_m\restriction[\xi_m, \xi_{m+1}))$:

$$\Big\langle\langle\lambda_{m+1}, \underset{\sim}{\alpha}_{m+1,m+1}, \lambda, \underset{\sim}{\beta}^{m+1}_{m+1}\rangle, \langle p_{m+1}\restriction[\xi_m, \xi_{m+1}), \underset{\sim}{\alpha}_{m+1,m}, \underset{\sim}{\beta}^{m+1}_m\rangle, \dots,$$
$$\langle\lambda_n, \underset{\sim}{\alpha}_{n,m+1}, \lambda, \underset{\sim}{\beta}^n_{m+1}\rangle, \langle p_n\restriction[\xi_m, \xi_{m+1}), \underset{\sim}{\alpha}_{n,n}, \underset{\sim}{\beta}^n_m\rangle\Big\rangle$$

i.e.

$$\Big\langle(\langle\lambda_{m+i}, \underset{\sim}{\alpha}_{m+i,m+1}, \lambda, \underset{\sim}{\beta}^{m+i}_{m+1}\rangle, \langle p_{m+i}\restriction[\xi_m, \xi_{m+1}), \underset{\sim}{\alpha}_{m+i,m}, \underset{\sim}{\beta}^{m+i}_m\rangle) :$$
$$1 \le i \le n-m\Big\rangle.$$

d) Player II choice of A_n, is $A_n = \alpha_{n,0}$ if $\lambda_n \ge \omega$ and $A_n = \{\alpha_{n,0}\}$ if $\lambda_n = 2$.

Player II can carry out his strategy easily, defining in stage n, first ξ_n, second $p_n\restriction[\xi_n, \alpha)$, $\underset{\sim}{\alpha}_{n,n}$ and $\underset{\sim}{\beta}^n_n$, third he defined by downward induction on $m < n$, $p_n\restriction[\xi_m, \xi_{m+1}]$, $\alpha_{n,m}, \underset{\sim}{\beta}^n_m$, and fourth play as in d).

2) The proof is left to the reader. (Compare 2.8).

3) Combining the proofs of 2.6(4) and part (1). $\square_{3.5}$

3.6 Definition.

For a forcing notion P, a P-name $\underset{\sim}{S}$ of a set of cardinals, an ordinal δ and a condition p we define the games $\mathrm{E}\eth^\delta_{\underset{\sim}{S}}(p, P)$, $\mathrm{R}\eth^\delta_{\underset{\sim}{S}}(p, P)$

(or $\mathrm{E}\eth^\delta(p, P, \underset{\sim}{S}), \mathrm{R}\eth^\delta(p, P, \underset{\sim}{S})$ respectively; E stands for essentially, R for really).

(1) In a play of the game $\mathrm{E}\eth^{\delta}_{\underset{\sim}{S}}(p, P)$ in the i-th move, player I chooses a cardinal λ_i and a P-name $\underset{\sim}{\beta}_i$ of an ordinal $< \lambda_i$ and player II has to find a set $A_i \subseteq \lambda_i$, $|A_i| < \lambda_i$, $(A_i \in V)$.

The play continues for δ moves. In the end player II wins if he can find a condition $p^\dagger \in P$, $p \leq p^\dagger$ such that for every $i < \delta$, $p^\dagger \Vdash_P$ "$\underset{\sim}{\beta}_i \in A_i$, or $\lambda_i \notin \underset{\sim}{S}$".

(2) In a play of the game $\mathrm{R}\eth^{\delta}_{\underset{\sim}{S}}(p, P)$ in the i-th move, player I chooses a condition q_i, $q_i \geq p_j$ for every $j < i$ and $q_i \geq p$, and a cardinal λ_i and a P-name $\underset{\sim}{\beta}_i$ of an ordinal $< \lambda_i$ and player II has to find a condition p_i and a set $A_i \subseteq \lambda_i$, $|A_i| < \lambda_i$, $(A_i \in V)$ such that

(A) $p_i \Vdash_P$ "$\underset{\sim}{\beta}_i \in A_i$ or $\lambda_i \notin \underset{\sim}{S}$",

(B) $p_i \geq q_i$.

The play continues for δ moves, and player II wins if $\{p\} \bigcup \{p_i : i < \delta\}$ has an upper bound.

Note: 3.6(1) is close to 3.1, 3.6(2) is stronger. Comparing Definition 3.6(2) with XIV Definition 2.1, the definition here is stronger when $\delta > \omega$.

3.7 Definition. The forcing P is essentially $(\underset{\sim}{S}, \delta)$-complete [really $(\underset{\sim}{S}, \alpha)$-complete] if player II wins in the game $\mathrm{E}\eth^{\delta}_{\underset{\sim}{S}}(p, P)$ [$\mathrm{R}\eth^{\delta}_{\underset{\sim}{S}}(p, P)$] for every $p \in P$.

3.8 Lemma.

(1) The parallels of 3.2, 3.4 hold.

(2) Let P be a forcing, B the corresponding Boolean algebra. Then P is essentially $(\underset{\sim}{S}, \alpha)$-complete iff $(B^P \setminus \{0\}, \geq)$ is $(\underset{\sim}{S}, \alpha)$-complete; and if $\alpha \geq \omega$, this implies P is $\underset{\sim}{S}$-semiproper. If P is complete (i.e. for any $\mathcal{I} \subseteq P$ there is p such that every $G \subseteq P$ generic over V: $p \in G$ iff $\mathcal{I} \cap G \neq \emptyset$ and $(\forall q \in \mathcal{I})(q \leq p)$) then P is $(\underset{\sim}{S}, \alpha)$-complete iff P is essentially $(\underset{\sim}{S}, \alpha)$-complete. If P is really $(\underset{\sim}{S}, \alpha)$-complete then P is $(\underset{\sim}{S}, \alpha)$-complete which implies essentially $(\underset{\sim}{S}, \alpha)$-complete.

Proof. Easy. $\square_{3.8}$

3.9 Theorem. (1) RCS-iteration strongly preserves the notions "essential (S, ω)-completeness" for $S \in \{\mathrm{S\underset{\sim}{C}ar}, \mathrm{R\underset{\sim}{C}ar}, \mathrm{RU\underset{\sim}{C}ar}\}$. Similarly for "real (S, ω)-completeness.

(2) Moreover, if the assumption holds for the iteration $\bar{Q}$, $\bar{Q}$ has limit length, and the cofinality of λ is in the set of cardinals mentioned above in each V^{P_i}, $i < \ell g\bar{Q}$, then it is in V^{P_δ}.

(3) RCS-iteration strongly preserves essential (S, ω)-completeness for $S \subseteq \{2, \aleph_0, \aleph_1\}$, if we restrict ourselves to $\bar{Q}$'s satisfying

$$(\forall i < \ell g(\bar{Q}))[(\exists n) \Vdash_{P_{i+n}} \text{“}|P_i| \leq \aleph_1\text{”}]$$

(or even without it).

Proof. Similar to previous ones. $\square_{3.9}$

3.10 Definition. For $\mathbf{W} \subseteq \omega_1$ we call a forcing notion pseudo $(*, \mathbf{W})$-complete if for each $p \in P$ in the following game player I has a winning strategy. The play lasts ω moves. *In the n'th move:* player I chooses an ordinal $\alpha_n < \omega_1$ such that $\bigwedge_{\ell<n} \beta_\ell < \alpha_n$ and a P-name $\underset{\sim}{\tau}_n$ of a countable ordinal. Player II chooses ordinals $\beta_n, \gamma_n < \omega_1$ such that $\alpha_n < \beta_n$, $\bigwedge_{l<n} \beta_l < \beta_n$. In the end player II wins the play iff (a) or (b) where

(a) $\bigcup_{n<\omega} \alpha_n \notin \mathbf{W}$.

(b) there is $q \in P$ satisfying: $p \leq q$ and $q \Vdash_P$ "$\underset{\sim}{\tau}_n = \underset{\sim}{\gamma}_n$ for $n < \omega$".

3.10A Remark. We can define games and completeness variations of the earlier notions in this section with length of game ω with a stationary $\mathbf{W} \subseteq \omega_1$ as a parameter as we have done to $(\{2\}, \omega)$ -completeness in 3.10 and the parallel theorems hold.

3.11 Claim. (1) Pseudo $(*, \mathbf{W})$-completeness is strongly preserved by RCS-iteration.

(2) If $\mathbf{W}$ is stationary (subset of ω_1) and P is $(*, \mathbf{W})$-complete *then* forcing with P preserves stationarity of subsets of $\mathbf{W}$ and adds no real.

3) If $\omega_1 \setminus \mathbf{W}$ is not stationary, P is pseudo $(*, \mathbf{W})$-complete *then* P is essentially $(\{\aleph_1\}, \omega)$-complete.

Proof. Left to the reader. $\square_{3.11}$

§4. Specific Forcings

We prove here for various forcings that they are semiproper and even (S, δ)-complete; of course, otherwise our previous framework will be empty. See [J], chapters 5-6, for a discussion of some of the large cardinals we use (which are standard).

Prikry forcing (adding an unbounded ω-sequence to a measurable cardinal without adding bounded subsets) satisfies all we can expect. But for our purposes, more important are forcings which change the cofinality of $\aleph_2$ to $\aleph_0$, without adding reals (or at least not collapsing $\aleph_1$). Namba [Nm] has found such a forcing, when CH holds.

However we do not know the answer to:

Problem. Is Namba forcing $\{\aleph_1\}$-semiproper? (But see XII §2).

However, Namba forcing is not necessarily $(\{2\}, \omega)$-complete; this is equivalent to "$\mathcal{D}^{\mathrm{cb}}_{\aleph_2}$ is Galvin" (see below).

We deal with a variant of Namba forcing, (for the original see XI 4.1), $\mathrm{Nm}'(\mathfrak{D})$ ($\mathfrak{D}$ a system of filters on sets of power $\aleph_2$, see below), and prove the relevant assertion (4.7). Then we prove that if each filter in $\mathfrak{D}$ has the $(\{2, \aleph_0, \aleph_1, \}, \omega)$-Galvin property (see 4.9, 4.9A), *then* $\mathrm{Nm}'(\mathfrak{D})$ is semiproper, moreover is $(\{\aleph_0, \aleph_1, 2\}, \omega)$-complete. The point is that when a large cardinal is collapsed to $\aleph_2$, if D was originally a normal ultrafilter, then after the collapse it may well have some largeness property like the Galvin property.

4.1 Definition. If D is a complete normal ultrafilter on κ, then the D-Prikry forcing, $PF(D)$, is:

$\{(f, A) : f$ a function, with domain $n < \omega, f$ is increasing, $(\forall i < n) f(i) < \kappa$, and A belongs to $D\}$.

$(f_1, A_1) \leq (f_2, A_2)$ iff $f_1 \subseteq f_2$, $A_1 \supseteq A_2$, and for $i \in \mathrm{Dom}(f_2) \backslash \mathrm{Dom}(f_1), f_2(i) \in A_1$.

Prikry defined this notion and proved [Pr] in fact that:

4.2 Theorem. For any normal ultrafilter D over κ, $P = PF(D)$ is $(\mathrm{RCar}^P, \lambda)$-complete for every $\lambda < \kappa$, and changes the cofinality of only one cardinal, κ (to $\aleph_0$). (So remembering the notation introduced before 3.2, $(\mathrm{RCar}^P,\ \lambda)$-complete really means $(\mathrm{Car} \setminus \{\kappa\}, \lambda)$-complete).

4.3 Definition. (1) A filter-tagged tree is a pair $(T, \mathfrak{D})$ such that:

(a) T is a nonempty set of finite sequences of ordinals, closed under taking initial segments, and there is some maximal $\eta_0 \in T$ for which $[\nu \in T$, $\ell g(\nu) \leq \ell g(\eta_0) \Rightarrow \nu = \eta_0 \restriction \ell g(\nu)]$; we call η_0 the trunk of T, $\eta_0 = tr(T)$.

(b) $\mathfrak{D}$ is a function such that for every $\eta \in T$, $\mathfrak{D}_\eta = \mathfrak{D}(\eta)$ is a filter on some set $\subseteq \{\eta\char`^\langle\alpha\rangle : \alpha$ an ordinal$\}$ and if $\mathrm{tr}(T) \trianglelefteq \eta \in T$ *then* $\mathrm{Suc}_T(\eta) \stackrel{\mathrm{def}}{=} \{\nu \in T : \ell g(\nu) = \ell g(\eta) + 1, \nu \restriction \ell g(\eta) = \eta\} \neq \emptyset \operatorname{mod} \mathfrak{D}_\eta$.

(2) We call $(T, \mathfrak{D})$ normal if $\mathrm{Dom}(\mathfrak{D}) = \{\eta \in T : \mathrm{tr}(T) \trianglelefteq \eta\}$ and for every such η, $\mathfrak{D}_\eta$ is a filter over $\mathrm{Suc}_T(\eta)$ (see below). For $\eta \in T$, $(T, \mathfrak{D})_{[\eta]} = (T_{[\eta]}, \mathfrak{D}) \stackrel{\mathrm{def}}{=} (\{\nu \in T : \nu \trianglelefteq \eta$ or $\eta \trianglelefteq \nu\}, \mathfrak{D})$.

(3) We call $(T, \mathfrak{D})$ λ-complete if each $\mathfrak{D}_\eta (\eta \in T)$ is λ -complete.

4.4 Definition. For filter-tagged trees $(T_1, \mathfrak{D}_1)$, $(T_2, \mathfrak{D}_2)$:

(1) We define: $(T_1, \mathfrak{D}_1) \leq (T_2, \mathfrak{D}_2)$ iff

(a) $T_2 \subseteq T_1$,

(b) For every $\eta \in T_2$, if $\eta \trianglerighteq \mathrm{tr}(T_2)$ then $\mathrm{Suc}_{T_2}(\eta) \neq \emptyset \operatorname{mod} \mathfrak{D}_1(\eta)$ and $\mathfrak{D}_1(\eta) \restriction \mathrm{Suc}_{T_2}(\eta) = \mathfrak{D}_2(\eta) \restriction \mathrm{Suc}_{T_2}(\eta)$ where for a filter D over I, and $J \subseteq I$, $J \neq \emptyset \operatorname{mod} D$ we let:

$$D \restriction J = \{A \cap J : A \in D\}.$$

(2) We define: $(T_1, \mathfrak{D}_1) \leq_{pr} (T_2, \mathfrak{D}_2)$ ("pure extension") if in addition $\mathrm{tr}(T_1) = \mathrm{tr}(T_2)$.

(3) We define: $(T_1, \mathfrak{D}_1) \leq_n (T_2, \mathfrak{D}_2)$ if in addition (to (2)) for η of length $\leq n$, $\eta \in T_1 \Leftrightarrow \eta \in T_2$.

(4) $\mathrm{Nm}'(T^*, \mathfrak{D}^*) \stackrel{\text{def}}{=} \{(T, \mathfrak{D}) : (T^*, \mathfrak{D}^*) \leq (T, \mathfrak{D})\}$ ordered by $\leq$. We write $\eta \in (T, \mathfrak{D})$ for $\eta \in T$. If $p = (T^*, \mathfrak{D}^*)$ we write T_p for T^*, $\mathfrak{D}_p$ for $\mathfrak{D}^*$. Instead of T_p, $T_{p'}$, T_{p_1}, T_{p^k}, etc, we usually write just T, T', T_1, T^k, etc.

4.4A Remark. For every filter-tagged tree $(T, \mathfrak{D})$ for a unique normal $(T, \mathfrak{D}^\dagger)$ we have $(T, \mathfrak{D}) \leq (T, \mathfrak{D}^\dagger) \leq (T, \mathfrak{D})$.
2)So we can restrict ourselves to normal members of $\mathrm{Nm}'(T^*, \mathfrak{D}^*)$.

4.5 Claim. 1) If $\eta \in T \Rightarrow |\mathrm{Suc}_T(\eta)| \leq \aleph_2$ then $\mathrm{Nm}'(T, \mathfrak{D})$ is $(\{\lambda : \lambda = \mathrm{cf}\lambda > \aleph_2\}, \omega)$-complete.
2) Moreover, in the cases where we shall prove that Nm' is (S, ω)-complete, $S \subseteq \{2, \aleph_0, \aleph_1\}$, we could prove it is $(S \cup \{\lambda : \lambda = \mathrm{cf}\lambda > \aleph_2\}, \omega)$-complete (see 4.12).

Proof of 1). It is enough to prove:
$(*)$ if $p \in P = \mathrm{Nm}'(T, \mathfrak{D})$, $n < \omega$, $\underset{\sim}{\tau}$ a P-name of an ordinal, *then* there is $q \in P$, $p \leq_n q$ and a set A of ordinals, $|A| \leq \aleph_2$, $q \Vdash$ "$\underset{\sim}{\tau} \in A$".

Proof of $(*)$: Let
$T_0^* = \{\eta \in p : \ell g(\eta) \geq n, \mathrm{tr}(p) \trianglelefteq \eta$, and $(p)_{[\eta]}$ has a pure extension deciding the value of $\underset{\sim}{\tau}\}$,
$T_1^* = \{\eta \in T_0^*$: there is no $\nu \triangleleft \eta$, $\nu \in T_0^*\}$.

4.5A Subfact. T_1^* is a front of r for some r satisfying $p \leq_n r$; i.e every ω-branch of r contains one and only one element of T_1^*.

Proof of the Subfact. Clearly without loss of generality $\mathrm{tr}(p)$ has the length $\geq n$. By a partition theorem in [RuSh:117] (or see here XI 3.5 or XV 2.6B(2), and if CH see 4.6 below) there is $r \in \mathrm{Nm}'(T, \mathfrak{D})$, $p \leq_{pr} r$, such that:

either (a) for every $\eta \in \lim(r)$, $(\exists n)$ $[\eta \restriction n \in T_1^*]$

or (b) for no $\eta \in \lim(r)$, $(\exists n)$ $[\eta \restriction n \in T_1^*]$.

If (b) holds, then we can find p' and γ such that: $r \le p'$ and $p' \Vdash$ "$\underset{\sim}{\tau} = \gamma$". But then let $\nu \in p'$, $\ell g \nu \ge n$, $\ell g(\mathrm{tr}(p'))$. Then $\nu \in T_0^*$ (witnessed by $(p')_{[\nu]}$) hence for some $k \le n$, $\nu \restriction k \in T_1^*$. But this is a contradiction to (b), as $\nu \in r$. Hence (a) holds, hence T_1^* is a front of r, and $p \le_n r$ because $p \le_{\mathrm{pr}} r$ and $\ell g(\mathrm{tr}(p)) \ge n$. $\square_{4.5A}$

Continuation of the proof of 4.5: Let, for $\nu \in T_1^*$, q^ν be a pure extension of $(r)_{[\nu]}$ satisfying

$$q^\nu \Vdash \text{“}\underset{\sim}{\tau} = \gamma^\nu\text{”}.$$

Then $q = \cup\{q^\nu : \nu \in T_1^*\}$ is a condition (i.e. $T_q = \bigcup_{\nu \in T_1^*} T_{q^\nu}$ and $\mathfrak{D}_q = \mathfrak{D}_p$) such that $p \le_n r \le_n q$ and

$$q \Vdash \text{“}\underset{\sim}{\tau} \in \{\gamma^\nu : \nu \in T_1^*\}.$$

So $(*)$ is proved.

2) Check the proof of part (1). $\square_{4.5}$

4.5B Remark.1) In 4.5(1), (2) we can replace $\aleph_2$ by any $\mu > \aleph_2$.

2) As in the proof of 4.5(1) we prove $(*)$ we can (e.g. in 5.5) use the preservation of RUCar-properness (3.5(1)) instead of 3.5(2).

4.6 Lemma. If $(T, \mathfrak{D})$ is a filter-tagged tree, which is λ^+-complete (i.e., each $\mathfrak{D}_\eta$ is a λ^+-complete filter) and $H : T \to \lambda$ and $\lambda^{\aleph_0} = \lambda$, *then* there is $(T^\dagger, \mathfrak{D}^\dagger)$, $(T, \mathfrak{D}) \le_{\mathrm{pr}} (T^\dagger, \mathfrak{D})$ such that $H(\eta)$ depends only on $\ell g(\eta)$, for $\eta \in T^\dagger$.

Remark. See Rubin and Shelah [RuSh:117] p. 47 – 48 on the history of this and such theorems there.

Proof. For any sequence $\bar\alpha = \langle \alpha_n : n < \omega \rangle$, $\alpha_n < \lambda$, we define a game $\partial_{\bar\alpha}$:

Let η_0 be the trunk of T.

In move 0 player I chooses $A_1 \subseteq \mathrm{Suc}_T(\eta_0)$, $A_1 = \emptyset \bmod \mathfrak{D}_{\eta_0}$, and player II chooses $\eta_1 \in \mathrm{Suc}_T(\eta_0) \setminus A_1$.

In move n, player I chooses $A_{n+1} \subseteq \mathrm{Suc}_T(\eta_n)$, $A_{n+1} = \emptyset \operatorname{mod} \mathfrak{D}_{\eta_n}$ and player II chooses $\eta_{n+1} \in \mathrm{Suc}_T(\eta_n) \setminus A_{n+1}$.

In the end, player II wins the play if for every n we have $H(\eta_n) = \alpha_n$. Now we prove:

(∗) For some $\bar{\alpha} = \langle \alpha_n : n < \omega \rangle$, $\alpha_n < \lambda$, player II wins the game (i.e., has a winning strategy).

Clearly the game is closed, hence it suffices to prove that for some $\bar{\alpha}$, player I does not have a winning strategy. So assume that for every $\bar{\alpha}$ player I has a winning strategy $F_{\bar{\alpha}}$ in the game $\partial_{\bar{\alpha}}$, and we shall get a contradiction. A winning strategy is a function which, given the previous moves of the opponent ($\eta_1, \dots, \eta_{n-1}$ in our case), gives a move to the player, so that in any play in which he uses the strategy he wins the play.

Now define by induction on n, $\eta_n \in T$ such that $\ell g(\eta_n) = \ell g(\mathrm{tr} T) + n$ and $\eta_{n+1} \restriction n = \eta_n$:

η_0 is the trunk of T

$\eta_{n+1} \in \mathrm{Suc}_T(\eta_n) \setminus \bigcup_{\bar{\alpha}} F_{\bar{\alpha}}(\langle \eta_1, \dots, \eta_n \rangle)$.

Why does η_{n+1} exist? For every $\bar{\alpha}$, $F_{\bar{\alpha}}(\langle \eta_1, \dots, \eta_n \rangle) = \emptyset \operatorname{mod} \mathfrak{D}_{\eta_n}$, $\mathfrak{D}_{\eta_n}$ is λ^+-complete and the number of $\bar{\alpha}$'s is $\lambda^{\aleph_0} = \lambda < \lambda^+$. So $\bigcup_{\bar{\alpha}} F_{\bar{\alpha}}(\langle \eta_1, \dots, \eta_n \rangle) = \emptyset \operatorname{mod} \mathfrak{D}_{\eta_n}$, and so η_{n+1} exists as $\mathrm{Suc}_T(\eta_n) \neq \operatorname{mod} \mathfrak{D}_{\eta_n}$ by Definition 4.3(1) clause (b).

But let $\alpha_n^* \stackrel{\text{def}}{=} H(\eta_n)$ and $\bar{\alpha}^* = \langle \alpha_n^* : n < \omega \rangle$, so

$$F_{\bar{\alpha}^*}(<>), \eta_1, \dots, F_{\bar{\alpha}^*}(\langle \eta_1, \dots, \eta_n \rangle), \eta_{n+1}, \dots$$

is a play of $\partial_{\bar{\alpha}^*}$ in which player I uses his strategy $F_{\bar{\alpha}^*}$, but he lost: contradiction, hence (∗) holds.

Proof of the Lemma from (∗). Let $\langle \alpha_n : n < \omega \rangle$ be as in (∗), and W be the winning strategy of player II.

Let $T_0 = \{\eta \in T$: for some n, we have: $\ell g(\eta) = \ell g(\eta_0) + n$, and for some $A_1, \dots, A_n$, for every $0 < \ell \le n$ we have $\eta \restriction (\ell g \eta_0 + \ell) = W(\langle A_1, \dots, A_\ell \rangle)\} \cup \{\eta_0 \restriction \ell : \ell \le \ell g \eta_0\}$.

It is clear that T_0 is closed under initial segments. Now if $\eta \in T_0$, $\eta_0 \trianglelefteq \eta \in T_0$ then $\mathrm{Suc}_{T_0}(\eta) \neq \bmod \mathfrak{D}_\eta$, for otherwise if $n = \ell g(\eta) - \ell g(\eta_0)$, and $A_1, \dots, A_n$ are "witnesses for $\eta \in T_0$", then player I could have chosen $A_{n+1} = \mathrm{Suc}_{T_0}(\eta)$, and then by definition $W(A_1, \dots, A_{n+1}) \in T_0$ and also $W(A_1, \dots, A_{n+1}) \notin \mathrm{Suc}_{T_0}(\eta) = A_{n+1}$ but $W(A_1, \dots, A_{n+1}) \in \mathrm{Suc}_T(\eta)$ and $\mathrm{Suc}_{T_0}(\eta) = T_0 \cap \mathrm{Suc}_T(\eta)$, contradiction.

So $(T_0, \mathfrak{D}) \le_{pr} (T, \mathfrak{D})$ and $(T, \mathfrak{D})$ is as required. $\square_{4.6}$

4.7 Theorem. Suppose $(T^*, \mathfrak{D}^*)$ is an $\aleph_2$-complete filter-tagged tree. Let $P = \mathrm{Nm}'(T^*, \mathfrak{D}^*)$ *then*

(1) (CH) P does not add reals.

(2) If for every $(T, \mathfrak{D}^*) \in P$ for some $\eta \in T, \mathrm{tr}(T) \trianglelefteq \eta$ and for some $A \subseteq \mathrm{Suc}_\eta(T)$ and function $F : A \to \lambda$ we have $(\forall \alpha < \lambda)[F^{-1}(\{i : i < \alpha\}) \equiv \emptyset \bmod \mathfrak{D}^*_\eta]$ and $A \neq \emptyset \bmod \mathfrak{D}^*_\eta$ *then* $\Vdash_P$ "$\mathrm{cf}(\lambda) = \aleph_0$".

(3) P does not collapse $\aleph_1$ (and if $\mathfrak{D}^*$ is λ^+-complete, $\mathrm{cf}(\lambda) > \aleph_0$ then $\Vdash_P$ "$\mathrm{cf}(\lambda) > \aleph_0$").

4.7A Remark. If we waive CH, P may add reals but it does not collapse $\aleph_1$; sometimes it satisfies the $\aleph_4$-c.c. even though $2^{\aleph_1} > \aleph_4$ (see XI 4.3).

4.7B Notation. If $\mathrm{Dom}(\mathfrak{D}^*) = T$ let $\mathrm{Nm}'(\mathfrak{D}^*) = \mathrm{Nm}'(T, \mathfrak{D}^*)$, and if $T = {}^{\omega >}(\omega_2)$, $\mathfrak{D}^*(\eta) = \{\{\eta\hat{\ } < \alpha >: \alpha \in A\} : A \in D\}$, we let $\mathrm{Nm}'(T, D) = \mathrm{Nm}'(D) = \mathrm{Nm}'(\mathfrak{D}^*)$.

4.7C Remark. So if $P = \mathrm{Nm}'(D^*)$, $D^* \supseteq \mathcal{D}^{cb}_{\aleph_2} (\overset{\mathrm{def}}{=} \{A \subseteq \aleph_2 : A$ co-bounded$\})$, $G \subseteq P$ is generic, *then* $\bigcup \{\eta : \eta \in (T, D^*)$ for every $(T, D^*) \in G\}$ is a member of ${}^{\omega}(\omega_2)$ (in $V[G]$) and as $D^* \supseteq \mathcal{D}^{cb}_{\aleph_0}$, it is unbounded in ω_2 so $\Vdash_{\mathrm{Nm}'(D^*)}$ "$\mathrm{cf}(\aleph_2^V) = \aleph_0$".

Proof of 4.7.

(1) Now suppose $\underset{\sim}{\tau}$ is a name of an ω-sequence from ω_1, and let $(T, \mathfrak{D}^*) \in P$. It is easy to define by induction $(T_n, \mathfrak{D}^*)$ such that:

(a) $(T_0, \mathfrak{D}^*) = (T, \mathfrak{D}^*)$,

(b) $(T_n, \mathfrak{D}^*) \leq_n (T_{n+1}, \mathfrak{D}^*)$ and $(T_n, \mathfrak{D}^*) \leq_{pr} (T_{n+1}, \mathfrak{D}^*)$,

(c) for every $\eta \in T_{n+1}$, if $\ell g(\eta) = n+1$, then for some $\bar{\alpha}_\eta$ and $\ell \leq n$ we have: $(T_{n+1}, \mathfrak{D}^*)_{[\eta]} \Vdash_P$ "$\underset{\sim}{\tau}\restriction\ell = \bar{\alpha}_\eta$", and ℓ is maximal, i.e., either $\ell = n$, or there are no $T^\dagger, \alpha$ such that $\alpha < \omega_1$ and $(T^\dagger, \mathfrak{D}^*) \Vdash$ "$\underset{\sim}{\tau}(\ell) = \alpha$" and $(T_{n+1}, \mathfrak{D}^*)_{[\eta]} \leq_{pr} (T^\dagger, \mathfrak{D}^*)$.

Clearly $(\bigcap_{n<\omega} T_n, \mathfrak{D}^*) \in P$ and $(T_n, \mathfrak{D}^*) \leq (\bigcap_{n<\omega} T_n, \mathfrak{D}^*)$.

Now use Lemma 4.6 on $(\bigcap_{n<\omega} T_n, \mathfrak{D}^*)$, and $H, H(\eta) = \bar{\alpha}_\eta$ and get $(T^\dagger, \mathfrak{D}^*)$, $(\cap_{n<\omega} T_n, \mathfrak{D}^*) \leq_{pr} (T^\dagger, \mathfrak{D}^*)$, $H(\eta) = \bar{\alpha}^n$ for $\eta \in T^*$, $\ell g(\eta) = n+1$. Now for each ℓ, there is $(T'', \mathfrak{D}^*)$, $(T^\dagger, \mathfrak{D}^*) \leq (T'', \mathfrak{D}^*)$ and $\bar{\alpha}$ such that $(T'', \mathfrak{D}^*) \Vdash_P$ "$\underset{\sim}{\tau}\restriction\ell = \bar{\alpha}$", and let $\eta_0 \in T''$ be the trunk of T''; w.l.o.g. $\ell + 1 < \ell g(\eta_0)$. By the choice of $\bar{\alpha}_{\eta_0}, \ell \leq \ell g(\bar{\alpha}_{\eta_0})$ hence $\bar{\alpha} = \bar{\alpha}^k\restriction\ell$ for $k = \ell g(\eta_0)$, hence for every $\eta \in T^\dagger$, $\ell g(\eta) = \ell g(\eta_0)$ implies $\bar{\alpha}_\eta\restriction\ell = \bar{\alpha}_{\eta_0}\restriction\ell$, hence $(T^\dagger, \mathfrak{D}^*)_{[\eta]} \Vdash$ "$\underset{\sim}{\tau}\restriction\ell = \bar{\alpha}_{\eta_0}\restriction\ell = \bar{\alpha}^\ell$". But $(T^\dagger, \mathfrak{D}^*) \Vdash$ " for some $\eta \in T^\dagger$, $\ell g(\eta) = \ell$ and $(T^\dagger, \mathfrak{D}^*)_{[\eta]}$ belongs to $\underset{\sim}{G}_P$ (the generic subset of P)." So clearly $(T^\dagger, \mathfrak{D}^*) \Vdash$ "$\underset{\sim}{\tau}\restriction\ell = \bar{\alpha}^{\ell g(\eta_0)}\restriction\ell$", and as this holds for every ℓ we have $(T^\dagger, \mathfrak{D}^*) \Vdash$ "$\underset{\sim}{\tau} = \langle \bar{\alpha}^{m(n)}(n) : n < \omega\rangle$" when we choose the numbers $m(n)$ large enough, i.e., such that $n < \ell g(\bar{\alpha}^{m(n)})$.

(2) Clearly the following is a dense open subset of P, $\mathcal{I}_0 = \{(T, \mathfrak{D}) : (T, \mathfrak{D}) \in P$ and for every $\eta \in T$, *if* there are $A_i \subseteq \mathrm{Suc}_T(\eta)$, for $i < \lambda$, $\bigcup_{j<i} A_j = \emptyset \bmod \mathfrak{D}_\eta$, $\bigcup_{i<\lambda} A_i \neq \emptyset \bmod \mathfrak{D}_\eta$ *then* there is $F_\eta : \mathrm{Suc}_T(\eta) \to \lambda$ such that $\bigwedge_{\alpha<\lambda}\{\nu : \nu \in \mathrm{Suc}_T(\eta), F(\nu) < \alpha\} = \emptyset \bmod \mathfrak{D}_\eta\}$. Now for each $(T, \mathfrak{D}) \in \mathcal{I}_0$ let $B(T, \mathfrak{D}) = \{\eta \in T :$ there is F_η as above$\}$. Note: $(T_1, \mathfrak{D}) \leq (T_2, \mathfrak{D}) \Rightarrow B(T_1, \mathfrak{D}) \cap T_2 = B(T_2, \mathfrak{D})$; by [RuSh:117] or XI 3.5 or XV 2.6B(2) here, for every $(T, \mathfrak{D}) \in \mathcal{I}_0$ there is $(T', \mathfrak{D})$, $(T, \mathfrak{D}) \leq_{\mathrm{pr}} (T', \mathfrak{D})$ such that:

(a) $(\forall \eta \in \lim T')(\exists^{\aleph_0} n)[\eta\restriction n \in B(T, \mathfrak{D})]$, or

(b) $(\forall \eta \in \lim T')(\exists^{<\aleph_0} n)[\eta\restriction n \in B(T, \mathfrak{D})]$, In the second case, applying again the partition theorem mentioned above we get a constant bound n to $\{\ell g(\eta) : \eta \in B(T, \mathfrak{D})\}$, and increasing the trunk contradict the hypothesis (of 4.7(2)). So $\mathcal{I}_1 = \{(T, \mathfrak{D}) \in \mathcal{I}_0$ and (a) holds$\}$ is dense open subset of P. Fix $(T, \mathfrak{D}) \in \mathcal{I}_1$. For $\eta \in B(T, \mathfrak{D})$, let F_η be as required above; and let $\underset{\sim}{\tau}$ be the unique ω-sequence such that for every $p \in \underset{\sim}{G}_P$, and $n < \omega$, $\underset{\sim}{\tau}{\restriction}n \in p$. Then $\lambda \cap \{F_{\underset{\sim}{\tau}\restriction n}(\underset{\sim}{\tau}(n)) : n < \omega$ (and $F_{\underset{\sim}{\tau}\restriction n}$ is well defined) $\}$ is a countable unbounded subset of λ.

3) Similar to part (1) using XI 3.7 instead of 4.6 (but not used here). $\square_{4.7}$

4.8 Problem. Is the forcing semiproper? (See XII.)

4.9 Definition. For a filter D on a set I, and a set S of cardinals, we call D an (S, α)-Galvin filter (and the dual ideal a Galvin ideal) if player II has a winning strategy in the following game, for every $J \subseteq I$, $J \neq \emptyset \operatorname{mod} D$ (we call the game the (S, α)-Galvin game for (D, J)):

In the ith move player I chooses a function F_i from I to some $\lambda \in S$ and player II chooses $A_i \subseteq J \cap \bigcap_{j<i} A_j$ such that $|F_i(A_i)| < \lambda$. Player II wins if $\bigcap_{i<\alpha} A_i \neq \emptyset \operatorname{mod} D$. For simplicity we can say J was chosen by player I in his first move.

4.9A Remark. Galvin suggests this game for $D^{cb}_{\omega_2}$ = the co-bounded subset of κ for a cardinal κ, $\alpha = \omega$ and $S = \{2\}$. So for $\alpha = \omega$, $S = \{2\}$ we omit (S, α). Note that only $S \cap (|I| + 1)$ is relevant for the game.

4.10 Definition. A filter D on κ has the Laver (or $\aleph_1$-Laver) property, if there is a family W of subsets of κ, $A \in W \Rightarrow A \neq \emptyset \operatorname{mod} D, W$ is dense [i.e. $\forall A \subseteq \kappa$, $A \neq \emptyset \operatorname{mod} D \rightarrow (\exists B \in W)(B \subseteq A \operatorname{mod} D)$], and W is closed with respect to countable intersections of descending chains.

Related to this property is the following game:

In the n'th move, player I chooses a set A_n and player II chooses a set B_n, such that for all n $A_n \supseteq B_n \supseteq A_{n+1} \neq \emptyset \operatorname{mod} D$, II wins iff $\bigcap_{n<\omega} A_n \neq \emptyset \operatorname{mod} D$.

Clearly, if D has the Laver property, then player II wins.

Galvin, Jech and Magidor [GJM] and Laver independently proved the following.

4.11 Theorem. If we start with a universe V, $V \models$ "$G.C.H. + \kappa$ is measurable" and use Levy collapsing of κ to $\aleph_2$ (so every λ, $\aleph_1 \le \lambda < \kappa$ now will have cardinality $\aleph_1$) *then* in the new universe $V[G]$, $\mathcal{D}^{cb}_{\omega_2}$ is a Galvin filter, in fact $(\mathrm{Car} \setminus \{\aleph_2\}, \omega)$-Galvin filter. Moreover if $D \in V$ was a normal ultrafilter on κ, then in $V[G]$ the filter D has the Laver property. [We identify here D with the filter it generates in $V[G]$ which is normal.]

More exactly, [GJM] proved that player II has a winning strategy in the play above for D a normal filter on λ, Laver proved the $\aleph_1$-Laver property in that context, but the difference is not essential in our context.
(We shall not prove it here.)

The relevance of this is:

4.12 Theorem. Let $S \subseteq \mathrm{SCar}$.

(1) If P is $\mathrm{Nm}'(T^\dagger, \mathfrak{D}^*)$ (see 4.4(4)), each $\mathfrak{D}^*_\eta$ is an (S,ω)-Galvin, $\aleph_2$-complete filter *then* P is S-semiproper and even (S,ω)-complete (and we can add all λ, $\mathrm{cf}(\lambda) > |T^\dagger|$ to S).

(2) We can strengthen the hypothesis in (1) by "$\mathfrak{D}^*_\eta$ is $|\alpha|^+$-Laver" and then get even "(S,α)-complete for pure extensions" (see XIV).

Proof. (1) Also easy, but we shall do it. By 3.4(4) it suffices to prove (S,ω)-completeness. Let $p^* \in P$ and we shall prove that the second player wins in $\partial^\omega_S(p,P)$. For every $\eta \in T^* \setminus \{\nu : \nu \lhd \mathrm{tr}(T^*)\}$, let H_η be a winning strategy of player II in the (S,ω)-Galvin game for $(\mathfrak{D}_\eta, \mathrm{Suc}_{T^*}(\eta))$.

We first prove

4.13 Fact. Suppose $p \in P$, $(P = \mathrm{Nm}'(T^\dagger, \mathfrak{D}^*), \mathfrak{D}^*$ is $\aleph_2$-complete), $\lambda \in S$, and $\underset{\sim}{\alpha}$ is a P-name of an ordinal, $p \Vdash$ "$\underset{\sim}{\alpha} < \lambda$" and

(*) $X \in \mathfrak{D}^+_\eta$, $F : X \to \lambda \Rightarrow (\exists\alpha)[\{\nu \in X : F(\nu) < \alpha\} \neq \emptyset \mod \mathfrak{D}]$ (this follows from $\mathfrak{D}_\eta$ being (S,ω)-Galvin).

Then there are $p^\dagger$ and $\alpha < \lambda$ such that $p \leq_{pr} p^\dagger \in P$ and $p^\dagger \Vdash$ "$[\lambda \geq \aleph_0 \Rightarrow \underset{\sim}{\alpha} < \alpha]$ and $[\lambda = 2 \Rightarrow \underset{\sim}{\alpha} = \alpha]$".

4.13A Remark. In the proof of the fact we do not use the Galvin property assumption; also the $\aleph_2$-completeness can be waived, see 4.14 below.

Proof of the fact 4.13. For notational simplicity only we assume, $2 \notin S$. Easily we can find $p_1 = (T_1, \mathfrak{D}^*)$, $p \leq_{pr} p_1$, such that for every $\eta \in T_1$, if there are $\beta < \lambda$, and q, $(T_1, \mathfrak{D}^*)_{[\eta]} \leq_{pr} q$, $q \Vdash$ "$\underset{\sim}{\alpha} < \beta$", then $(T_1, \mathfrak{D}^*)_{[\eta]} \Vdash$ "$\underset{\sim}{\alpha} < \beta$" for some β. For each $\eta \in T_1$, let β_η be such that $(p_1)_{[\eta]} \Vdash$ "$\underset{\sim}{\alpha} < \beta_\eta$", β_η may be undefined for some η but if $\eta \vartriangleleft \nu \in T_1$, β_η defined, then β_ν is defined and equal to β_η. So for every $\eta \in \lim T_1$, $\{\ell : \beta_{\eta\restriction\ell}$ not defined$\}$ is an initial segment of ω. By the $\aleph_2$-completeness and 4.6 if CH and XI 3.5 in general, there is T_2, $(T_1, \mathfrak{D}^*) \leq_{pr} (T_2, \mathfrak{D}^*)$ and a set $A \subseteq \omega$ such that $\forall\eta \in T_2$ β_η is defined iff $\ell g(\eta) \in A$ (A is an endsegment or the empty set (so there are only countably many possibilities, this is why XI 3.5 can be applied)). But $A = \emptyset$ is impossible by density. So for some n β_η is defined for every $\eta \in T_2$, $\ell g(\eta) = n$. We can (by induction on n using (*) in the assumption, see below for a similar argument or again by XI 3.5) define p_3, $(T_2, \mathfrak{D}^*) \leq_{pr} p_3$, and $\beta < \lambda$ such that $[\eta \in p_3$ & β_η defined $\Rightarrow \beta_\eta < \beta]$ this implies $p_3 \Vdash$ "$\underset{\sim}{\alpha} < \beta$", so the Fact holds. $\square_{4.13}$

Continuation of the proof of 4.12(1): Remember $p^* = (T^*, \mathfrak{D}^*)$ is given; w.l.o.g. the trunk of T^* is $<>$.

In the first move player I chooses $\lambda_0 \in S$ and a P-name $\underset{\sim}{\beta}_0$ of an ordinal $< \lambda$.

Player II chooses $\beta_0 < \lambda$ and $p_0 \in P$ such that $p^* \leq_{pr} p_0$, $p_0 \Vdash_P$ "$\underset{\sim}{\beta}_0 < \beta$" (possible by the Fact 4.13 above).

However if player II continues to play like this, he may loose as maybe $\bigcap_n T_n$ (where $p_n = (T_n, \mathfrak{D}^*)$) will be $\{<>\}$.

So he is thinking how to make $\mathrm{Suc}_{\bigcap_{n<\omega} T_n}(<>) \neq \emptyset \bmod \mathfrak{D}^*_{<>}$. If he, on the other hand, will demand $p_0 \leq_1 p_{n+1}$, he will have $\mathrm{Suc}_{\bigcap_{n<\omega} T_n}(<>) \notin \emptyset \bmod \mathfrak{D}^*$, but it will be hard (and in fact impossible) to do what is required when, e.g., $\lambda_n = \aleph_1$. So what he will do is to decrease $\mathrm{Suc}_{T_n}(<>)$, but do it using his winning strategy $H_{<>}$ for the (S,ω)-Galvin game for $\mathfrak{D}_{<>}$. So in the second move player I chooses a cardinal $\lambda_1 \in S$ and P-name $\underset{\sim}{\beta}_1$ of an ordinal $< \lambda_1$. Player II, first for each $\eta \in p_0$, $\ell g(\eta) = 1$, chooses $p_1^\eta = (T_1^\eta, \mathfrak{D}^*)$ such that $(p_0)_{[\eta]} \leq_{pr} p_1^\eta$ and $p_1^\eta \Vdash_P$ "$\underset{\sim}{\beta}_1 \leq \beta_\eta$", this is possible by 4.13. This defines a function from $\mathrm{Suc}_{T_0}(<>)$ to λ_1, so player II consults the winning strategy $H_{<>}$, gets $A^0_{<>} \subseteq \lambda_1$, $|A^0_{<>}| < \lambda_1$, and lets $T_1 = \bigcup\{T_1^\eta : \beta_\eta \in A^0_{<>}\}$. Now at last player II actually plays: the condition $(T_1, \mathfrak{D}^*)$ and the ordinal $\sup A^0_{\langle\rangle}$.

In the third move, player II tries also to insure that $\{\eta \in \bigcap_n T_n : \ell g(\eta) = 2\}$ will be as required. Now player I chooses $\lambda_2 \in S$ and a P-name $\underset{\sim}{\beta}_2$. Player II chooses for every $\eta \in T_1$, $\ell g(\eta) = 2$ a condition p_2^η such that $(p_1)_{[\eta]} \leq_{pr} p_2^\eta$ and $p_2^\eta \Vdash_P$ "$\underset{\sim}{\beta}_2 \leq \beta_\eta$". So for every $\eta \in T_1$, $\ell g(\eta) = 1$, we have a function from $\mathrm{Suc}_\eta(T_1)$ to λ_2, so consulting the strategy H_η, player II chooses $A^1_\eta \subseteq \lambda_2$, $|A^1_\eta| < \lambda$. We can assume that each A^1_η is a proper initial segment (i.e., an ordinal) and for $\lambda = 2$, a singleton. So the number of possible A^1_η is λ. So now the function $\eta \mapsto A^1_\eta$ $(\eta \in \mathrm{Suc}_{T_1}(<>))$ is a function whose domain is $\mathrm{Suc}_{T_1}(<>)$. So player II can consult again the strategy $H_{<>}$, and find $A^2_{<>}$, and let $T_2 = \bigcup\{T_2^\eta : \ell g(\eta) = 2, \eta \in T_1, \beta_\eta \in A^1_{\eta\restriction 1}$ and $A^1_{\eta\restriction 1} \subseteq A^2_{\eta\restriction 0} = A^2_{<>}\}$. Now at last player II plays: the condition $(T_2, \mathfrak{D}^*)$ and the ordinal $\sup A^2_{\langle\rangle}$.

The rest should be clear (compare with the proof of 6.2).

(2) By 4.13 it should be clear $\square_{4.12}$

4.14 Remark. Really in 4.12(1) we can replace $\aleph_2$-completeness by $\aleph_1$-completeness by using XI 3.5 instead of 4.6. In fact even this can be waived. We use $\aleph_2$-completeness only in the proof of Fact 4.13; but we now give a proof which eliminate it. Instead of choosing T_2, we let $H : \lim T^\dagger \to 2$ be defined by $H(\eta) = 0$ iff $(\exists n)[\beta_{\eta\restriction n}$ is defined], and so there is T' such that $(T^\dagger, \mathfrak{D}^*) \leq_{pr} (T', \mathfrak{D}^*)$ and H is constant on $\lim T'$ (by XI 3.5 which does not need any completeness). Now on $B = \{\eta \in T' : (\forall \ell < \ell g(\eta))\ [\beta_{\eta \restriction \ell}$ is not defined$]\}$, we can define a rank:

$$\mathrm{rk}(\eta) = \bigcup \{\gamma + 1 : \{\nu : \nu \in B \cap \mathrm{Suc}_{T'}(\eta) \text{ and } \mathrm{rk}(\nu) \geq 0\} \neq \emptyset \bmod \mathfrak{D}_\eta\}.$$

If for some η, $\mathrm{rk}(\eta) = \infty$ we let $T'' \stackrel{\text{def}}{=} \{\nu \in T' : \nu \trianglelefteq \eta$ or $\eta \triangleleft \nu \& \bigwedge_{\ell g \eta \leq \ell \leq \ell g \nu} \mathrm{rk}(\nu \restriction \ell) = \infty\}$; we get T'', $(T', \mathfrak{D}) \leq (T'', \mathfrak{D}), T'' \subseteq T'$, contradiction to the choice of T'. Otherwise (i.e. $\eta \in T' \Rightarrow \mathrm{rk}(\eta) < \infty$) we can prove by induction on its $\mathrm{rk}(\mathrm{tr} T')$ that we can find $p^\dagger$ as required.

(Note we are not assuming CH).

§5. Chain Conditions and Abraham's Problem

Chain conditions are very essential for iterated forcing. In Solovay and Tannenbaum [ST] this is the point, but even when other conditions are involved, we have to finish the iteration and exhaust all possibilities, so some chain condition is necessary to "catch our tail." In our main line we want to collapse some large κ to $\aleph_2$, in an iterated forcing of length (and power) κ, each P_i of power $< \kappa$. So we want that κ stays a regular cardinal, and the obvious way to do this is by the κ-chain condition. We prove it by the traditional method of the Δ-system. For general RCS iteration, we have to assume κ is Mahlo (i.e., $\{\lambda < \kappa$ strongly inaccessible$\}$ is stationary) and for iteration of semiproper forcing we ask for less.

Now we are able to answer the following problem of U. Abraham:

Problem. Suppose G.C.H. holds in V. Is there a set $A \subseteq \aleph_1$ so that every ω-sequence from $\aleph_2$, belongs to $L[A]$?

To construct a model where the answer is "no" we shall collapse some inaccessible κ, which is the limit of measurable cardinals, changing the cofinalities of arbitrarily large measurables $< \kappa$ to $\aleph_0$.

5.1 Definition.

(1) For any iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ and a set $S \subseteq \alpha$ we call $\bar{p} = \langle p_i : i \in S \rangle$ a Δ-system if, for $i < j$ in $S, p_i \restriction i = p_j \restriction j$ and $p_i \in P_j$. We call $p_i \restriction i$ the heart of the Δ-system, $hr(\bar{p})$.

(2) For a forcing P, we call $\bar{p} = \langle p_i : i \in S \rangle$ a μ-weak Δ-system if $p_i \in P$, $\bigcup_{i \in S} i$ is a regular cardinal κ, and there is a condition $q = hr(\bar{p})$ (the heart of $\bar{p}$) such that for every r, $q \leq r \in P$ there is $\alpha < \kappa$ satisfying : if $\alpha < \alpha_i \in S$ for $i < \mu_1 < \mu$ then $\{r\} \bigcup \{p_{\alpha_i} : i < \mu_1\}$ has an upper bound in P.

5.2 Claim. Any Δ-system in an RCS iteration as in Definition 5.1 (1), with $\alpha = \sup S$ and $P_\alpha = \bigcup_{i<\alpha} P_i$ is an $\aleph_1$-weak Δ-system.

Proof. Easy. $\square_{5.2}$

5.3 The Chain Condition Lemma.

(1) Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \kappa \rangle$ is an RCS iteration, κ regular, $|P_i| < \kappa$ for $i < \kappa$ and let $A = \{\lambda < \kappa : \lambda \text{ strongly inaccessible}\}$. *Then* for every sequence $\bar{p} = \langle p_j : j \in B \subseteq A \rangle$, we can find a closed unbounded $C \subseteq \kappa$ and a pressing down function h on $C \cap B$ (i.e., $h(j) < j$) such that for any α, $\langle p_j : j \in B \cap C, h(j) = \alpha \rangle$ is a Δ-system. (So in the non trivial case κ is strongly inaccessible Mahlo cardinal.)

(2) Assume $A^\dagger = \kappa$ and: $\underset{\sim}{Q}_i$ is RṴCar-semiproper (for all i) *or* semiproper, $\Vdash_{P_i + n_i}$ " $(2^{|P_i|})^V$ has cardinality $\aleph_1$" (for all i) *or* even $\bar{Q}$ as in 2.6(3), $\bar{Q}$ as in 2.6(4) and $A^\dagger = \{i : \emptyset \Vdash_{P_i}$ "$\bigcup_{j<i} P_j$ is dense in P_i"$\}$. *Then* in (1), we can replace A by $A^\dagger$ (We know that if each Q_i is semiproper (or just P_j / P_{i+1}) then $[\mathrm{cf}^V(i) = \aleph_1 \Rightarrow i \in A^\dagger]$ and also:$[i$ limit, 2.6(2) or 2.6(3) apply to $\bar{Q} \restriction i$ and $\Vdash_{P_j}$ "cf$i > \aleph_0$" for every $j < i\} \Rightarrow i \in A^\dagger])$.

(3) If we agree to weaken the conclusion to "$\aleph_1$-weak Δ-system", we can replace "$|P_i| < \kappa$ for $i < \kappa$" by "$d(P_i) < \kappa$ for $i < \kappa$" or even, for any $A \subseteq \kappa$, "each $P_i (i < \kappa)$ satisfies the conclusion of (1) for A". In (2) we can assume just each Q_i is semiproper.

Before we prove the lemma note:

5.4 Corollary.

(1) If in 5.3(1), A is stationary or in 5.3(2), $A^\dagger$ is stationary, *then* $P_\kappa = \mathrm{Rlim}\bar{Q}$ satisfies the κ-chain condition.

(2) If D is a normal ultrafilter on κ, $B \in D$, $B \subseteq A$ then (in 5.3(1)) for some $B^\dagger \in D$, $\langle p_j : j \in B^\dagger \rangle$ is a Δ-system.

Proof of 5.3. (1) If B is not stationary (as a subset of κ), the conclusion is trivial, so suppose B is stationary. Necessarily κ is strongly inaccessible (as κ is regular, every member of A is strongly inaccessible and $B \subseteq A$), hence by 1.6, $P_\kappa \stackrel{def}{=} \mathrm{Rlim}\bar{Q} = \bigcup_{i<\kappa} P_i$. As $|P_i| < \kappa$ for every $i < \kappa$, there is a one to one function H from P_κ onto κ. Again as $|P_i| < \kappa$ for $i < \kappa$, clearly

$$C = \{i : H \text{ maps } \textstyle\bigcup_{j<i} P_j \text{ onto } i \text{ and for } j < i\text{: if } j \in B \text{ then } p_j \in P_i\}.$$

is a closed unbounded subset of κ. We now define the function h with domain $B \cap C$: $h(i) = H(p_i{\restriction}i)$.

We first prove that h is pressing down. Clearly $p_i{\restriction}i \in P_i$, and if $i \in B \cap C$ then i is strongly inaccessible and $(\forall j < i)[|P_j| < i]$ hence by 1.6, $P_i = \bigcup_{j<i} P_j$, hence $p_i{\restriction}i \in \bigcup_{j<i} P_j$, so $h(i) < i$. Now looking at the definitions of h and C we see that $\langle p_j : j \in B \cap C, h(j) = \alpha \rangle$ is a Δ-system, for any α.

(2) The proof is similar, using 2.7 instead of 1.6.

(3) Left to the reader. $\square_{5.3}$

5.5 Theorem. Suppose Con (ZFC + "there is an inaccessible cardinal κ which is the limit of measurable cardinals"). *Then* the following theory is consistent: ZFC + G.C.H. $+(\forall A \subseteq \aleph_1)(\exists \bar{\alpha})$ ($\bar{\alpha}$ an ω-sequence of ordinals $< \aleph_2$, $\bar{\alpha} \notin L[A]$).

Proof. We start with a model V of ZFC + G.C.H + "κ is strongly inaccessible, and limit of measurables". We define an RCS iterated forcing $\langle P_i, \underset{\sim}{Q}_i : i < \kappa \rangle$, such that $|P_i| < \kappa$. We do it by induction on i, and clearly (see 1.4(6) for i limit) the induction hypothesis $|P_i| < \kappa$ continues to hold. If $\bar{Q}^i = \langle P_j, \underset{\sim}{Q}_j : j < i \rangle$ is defined, let κ_i be the first measurable cardinal $> |P_i|$, where $P_i = \mathrm{Rlim}\bar{Q}_i$. It is known (see e.g. [J]) that κ_i is measurable in V^{P_i}, and any normal ultrafilter on it from V is an ultrafilter (and normal) in V^{P_i}, too. As $|P_i| < \kappa$, by hypothesis

$\kappa_i < \kappa$. So let $Q_{i,0}$ be $PF(D_i)$ (see 4.1) where $D_i \in V$ is any normal ultrafilter on κ_i, and let $\underset{\sim}{Q}_{i,1}$ be $P_i * \underset{\sim}{Q}_{i,0}$-name of the Levy collapse of κ_i^+ to $\aleph_1$ (i.e. $Q_{i,1} = \{f : \mathrm{Dom}(f)$ is an ordinal $< \aleph_1$, and $\mathrm{Rang}(f) \subseteq \kappa_i^+\}$, with inclusion as order). We let $Q_i = Q_{i,0} * \underset{\sim}{Q}_{i,1}$.

Now by 4.2, $Q_{i,0} = PF(D_i)$ is $(\mathrm{Car}^V \setminus \{\kappa_i\}, \omega)$-complete, $Q_{i,1}$ is (Car^V, ω)-complete trivially (by 3.4(1)) hence by 3.5 Q_i is $(\mathrm{Car}^V \setminus \{\kappa_i\}, \omega)$-complete.

Hence by 3.5(2), $P_\kappa = \mathrm{Rlim}\langle P_i, \underset{\sim}{Q}_i : i < \kappa\rangle$ is $(\{2, \aleph_0, \aleph_1\}, \omega)$-complete, hence it does not add reals and does not change the cofinality of $\aleph_1$. By 3.4(4) P_κ is semiproper. By 5.3(2) P_κ satisfies the κ-chain condition, so clearly if $G_\kappa \subseteq P_\kappa$ is generic then $\aleph_1^{V[G_\kappa]} = \aleph_1^V, \aleph_2^{V[G_\kappa]} = \kappa$, $V[G_\kappa]$ have the same reals as V, and $V[G_\kappa]$ satisfies the G.C.H.

Now if $A \subseteq \omega_1$, then as P_κ satisfies the κ-chain condition, A is determined by $G_i = G_\kappa \cap P_i$ for some $i < \kappa$. By 1.1(D), G_i is generic for P_i, so $L[A] \subseteq V[G_i]$, but in $V[G_i]$ an ω-sequence from $\aleph_2^{V[G_\kappa]}$ is missing: the Prikry sequence we shot through κ_{i+1} which was measurable in $V[G_i]$. $\square_{5.5}$

§6. Reflection Properties of S^2_0: Refining Abraham's Problem and Precipitous Ideals

In the previous section we have collapsed a large cardinal κ to $\aleph_2$, such that to "many" measurable cardinals $< \kappa$ we add an unbounded ω-sequence. However, "many" was interpreted as "unbounded set". This is very weak and we often desire for more, e.g. in 6.4, we would like to change cofinalities on a stationary set.

Notice that it is known that if we collapse a large cardinal by $\aleph_1$-complete forcing then $S^2_1 \overset{\mathrm{def}}{=} \{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1\}$ has reflection and bigness properties, e.g., those from Definition 4.10. However, for S^2_0, we get nothing as it is equal to $\{\delta < \aleph_2 :$ in the universe before the collapse, $\mathrm{cf}(\delta) = \aleph_0\}$ and it is known, e.g., that on such a set there was no normal ultrafilter.

So we can ask whether S^2_0 can have some "large cardinal properties". The natural property to consider is precipitous normal filters D on $\aleph_2$ such that $S^2_0 \in D$. Such filters were introduced in Jech and Prikry [JP1] and studied in Jech and Prikry [JP2], Jech, Magidor, Mitchell and Prikry [JMMP].

Their important property is that if we force by $PP(D)$ (which is $\{A \subseteq \kappa : A \neq \emptyset \bmod D\}$ ordered by an inverse inclusion), G is generic, the domain of D is I, and in $V[G]$, $E \supseteq D$ is the ultrafilter G generates (on old sets) then V^I/E (taking only old $f : I \to V$) is well-founded. Jech, Magidor, Mitchell and Prikry [JMMP] proved that the existence of a precipitous filter on $\aleph_1$ is equiconsistent with the existence of a measurable cardinal, and also proved the consistency of "$\mathcal{D}_{\aleph_1}$ (= the filter of closed unbounded sets) is precipitous". (Notice that the Laver property is stronger). Magidor asked:

Problem I. Is ZFC + G.C.H.+ there is a normal precipitous filter D on $\aleph_2$, $S^2_0 \in D$ consistent?

We answer positively, by collapsing suitably some κ to $\aleph_2$. Letting D be a normal ultrafilter on κ in V, provided that $A = \{\lambda < \kappa$: in the old universe λ is measurable $\} \in D$. We will force that in the new universe, D generates a normal precipitous filter (which we also call D) such that S^2_0 belongs to it.

This was proved previously and independently, using supercompact cardinals, by Gitik.

We can also consider the following strengthening of Abraham's problem:

Problem II. If V satisfies G.C.H., does there exist $A \subseteq \aleph_2$ such that, for every $\delta < \aleph_2$, every ω-sequence from δ belongs to $L[A \cap \delta]$?

Again we have to change the cofinality on a stationary set, and to iterate forcing such that stationarily often we change the cofinality of $\aleph_2$ to $\aleph_0$.

When we do this the first time, in stage λ for example, the forcing so far P_λ is just Levy's collapse Levy$(\aleph_1, < \lambda)$ so by 4.11, 4.12 we have a $(\mathrm{Car}^V \setminus \{\aleph_2\}, \omega)$-complete forcing Q_λ doing this; but later the collapse P_λ is not even $\aleph_1$-complete. We have two ways to cope with this. One way is to look again at theorem 3.5 on iterated (S, ω)-complete forcing (for various S), from which we see that less is needed. If $P_\lambda = \mathrm{Rlim}\, \bar{Q}, \bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \lambda \rangle$ collapses λ to $\aleph_2$, it suffices that $(\mathrm{Rlim}\bar{Q}/P_{i+1}) * \underset{\sim}{Q}_\lambda$ is $(\{2, \aleph_0, \aleph_1\}, \omega)$-complete. We will show that we can achieve this by using Namba forcing as Q_i and our induction hypothesis there. The second possibility is to demand e.g. each Q_i is quite pseudo-complete and prove that in V^{P_λ} we get a large ideal in λ. We use the first approach (but see 6.1A). For clarity of exposition, we first prove a weaker lemma.

6.1 Lemma. Suppose D is a normal ultrafilter on λ, $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \lambda \rangle$ an RCS iteration and for all $i < \lambda$, $|P_i| < \lambda$. Suppose further $P_\lambda = \mathrm{Rlim}\bar{Q}$ is $(\{2, \aleph_0, \aleph_1\}, \omega)$-complete and collapses λ to $\aleph_2$. Consider the following game $\partial(p_0, \underset{\sim}{A}_0) = \partial^\omega(p_0, \underset{\sim}{A}_0, P_\lambda, D)$, for $p_0 \in P_\lambda$, $\underset{\sim}{A}_0$ a P_λ-name of a subset of λ such that $p_0 \Vdash_{P_\lambda}$ "$\underset{\sim}{A}_0 \neq \emptyset \bmod D$". (The game is played in V.)

In the first move:

Player I chooses P_λ-names $\underset{\sim}{\beta}_1$ (of an ordinal $< \aleph_1$) and $\underset{\sim}{F}_1$ (a function from λ to $\aleph_1$).

Player II has to choose $p_1 \in P_\lambda$, $p_0 \leq p_1$ and $\gamma_1 < \omega_1$ and $\beta_1 < \omega_1$ such that $p_1 \Vdash_{P_\lambda}$ "$\underset{\sim}{A}_1 = \underset{\sim}{A}_0 \cap \underset{\sim}{F}_1^{-1}(\{\gamma_1\}) \neq \emptyset \bmod D$, and $\underset{\sim}{\beta}_1 = \beta_1$".

In the n-th move, player I chooses P_λ-names $\underset{\sim}{\beta}_n < \omega_1, \underset{\sim}{F}_n : \lambda \to \aleph_1$, and player II chooses p_n, $p_{n-1} \leq p_n$ and $\gamma_n < \omega_1$ and $\beta_n < \omega_1$ such that $p_n \Vdash_{P_\lambda}$ "$\underset{\sim}{A}_n = \underset{\sim}{A}_{n-1} \cap \underset{\sim}{F}_n^{-1}(\{\gamma_n\}) \neq \emptyset \bmod D$, and $\underset{\sim}{\beta}_n = \beta_n$".

In the end, player II wins if $\{p_n : n < \omega\}$ has an upper bound $p \in P_\lambda$ such that $p \Vdash_{P_\lambda}$ "$\bigcap_{n<\omega} \underset{\sim}{A}_n \neq \emptyset \bmod D$".

Our conclusion is that player II wins the game.

6.1A Remark. *If* $\{i : P_\lambda/P_i$ is $\{(\{2, \aleph_0, \aleph_1\}, \omega)$-complete $\} \in D$ (i.e. for almost all i, for every $G_i \subseteq P_i$ generic over V, in $V[G_i]$, P_λ/G_i is $(\{2, \aleph_0, \aleph_1\}, \omega)$-complete) *then* in V^{P_λ}, D is a $\{2, \aleph_0, \aleph_1\}$-Galvin filter. Similarly for 6.2 (see XIII 1.9).

Proof. Let $p_0 \in P_\lambda$, $\underset{\sim}{A}_0$ a P_λ-name, $p_0 \Vdash_{P_\lambda}$ "$\underset{\sim}{A}_0 \neq \emptyset \bmod D$". We shall describe the winning strategy of player II in the game $\partial(p_0, \underset{\sim}{A}_0)$. Let the winning strategy of player II in $\Game^{\omega}_{\{2,\aleph_0,\aleph_1\}}(p, P_\lambda)$ be $H[p]$. By 3.2(2), we can assume that player II really determined the value of the P_λ-names of countable ordinals given to him. We can also assume player II is given by player I a pair of names of countable ordinals (instead of one).

Let $B_0 = \{i < \lambda$: there is $p \geq p_0$, $p \Vdash_{P_\lambda}$ "$i \in \underset{\sim}{A}_0$"$\}$. Now $B_0 \in D$ because otherwise, as D is an ultrafilter in V we have $B_0 = \emptyset \bmod D$, but since $p_0 \Vdash_{P_\lambda}$ "$\underset{\sim}{A}_0 \subseteq B_0$" (by B_0's definition) we have $p_0 \Vdash_{P_\lambda}$ "$\underset{\sim}{A}_0 = \emptyset \bmod D$", contradiction.

Now for every $i \in B_0$, there is $p_{0,i} \in P_\lambda$, $p_0 \leq p_{0,i}$ such that $p_{0,i} \Vdash_{P_\lambda}$ "$i \in \underset{\sim}{A}_0$".

So let player I's first move in $\Game(p_0, \underset{\sim}{A}_0)$ be choosing $\underset{\sim}{\beta}_1$ (a P_λ-name of an ordinal $< \aleph_1$), and $\underset{\sim}{F}_1 : \lambda \to \aleph_1$, $\underset{\sim}{F}_1$ a P_λ-name. Now for each $i \in B_0$, player II simulates a play of the game $\Game_i = \Game^{\omega}_{\{2,\aleph_0,\aleph_1\}}(p_{0,i}, P_\lambda)$. He plays $(\underset{\sim}{\beta}_1, \underset{\sim}{F}_1(i))$ (i.e., a pair of names of ordinals $< \aleph_1$) for player I_i, and by the strategy $H[p_{0,i}]$ gets a move for player II$_i$: $p_{1,i} \in P_\lambda$, $p_{0,i} \leq p_{1,i}$, and $\alpha_{1,i} < \aleph_1$, $\varepsilon_{1,i} < \aleph_1$ such that $p_{1,i} \Vdash_{P_\lambda}$ "$\underset{\sim}{\beta}_1 = \alpha_{1,i}$ and $\underset{\sim}{F}_1(i) = \varepsilon_{1,i}$". Now for some $B_1 \subseteq B_0$, $B_1 \in D$, and $\langle p_{1,i} : i \in B_1 \rangle$ is a Δ-system with heart p_1 (see 5.4(2)), and we can also make $\langle \alpha_{1,i}, \varepsilon_{1,i} \in B_1 \rangle$ constantly $(\alpha_1, \varepsilon_1)$ (for $i \in B_1$) since there are only $\aleph_1$ many possibilities.

Now player II can make his move in $\Game(p_0, \underset{\sim}{A}_0)$: he chooses p_1, α_1 and ε_1. It is easy to check that this is a legitimate move. (Use 5.2 to show $p_1 \Vdash$ "$\underset{\sim}{\beta}_1 = \alpha_1$".)

So player II continues to play such that after the n-th move:

$(*)_n$ there are $B_n \subseteq B_{n-1} \subseteq \ldots \subseteq B_1 \subseteq B_0$ all in D, $p_{\ell,i} \in P_\lambda$, for $0 \leq \ell \leq n$, $i \in B_\ell$, $p_{0,i} \leq p_{1,i} \leq \ldots \leq p_{\ell,i}$, $\langle p_{\ell,i} : i \in B_\ell \rangle$ is a Δ-system with heart p_ℓ (for $0 < \ell \leq n$) $p_0 \leq p_1 \leq \ldots \leq p_n$, and at the ℓ-th move player I chooses $\underset{\sim}{\beta}_\ell, \underset{\sim}{F}_\ell$, and player II chooses $p_\ell, \alpha_\ell, \varepsilon_\ell$ and (for $1 \leq \ell \leq n$ and $i \in B_\ell$) $p_{\ell,i} \Vdash$ "$\alpha_\ell = \underset{\sim}{\beta}_\ell$ and $\underset{\sim}{F}_\ell(i) = \varepsilon_\ell$". Also for each $\ell \leq n$, $\ell > 0$ and each $i \in B_\ell$, the following is an initial segment of a play of a game $\Game^{\omega}_{\{2,\aleph_0,\aleph_1\}}(p_{0,i}, P_\lambda)$, in which player II$_i$ uses the winning strategy $H[p_{0,i}]$:

$$\langle \underset{\sim}{\beta}_1, \underset{\sim}{F}_1(i) \rangle, \langle p_{1,i}, \alpha_1, \varepsilon_1 \rangle, \langle \underset{\sim}{\beta}_2, \underset{\sim}{F}_2(i) \rangle, \langle p_{2,i}, \alpha_2, \varepsilon_2 \rangle, \ldots, \langle \underset{\sim}{\beta}_\ell, \underset{\sim}{F}_\ell(i) \rangle, \langle p_{\ell,i}, \alpha_\ell, \varepsilon_\ell \rangle.$$

It is easy to check that player II can use this strategy; moreover, by the choice of $H[p_{0,i}]$, for every $i \in \bigcap_n B_n$ the set $\{p_{n,i} : n < \omega\} \subseteq P_\lambda$ has an upper bound, say q_i; as $B_n \in D$, $\cap_{n<\omega} B_n \in D$ and clearly, by 5.4(2), for some $B_\omega \in D$, $B_\omega \subseteq \bigcap_{n<\omega} B_n$ and $\langle q_i : i \in \bigcap_n B_n \rangle$ is a Δ-system with heart p, clearly $p_n \leq p$ for each n, and so by 5.2 $p \Vdash_{P_\lambda}$ " $\{i \in \bigcap_n B_n$: $q_i \in \underset{\sim}{G}_{P_\lambda}$ hence for every ℓ $\underset{\sim}{\beta}_\ell = \alpha_\ell$ & $\underset{\sim}{F}_\ell(i) = \varepsilon_\ell\} \neq \emptyset \bmod D$". So clearly player II has won the play, hence the game.

$\square_{6.1}$

6.1B Remark. We could have used any $S, S \subseteq \{2, \aleph_0, \aleph_1\}$ instead of $\{2, \aleph_0, \aleph_1\}$ and would have obtained a parallel result. The same holds for 6.2 and 7.2 . Also in both we can replace completeness by essential completeness.

6.2 Lemma. Suppose λ is measurable, D a normal ultrafilter over λ, $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \lambda \rangle$ an RCS iteration, each P_j, P_j/P_{i+1} is $(\{2, \aleph_0, \aleph_1\}, \omega)$-complete and $|P_i| < \lambda$ for $i < j < \lambda$.
Then, letting $\underset{\sim}{Q}_\lambda = \mathrm{Nm}'(D)$ in the universe $V^{P_\lambda * \underset{\sim}{Q}}$ the forcing notion $P_\lambda * \underset{\sim}{Q}_\lambda$ is $(\{2, \aleph_0, \aleph_1\}, \omega)$-complete.

Proof. Just combine the proofs of 6.1 and 4.12(1) (so now we will have a tree of conditions instead $p_{\ell,i}, i \in B_\ell$)). Let us give the details. We will only prove *essential* $(\{2, \aleph_0, \aleph_1\}, \omega)$- completeness (which is enough for all practical purposes) and indicate modifications for $(\{2, \aleph_0, \aleph_1\}, \omega)$-completeness (if we like to use only the essential version, naturally we should then also assume only that P_j, P_j/P_{i+1} are essentially $(\{2, \aleph_0, \aleph_1\}, \omega)$-complete; remember the implications from 3.8(2)). So let $S = \{2, \aleph_0, \aleph_1\}$, $r^* = (p^*, q^*) \in P_\lambda * \underset{\sim}{Q}_\lambda$ and we shall describe a winning strategy for player II in the game $\mathrm{E}\eth^\omega_S(r^*, P_\lambda * \underset{\sim}{Q}_\lambda)$ (see 3.6). As $S = \{2, \aleph_0, \aleph_1\}$, by 3.2(2) without loss of generality player II has to give actual values.

Without loss of generality $p^* \Vdash_{P_\lambda}$ "$\mathrm{tr}(q^*) = \eta^*$", $\eta^* \in {}^{\omega>}\lambda$. For notational convenience only (or considering $\underset{\sim}{Q}'_\lambda = \{T \in \underset{\sim}{Q}_\lambda : \mathrm{tr}(T) \trianglerighteq \eta^*\} \simeq \underset{\sim}{Q}_\lambda$) we may assume $\eta^* =<>$. In the n'th move ($n \geq 0$) player I will choose a $P_\lambda * \underset{\sim}{Q}_\lambda$-name

$\underset{\sim}{\beta}^n$ of a countable ordinal, and player II will choose a countable ordinal γ_n [and a condition for the "real" game $\Game^\omega_S(r^*, P_\lambda * \underset{\sim}{Q}_\lambda)$].

To make his choice, player II plays on the side also trees $T_n \subseteq {}^{n\geq}\lambda$, ordinals $\langle i_\eta : \eta \in T_n \cap {}^n\lambda\rangle$, and P_λ-names of $\underset{\sim}{Q}_\lambda$-conditions $\langle \underset{\sim}{q}^n_\eta : \eta \in T_n \cap {}^n\lambda\rangle$ and conditions $\langle p^n_\eta : \eta \in T_n\rangle$ in certain forcings appearing in the iteration $\bar{Q}$ and names of ordinals $\langle \underset{\sim}{\beta}^n_\eta : \eta \in T_n \cup \{\langle\rangle^-\}\rangle$, preserving the following:

(A)(1) $T_n \subseteq {}^{n\geq}\lambda$, each $\eta \in T_n$ is strictly increasing.

(2) If $\eta \in T_n$, $\ell < \ell g(\eta)$, then $\eta{\restriction}\ell \in T_n$.

(3) If $\eta \in T_n$, $\ell g(\eta) < n$, then $\{i > \lambda : \eta\hat{\ }\langle i\rangle \in T_n\} \in D$.

(4) $T_{n+1} \cap {}^{n\geq}\lambda \subseteq T_n$.

(B)(1) If $n \geq 1$, $\eta \in {}^n\lambda$, then we let $\eta^- = \eta{\restriction}(n-1)$.

(2) If $\eta \in T_n \cap {}^n\lambda$, then $i_\eta < \lambda$, i_η a successor ordinal $> \eta(n-1)$.

(3) If $\eta \in T_n \cap {}^n\lambda$, $n \geq 1$ then $i_\eta > i_{\eta^-}$

(4) $<>^-$ is not really defined, but we let $i_{<>^-} = 0$,

so $P_{i_{<>}}/P_{i_{<>^-}} = P_{i_{<>}}/P_0 = P_{i_{<>}}$.

(C)(1) For $\eta \in T_n$, $p^n_\eta \in P_{i_\eta}/P_{i_{\eta^-}}$. (So for $\eta = <>$, $p^n_{<>} \in P_{i_{<>}}$).

(2) For $\eta \in T_n \cap T_{n+1}$ we have $p^n_\eta \leq p^{n+1}_\eta$ (this is actually implied by (3) below).

(3) For $\eta \in T_n$, $\langle \underset{\sim}{\beta}^\ell_\eta, p^\ell_\eta, \underset{\sim}{\beta}^\ell_{\eta^-} : \ell g(\eta) < \ell \leq n\rangle$ is an initial segment of a play of $\Game^\omega_S[p^{\ell g(\eta)}_\eta, P_{i_\eta}/P_{i_{\eta^-}}]$ in which player II uses his winning strategy. So $\underset{\sim}{\beta}^\ell_\eta$ is a P_{i_η}-name for a countable ordinal and $\beta^\ell_{<>^-}$ is a real ordinal. Player II lets (in the actual play) $\gamma_\ell = \beta^\ell_{<>}$ (for the "purely" essential version we should just have p^ℓ_η be in the completion of $P_{i_\eta}/P_{i_{\eta^-}}$).

(4) For $p \in P_\alpha$, $p\prime \in P_\lambda/P_\alpha$, $p \cup p\prime$ is the element of P_λ corresponding to $(p, p\prime) \in P_\alpha * (P_\lambda/P_\alpha)$. We let $\bar{p}^n_\eta \overset{\text{def}}{=} \bigcup_{\ell \leq \ell g(\eta)} p^n_{\eta{\restriction}\ell} \in P_{i_\eta}$ (see C1).

(D)(1) $\underset{\sim}{q}^n_\eta$ is a P_λ-name for an element of $\underset{\sim}{Q}_\lambda = \mathrm{Nm}'(D)$ with trunk η (remember $\eta \in T_n \cap {}^n\lambda$).

(2) $(\emptyset, \underset{\sim}{Q}^n_\eta) \Vdash_{P_\lambda * \underset{\sim}{Q}_\lambda}$ "$\underset{\sim}{\beta}^n = \underset{\sim}{\beta}^n_\eta$" when $\eta \in T_n \cap {}^n\lambda$.

(3) For $\eta \in T_n \cap {}^n\lambda$, $\bar{p}^n_\eta \Vdash_{P_\lambda}$ "$[Q_\lambda \models q^{n-1}_{\eta^-} \leq q^n_\eta$ and $Q_\lambda \models [q^{n-1}_{\eta^-}]_{[\eta]} \leq_{\mathrm{pr}} q^n_\eta$"].

For the following, note that $\Vdash_{P_\lambda}$ "D is an $\aleph_2$-complete filter", so by 4.13 we have $\Vdash_{P_\lambda}$ " for every $q \in Q_\lambda$, every $\underset{\sim}{Q}_\lambda$-name $\underset{\sim}{\beta}$ of an ordinal $< \omega_1$, there is $\gamma < \omega_1$, and q', $q \leq_{\rm pr} q'$, $q \Vdash \underset{\sim}{\beta} = \gamma$". In move number 0, player I plays $\underset{\sim}{\beta}^0$. Player II finds $\underset{\sim}{q}^0_{<>} \in \underset{\sim}{Q}_\lambda$ and a P_λ-name $\underset{\sim}{\beta}^0_{<>}$ such that:

$\Vdash_{P_\lambda}$ "$q^* \leq_{\rm pr} q^0_{<>}$, so $\mathrm{tr}(q^0_{<>}) =<>$, and $q^0_{<>} \Vdash_{Q_\lambda}$ "$\underset{\sim}{\beta}^0 = \underset{\sim}{\beta}^0_{<>}$"".

But as $P_\lambda \models \lambda$-c.c., $\beta^0_{<>}$ is really a $P_{i_{<>}}$-name (for some successor ordinal $i_{<>} < \lambda$), so player II can find $p^0_{<>} \in P_{i_{<>}}$ such that $P_\lambda \models$ "$p^* \leq p^0_{\langle\rangle}$" and $p^0_{<>} \Vdash$ "$\underset{\sim}{\beta}^0_{<>} = \gamma_0$", for some $\gamma_0 < \omega_1$. Then player II lets $T_0 = \{<>\}$ and play γ_0.

In move $n+1$, player I plays a $P_\lambda * \underset{\sim}{Q}_\lambda$-name $\underset{\sim}{\beta}^{n+1}$. For each $\eta \in T_n \cap {}^n\lambda$, let (in V):

$B_\eta = \{\alpha < \lambda : \forall \ell < \ell g(\eta)[\alpha > \eta(\ell)]$ and $\exists p \in P_\lambda / P_{i_\eta}$, $\bar{p}^n_\eta \cup p \Vdash_{P_\lambda}$ "$\eta\hat{\ }\langle\alpha\rangle \in q^n_\eta$"$\}$.

6.2A Claim. $B_\eta \in D$.

Proof of the Claim. For each $\alpha < \lambda$ let $\underset{\sim}{t}^\alpha_\eta$ be the following P_{i_η}-name:
$\underset{\sim}{t}^\alpha_\eta = 0$ if $\Vdash_{P_\lambda / G_{P_{i_\eta}}}$ "$\eta\hat{\ }\langle\alpha\rangle \notin q^n_\eta$" and $\underset{\sim}{t}^\alpha_\eta = 1$ otherwise.
As there are (essentially) $< \lambda$ many possible such $P^{n+1}_{i_\eta}$-names [as $|P_{i_\eta}| < \lambda$, so $2^{|P_{i_\eta}|} < \lambda$], for some $A_\eta \in D$ and $\underset{\sim}{t}$, $\forall \alpha \in A_\eta : \underset{\sim}{t}^\alpha_\eta = \underset{\sim}{t}$. If $\bar{p}^n_\eta \Vdash$ "$\underset{\sim}{t} = 1$", then $A_\eta \subseteq B_\eta$ and we are done. Otherwise, there is $p' \geq \bar{p}^n_\eta$, $p' \in P_{i_\eta}$, $p' \Vdash$ "$\underset{\sim}{t} = 0$". But $\bar{p}^n_\eta \Vdash$ "$\{\alpha < \lambda : \eta\hat{\ }\langle\alpha\rangle \in q^n_\eta\} \neq \emptyset \bmod D$" and $p' \Vdash$ "$\forall \alpha \in A_\eta$, $\eta\hat{\ }\langle\alpha\rangle \notin q^n_\eta$", a contradiction (as $A_\eta \in D$). This ends the proof of the claim. $\square_{6.2A}$

Continuation of the proof of 6.2: For $\alpha \in B_\eta$, let $p^n_{\eta\hat{\ }\langle\alpha\rangle}$ be a p as in the definition of B_η. Then let $q^{n+1}_{\eta\hat{\ }\langle\alpha\rangle}$ be (a P_λ-name of a member of $\underset{\sim}{Q}_\lambda$) such that:

$$\bar{p}^{n+1}_{\eta\hat{\ }\langle\alpha\rangle} \Vdash_{P_\lambda} \text{“}\Big[[q^n_\eta]_{[\eta\hat{\ }\langle\alpha\rangle]} \leq_{\rm pr} q^{n+1}_{\eta\hat{\ }\langle\alpha\rangle} \text{ and } q^{n+1}_{\eta\hat{\ }\langle\alpha\rangle} \Vdash_{\underset{\sim}{Q}_\lambda} \text{“}\underset{\sim}{\beta}^{n+1} = \underset{\sim}{\beta}^{n+1}_{\eta\hat{\ }\langle\alpha\rangle}\text{”}\Big]\text{”},$$

where $\underset{\sim}{\beta}^{n+1}_{\eta\hat{\ }\langle\alpha\rangle}$ is a P_λ-name. Again by λ-c.c., for some large enough successor ordinal $i_{\eta\hat{\ }\langle\alpha\rangle} < \lambda$, $\underset{\sim}{\beta}^{n+1}_{\eta\hat{\ }\langle\alpha\rangle}$ is a $P_{i_{\eta\hat{\ }\langle\alpha\rangle}}$-name and $p^{n+1}_{\eta\hat{\ }\langle\alpha\rangle} \in P_{i_{\eta\hat{\ }\langle\alpha\rangle}} / P_{i_\eta}$ (and $i_\eta < i_{\eta\hat{\ }\langle\alpha\rangle}$) and $\alpha < i_{\eta\hat{\ }\langle\alpha\rangle}$. We can increase $p^{n+1}_{\eta\hat{\ }\langle\alpha\rangle}$ and $i_{\eta\hat{\ }\langle\alpha\rangle}$ such that

$\bar{p}^n_\eta \Vdash_{P_{i_\eta}}$ "$p^{n+1}_{\eta\hat{}\langle\alpha\rangle} \Vdash_{P_{i_\eta\hat{}\langle\alpha\rangle}/P_{i_\eta}}$ "$\underset{\sim}{\beta}^{n+1}_{\eta\hat{}\langle\alpha\rangle} = \underset{\sim}{\beta}^{n+1}_{\eta,\,\alpha}$"" for some P_{i_η}-name $\underset{\sim}{\beta}^{n+1}_{\eta,\,\alpha}$. As there are only $< \lambda$ many such names of countable ordinals, we can find $S^{n+1}_\eta \in D$, $S^{n+1}_\eta \subseteq B_\eta$, and a name $\underset{\sim}{\beta}^{n+1}_\eta$ such that for all $\alpha \in S^{n+1}_\eta$, $\underset{\sim}{\beta}^{n+1}_{\eta,\,\alpha} = \underset{\sim}{\beta}^{n+1}_\eta$. Now we can, for each $\eta = \nu\hat{}\langle\alpha\rangle \in T_n \cap {}^n\lambda$, play a step $[\underset{\sim}{\beta}^{n+1}_\eta,\ p^{n+1}_\eta,\ \underset{\sim}{\beta}^{n+1}_{\eta^-,\alpha}]$ in the game $\partial^\omega_S[p^n_\eta, P_{i_\eta}/P_{i_{\eta^-}}]$, to get $P_{i_{\eta^-}}$-name $\underset{\sim}{\beta}^{n+1}_{\eta^-,\alpha}$ (i.e. we play for player I_η the name $\underset{\sim}{\beta}^{n+1}_\eta$, and the winning strategy of player II_η gives us $p^{n+1}_\eta, \underset{\sim}{\beta}^{n+1}_{\eta^-,\alpha}$; α was the last element in η). Again, for each $\nu \in T_n$ of length $n-1$ there is a set $S^{n+1}_\nu \in D$, $S^{n+1}_\nu \subseteq \{\alpha : \nu\hat{}\langle\alpha\rangle \in T_n\}$, and for all $\alpha \in S^{n+1}_\nu$, $\bar{\beta}^{n+1}_{\nu\hat{}\langle\alpha\rangle} = \beta^{n+1}_\nu$.

We continue by downward induction (in each step k, $n \geq k \geq 0$ defining P_ν-names $\underset{\sim}{\beta}^{n+1}_{\nu,\alpha}$ for $\nu\hat{}\langle\alpha\rangle \in T_n \cap {}^k\lambda$, satisfying demand (C3) and then "uniformizing" using a set $S^{n+1}_\nu \in D$ as before). Finally, player II plays $\gamma_{n+1} = \beta^{n+1}_{<>}$, and define $T_{n+1} \subseteq {}^{n+1\geq}\lambda$ by:

(1) $T_{n+1} \cap {}^0\lambda = \{<>\}$

(2) for $\eta \in T_{n+1} \cap {}^k\lambda$, $\mathrm{Suc}_{T_{n+1}}(\eta) = \{\eta\hat{}\langle\alpha\rangle : \alpha \in S^{n+1}_\eta\}$, for $k \leq n$.

This completes the description of player II's strategy. Finally, define T as $T = \bigcup_{\ell<\omega}\bigcap_{n\geq\ell}(T_n \cap {}^\ell\lambda)$. Clearly $({}^{\omega>}\lambda, \mathfrak{D}) \leq_{\mathrm{pr}} (T, \mathfrak{D})$ (where $\mathfrak{D}_\eta = D$). For each $\eta \in T$ let $p_\eta \in P_{i_\eta}/P_{i_{\eta^-}}$ be $\geq p^n_\eta$ for every $n \geq \ell g(\eta)$ (i.e. this is $\Vdash_{P_{i_{\eta^-}}}$); (exists as we have used a winning strategy in $\partial^\omega_{\{2,\,\aleph_0,\,\aleph_1\}}[p^{\ell g(\eta)}_\eta,\ P_{i_\eta}/P_{i_{\eta^-}}]$). Let $\bar{p}_\eta = \bigcup_{\ell\leq\ell g(\eta)} p_{\eta\restriction\ell} \in P_{i_\eta}$. By repeated use of 5.4(2) we can find T', $(T, \mathfrak{D}) \leq_{\mathrm{pr}} (T', D)$ and $\langle p^+_\eta : \eta \in T\rangle$ such that for each $\eta \in T'$ we have $\langle p_{\eta\hat{}\langle\alpha\rangle} : \alpha < \lambda, \eta\hat{}\langle\alpha\rangle \in T\rangle$ is a Δ-system with heart $p^+_\eta \in P_\lambda/P_{i_\eta}$. Note: $\eta^- = \nu$, $\eta \in T' \Rightarrow p^+_\nu \leq p_\eta$.

It is easy to see that if $\langle p_\alpha : \alpha < \lambda\rangle$ is a Δ-system with heart p^+, then $p^+ \Vdash_{P_\lambda}$ "$\{\alpha : p_\alpha \in G_{P_\lambda}\} \neq \emptyset \bmod D$ in V^{P_λ}". Using this fact we can show that $p_{<>} \cup p^+_{<>} \Vdash$ "$\underset{\sim}{q} \stackrel{\mathrm{def}}{=} \{\eta \in T' : \bar{p}_\eta \cup p^+_\eta \in G_{P_\lambda}\} \in \underset{\sim}{Q}_\lambda = \mathrm{Nm}'(D)$ and $q^* \leq_{\mathrm{pr}} \underset{\sim}{q}$".

To finish the proof it is enough to show that $(p_{<>} \cup p^+_{<>},\ \underset{\sim}{q}) \Vdash$ "$(\forall n)\underset{\sim}{\beta}^n = \gamma_n$". Assume that this is false, then there is a witness, i.e. $(p', \underset{\sim}{q}') \geq (p_{<>} \cup p^+_{<>}, \underset{\sim}{q})$, $n \in \omega$ and $\alpha^* < \omega_1$, such that $(p', \underset{\sim}{q}') \Vdash$ "$\underset{\sim}{\beta}^n = \alpha^*$", but $\alpha^* \neq \gamma_n$. Without loss of generality $\underset{\sim}{q}'$ has a trunk of length $> n$, and also there is $\eta \in$

${}^{\omega>}\lambda$ such that $p' \Vdash$ "$\mathrm{tr}(\underset{\sim}{q}') = \eta$" and $\ell g(\eta) = m > n$. As $p' \Vdash$ "$\eta \in \underset{\sim}{q}$", without loss of generality $\forall \ell \leq \ell g(\eta) : \bar{p}_{\eta\restriction\ell} \cup p^+_{\eta\restriction\ell} \leq p'$. By (D), $p' \Vdash$ "$\underset{\sim}{q}' \geq \underset{\sim}{q} \geq \underset{\sim}{q}^{\ell}_{\eta\restriction\ell}$" for all $\ell \leq \ell g(\eta)$. So for $\ell = n$ we get $(p', \underset{\sim}{q}') \geq (\bar{p}^n_{\eta\restriction n}, \underset{\sim}{q}^n_{\eta\restriction n})$, so $(p', \underset{\sim}{q}') \Vdash$ "$\underset{\sim}{\beta}^n = \gamma_n$" and we are done.

If we want to play $\Game^{\omega}_{\{2,\,\aleph_0,\,\aleph_1\}}$ and not only $\mathrm{E}\Game^{\omega}_{\{2,\,\aleph_0,\,\aleph_1\}}$, we also have to give conditions r_n forcing $\underset{\sim}{\beta}^n = \gamma_n$ at each step n.

Without loss of generality we may assume that for each $\eta \in T_n$, of length $< n$ we have $\langle p^n_{\eta\hat{}\langle\alpha\rangle} : \eta\hat{}\langle\alpha\rangle \in T_n\rangle$ forms a Δ-system with heart $\hat{p}^n_{\eta}$. Let $p_n = p^n_{<>} \cup \hat{p}^n_{<>}$, $q_n = \{\eta \in {}^{\omega>}\lambda : \ell g(\eta) \leq n \Rightarrow \bar{p}^n_{\eta} \cup \hat{p}^n_{\eta} \in G_{P_\lambda}, \ell g(\eta) \geq n \Rightarrow \eta \in \underset{\sim}{q}^n_{\eta\restriction n}\}$.

Then as above we can prove $(p_n, \underset{\sim}{q}_n) \Vdash$ "$\underset{\sim}{\beta}^n = \gamma_n$", then we have to show $(p_n, \underset{\sim}{q}_n) \leq (p_{n+1}, \underset{\sim}{q}_{n+1})$, and finally that $(p_{<>} \cup p^+_{<>}, \underset{\sim}{q})$ (from the end of the proof for $\mathrm{E}\Game^{\omega}_{\{2,\,\aleph_0,\,\aleph_1\}}$) is $\geq (p_n, \underset{\sim}{q}_n)$ for all n (in $P_\lambda * \underset{\sim}{Q}_\lambda$, or at least in $P_\lambda * \underset{\sim}{Q}_\lambda/\simeq$). These details are left to the reader. $\square_{6.2}$

6.3 Definition. A filter D on a set I (in a universe V) is called precipitous if the following holds:

$\Vdash_{PP(D)}$ "there are no $f_n : I \to$ ordinals, $f_n \in V$, such that $f_{n+1} <_{\underset{\sim}{E}} f_n$ for each n", where

(i) $PP(D) = \{A \subseteq I : A \neq \emptyset \bmod D\}$ ordered by reverse inclusion.

(ii) $\underset{\sim}{E}$ is the filter generated by the generic set of $PP(D)$,

(iii) $f <_{\underset{\sim}{E}} g$ means $\{\alpha \in I : f(\alpha) < g(\alpha)\} \in \underset{\sim}{E}$.

6.3A Remark. The following is an equivalent definition: a filter D over I is precipitous if player I does not have a winning strategy in the following game $\Game_{\mathrm{prec}}(D)$.

First move

player I chooses $A_1 \subseteq I, A_1 \neq \emptyset \bmod D$,

player II chooses $B_1 \subseteq A_1, B_1 \neq \emptyset \bmod D$;

n-th move

player I chooses $A_n \subseteq B_{n-1}, A_n \neq \emptyset \bmod D$,
player II chooses $B_n \subseteq A_n, B_n \neq \emptyset \bmod D$.

Player II wins if $\bigcap_{n<\omega} A_n$ (which is $= \bigcap_{n<\omega} B_n$) is nonempty (not necessarily $\neq \emptyset \bmod D$).

See Jech and Prikry [JP2], and Jech, Magidor, Mitchell and Prikry [JMMP].

6.4 Theorem. Suppose "ZFC + G.C.H. + κ is strongly inaccessible and $A = \{\lambda < \kappa : \lambda$ measurable $\}$ is stationary" is consistent. *Then:*

(1) The following statement is consistent with ZFC + G.C.H.: for every $B \subseteq \aleph_2$ for some $\delta < \aleph_2$ (in fact for some club C of λ for every $\delta \in A \cap C \cap (S^2_0)^{V[G]}$), $\mathrm{cf}(\delta) = \aleph_0$, but in $L[B \cap \delta]$, δ is a regular cardinal $> \aleph_1$ (in $V[G]$, $\lambda = \aleph_2$).

(2) If in the hypothesis $A \in D$, D is a normal ultrafilter on κ, then there is a normal precipitous filter on $\aleph_2$ to which S^2_0 belongs.

Proof. So let V be a model of ZFC +G.C.H., and let κ be a strongly inaccessible cardinal, such that $A = \{\lambda < \kappa : \lambda \text{ measurable}\}$ is stationary.

We now define by induction in $i < \kappa$ forcing notions $P_i \in V, \underset{\sim}{Q}_i \in V^{P_i}$, such that $|P_i| < \kappa$, $\langle P_j, \underset{\sim}{Q}_j : j < \kappa\rangle$ is an RCS iteration. So by 1.5(1) it suffices to define $\underset{\sim}{Q}_i$ for a given $\langle P_j, \underset{\sim}{Q}_j : j < i\rangle$.

Case A. $i = \lambda$ is a measurable cardinal, such that for every $j < \lambda$, $|P_j| < \lambda$. In this case let D_λ be a normal ultrafilter over λ (in V), and $Q_\lambda = \mathrm{Nm}'(D_\lambda)$. (In V^{P_λ}, D_λ is not an ultrafilter any more, since we may have $\Vdash_{P_\lambda} \lambda = \aleph_2$, but it will still be "large", see 6.1, 6.2).

Case B. Not case A.

In this case let Q_i be the Levy collapse of $(2^{|P_i|} + |i|^+)^V$ to $\aleph_1$, i.e., $\{f \in V^{P_i} : f$ a countable function from ω_1 to $2^{|P_i|} + |i|^+\}$.

Now by 3.5 and 6.2 it is easy to see that $P_\kappa = \mathrm{Rlim}\langle P_i, \underset{\sim}{Q}_i : i < \kappa\rangle$ is $(\{2, \aleph_0, \aleph_1\}, \omega)$-complete (note: if for i Case A occurs, then for every $j < i$, in $V^{P_{j+1}}$, D is still a normal ultrafilter), and by 5.4 it satisfies the κ-chain condition.

So clearly in V^{P_κ} G.C.H. holds, every real is from V, and $\aleph_1 = \aleph_1^V$, $\aleph_2 = \kappa$. Also if $\lambda \in A$, then $(\forall i < \lambda)$ $|P_i| < \lambda$ (prove by induction on i for each λ). Let $G \subseteq P_\lambda$, be generic, and we shall prove that $V[G]$ satisfies the requirements.

Part 1. So let $B \subseteq \aleph_2$, and let $\underset{\sim}{B} \in V$ be a P_κ-name for it. Then $C_0 = \{\delta :$ for every $i < \delta$ we have $\underset{\sim}{B} \cap \{i\}$ has a P_j-name for some $j < \delta\}$ is a closed unbounded subset of κ, because P_κ satisfies the κ-chain condition (and $P_j \lessdot P_\kappa = \bigcup_{i<\kappa} P_i$ for $j < \kappa$) and obviously $C_0 \in V$.

Now if $\lambda \in C_0 \cap A$, then we can check that $|P_i| < \lambda$ for $i < \lambda$, so case A holds hence $Q_\lambda = \mathrm{Nm}'(D_\lambda)$, hence in $V[G]$, $\mathrm{cf}(\lambda) = \aleph_0$ by 4.7(2). On the other hand, clearly $G \cap P_\lambda$ is a generic subset of P_λ (as $P_\lambda \lessdot P_\kappa$), by 5.4 P_λ satisfies the λ-chain condition, so $\Vdash_{P_\lambda}$ "$\mathrm{cf}(\lambda) = \lambda$". Hence in $V[G \cap P_\lambda]$, $A \cap \lambda$ is present, but λ is a regular cardinal $> \aleph_1$. So also in $L[A \cap \lambda], \lambda$ is a regular cardinal $> \aleph_1$. Lastly as P_κ satisfies the κ-c.c. also in V^{P_λ}, A is a stationary subset of $\kappa = \aleph_2^{V^{P_\kappa}}$. Together we finish.

Part 2. The following implies the desired conclusion; it is essentially the same proof as [JMMP] who do it for the Levy collapse; and it suffices for (2) of the theorem. It follows from Magidor [Mg80] Theorem 2.1, and is included for completeness only. (By construction, $V^P \models$ "$A \subseteq S_2^0$ and $A \in D$").

6.5 Lemma. Suppose κ is measurable, D a normal ultrafilter over κ, $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \kappa \rangle$ an RCS iteration, $|P_i| < \kappa$ for $i < \kappa$, $P = P_\kappa = \mathrm{Rlim}\bar{Q}$.

Then in V^P, D is a precipitous filter.

Proof. If not, in V^P there is $A_0 \in PP(D)$, $A_0 \Vdash_{PP(D)}$ "$\langle \underset{\sim}{f}_n : n < \omega \rangle$ is an ω-sequence of functions from κ to ordinals which belong to V^P which is decreasing mod $\underset{\sim}{E}, \underset{\sim}{f}_n \in V^P$" where $\underset{\sim}{E}$ is as in clause (ii) of Definition 6.3.

So there is $p \in P$, a P-name $\underset{\sim}{A}_0$, and $P * PP(D)$-names $\underset{\sim}{f}^\dagger_n$ of the $\underset{\sim}{f}_n$ such that $p \Vdash_{P_\lambda}$ "$\underset{\sim}{A}_0$, $\underset{\sim}{f}^\dagger_n$ are as above".

Let $B_0 = \{\lambda < \kappa : \lambda$ is strongly inaccessible and for some $p^\dagger \geq p$, $p^\dagger \in P$, and $p^\dagger \Vdash_{P_\lambda}$ "$\lambda \in \underset{\sim}{A}_0$"$\}$.

Because D is normal, κ measurable, $\{\lambda < \kappa : \lambda$ strongly inaccessible$\} \in D$, hence $B_0 \in D$. For each $\lambda \in B_0$ choose $p_{\lambda,0}$, $p \leq p_{\lambda,0} \in P$, $p_{\lambda,0} \Vdash$ "$\lambda \in \underset{\sim}{A}_0$". By 5.3 there is $B^\dagger_0 \subseteq B_0$, $B^\dagger_0 \in D$ such that $\langle p_{\lambda,0} : \lambda \in B^\dagger_0 \rangle$ is a Δ-system with heart $p^\dagger$.

Now we define by induction on $n < \omega$, $p_{\lambda,n}, p_n, p^\dagger_n, B_n, B^\dagger_n, \underset{\sim}{A}_n, \underset{\sim}{A}^\dagger_n, \underset{\sim}{g}_n, \alpha_{\lambda,n}$ (for $\lambda \in B^\dagger_n$) ($\underset{\sim}{A}_n, \underset{\sim}{A}^\dagger_n, \underset{\sim}{g}_n$ are P-names) such that:

(1) $\langle p_{\lambda,n} : \lambda \in B^\dagger_n \rangle$ is a Δ-system of members of P with heart $p^\dagger_n$.

(2) $B_{n+1} \subseteq B^\dagger_n \subseteq B_n$, $B_{n+1} \in D$,

(3) $p_{n+1} \geq p^\dagger_n \geq p_n$ all in P,

(4) $p_{\lambda,n+1} \geq p_{\lambda,n}$ both in P, $\underset{\sim}{g}_n$ a P-name of a function from κ to Ord,

(5) $p_{\lambda,n} \Vdash_P$ "$\lambda \in \underset{\sim}{A}_n$ and $\underset{\sim}{g}_{n-1}(\lambda) = \alpha_{\lambda,n-1}$", $\alpha_{\lambda,n} < \alpha_{\lambda,n-1}$ for $n > 0$,

(6) $\underset{\sim}{A}^\dagger_n = \{\lambda \in B^\dagger_n : p_{\lambda,n}$ is in the generic set of $P\}$,

(7) $p_{n+1} \Vdash_P$ "$\underset{\sim}{A}_{n+1} \in PP(D)$ and $\underset{\sim}{A}_{n+1} \subseteq A^\dagger_n$ and $[\underset{\sim}{A}_{n+1} \Vdash_{PP(D)}$ "$\underset{\sim}{f}_n/\underset{\sim}{E} = \underset{\sim}{g}_n/\underset{\sim}{E}$"$]$ and $\underset{\sim}{A}_{n+1} \subseteq \{i < \kappa : \underset{\sim}{g}_n(i) < \underset{\sim}{g}_{n-1}(i)\}$",

(8) $B_{n+1} = \{\lambda \in B^\dagger_n$: there is $p^\dagger \geq p_{\lambda,n}, p^\dagger \geq p_{n+1}$, such that for some α, $p^\dagger \Vdash$ "$\lambda \in \underset{\sim}{A}_{n+1}$ and $\underset{\sim}{g}_n(\lambda) = \alpha$"$\}$.

The definition is easy: for $n > 0$, we first we define p_n, $\underset{\sim}{A}_n$ and $\underset{\sim}{g}_{n-1}$ (by 7), then B_n and $p_{\lambda,n}$ (by 5 and 8), then $B^\dagger_n$ and $p^\dagger_n$ (by (1), using 5.4), finally $\underset{\sim}{A}^\dagger_n$ by (6).

Now as $B^\dagger_n \in D$, $\bigcap_{n<\omega} B^\dagger_n \neq \emptyset$, and if λ belongs to the intersection, $\langle \alpha_{\lambda,n} : n < \omega \rangle$ is strictly decreasing sequence of ordinals, contradiction. $\square_{6.5,6.4}$

§7. Friedman's Problem

Friedman [Fr] asked the following.

7.0 Problem. Is there for every $S \subseteq S_0^\alpha$ $(= \{i < \aleph_\alpha : \mathrm{cf}(i) = \aleph_0\})$, a closed set of order type ω_1, included in S or in $S_0^\alpha \setminus S$? We call this statement $\mathrm{Fr}(\aleph_\alpha)$. Let $\mathrm{Fr}^+(\aleph_\alpha)$ means that every stationary $S \subseteq S_0^\alpha$ includes a closed set of order type ω_1.

Van Liere proved that $\mathrm{Fr}(\aleph_2)$ implies $\aleph_2$ is a Mahlo strongly inaccessible cardinal in L; and $\mathrm{Fr}(\aleph_\alpha)+$ not $\mathrm{Fr}(\aleph_2)$ ($\aleph_\alpha$ regular $> \aleph_2$) implies $0^\#$ exists (using squares). We prove the consistency of $\mathrm{Fr}(\aleph_2)$+G.C.H. with ZFC, modulo the consistency of a measurable cardinal of order 1 . We recall the well known:

7.1 Definition. We define by induction on n what are a measurable cardinal of order n and a normal ultrafilter of order n. For $n = 0$ those are just a measurable cardinal and a normal ultrafilter. For $n+1, D$ is a normal ultrafilter of order $n+1$ on κ if $\{\lambda < \kappa : \lambda$ is measurable of order $n\} \in D$ and it is a normal ultrafilter. We call κ measurable of order $n+1$ if there is an ultrafilter of order $n+1$ on it.

7.2 Lemma. Suppose D is a normal ultrafilter on κ, $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \kappa \rangle$ an RCS iteration and $|P_i| < \kappa$ for every $i < \kappa$.

Suppose further that $G \subseteq P_\kappa$ is generic, $S \subseteq (S_0^\kappa)^{V[G]}$, $S \in V[G]$ stationary and even $\neq \emptyset \bmod D$, and (in $V[G]$) let

$Q_\kappa = \{f :$ the domain of f is some successor ordinal $\alpha < \aleph_1$, f is into S and it is increasing and continuous $\}$

So let $\underset{\sim}{S}, \underset{\sim}{Q}_\kappa$ be P_κ-names for them and $\Vdash_{P_\kappa}$ "$\underset{\sim}{S} \neq \emptyset \bmod D$ and $S \subseteq S_0^\kappa$". We then conclude:

(1) If P_κ is $\{\aleph_1\}$-semiproper, *then* so is $P_\kappa * \underset{\sim}{Q}_\kappa$,

(2) If P_κ is essentially $(\{2, \aleph_0, \aleph_1\}, \omega)$-complete, then so is $P_\kappa * \underset{\sim}{Q}_\kappa$.

Proof. (1) The problem is that Q_κ may destroy a stationary subset (of ω_2), so it is not proper, though it obviously does not add ω-sequences. So let $\underset{\sim}{S}, \underset{\sim}{Q}_\kappa$ be P_κ-names for S, Q_κ. Let $A = \{\alpha < \kappa : \alpha$ a strongly inaccessible cardinal and $(\forall i < \alpha)(|P_i| < \alpha)$.

Let λ be regular, big enough, $\bar{Q}, \underset{\sim}{Q}_\kappa$, $\underset{\sim}{S} \in H(\lambda)$, let $<^*$ be a well ordering of $H(\lambda)$ and let $N \prec (H(\lambda), \in, <^*)$ be countable, $p, q, \bar{Q}, \underset{\sim}{S}, \underset{\sim}{Q}_\kappa \in N$, $(p, \underset{\sim}{q}) \in P_\kappa * \underset{\sim}{Q}_\kappa$, and we shall prove the existence of an $\{\aleph_1\}$-semi $(N, P_\kappa * \underset{\sim}{Q}_\kappa)$- generic condition $\geq (p, \underset{\sim}{q})$. In V (hence in $H(\lambda)$), we let

$$S_0 = \{\lambda \in A : \text{ there is } p^\dagger \in P_\kappa \text{ such that } p \leq p^\dagger \text{ and } p^\dagger \Vdash \text{“}\lambda \in \underset{\sim}{S}\text{”}\}.$$

As in previous proofs $S_0 \in D$, and for each $\lambda \in S_0$ let $p_{\lambda,0} \in P_\kappa$, $p_{\lambda,0} \geq p$, $p_{\lambda,0} \Vdash$ “$\lambda \in \underset{\sim}{S}$” and for some $S_1 \subseteq S_0$, $S_1 \in D$, and $\langle p_{\lambda,0} : \lambda \in S_1 \rangle$ is a Δ-system (see 5.4(2)). As N was an elementary submodel we can assume S_0, S_1, $\langle p_{\lambda,0} : \lambda \in S_1 \rangle$ and its heart p_0 belongs to N (but of course not all included in N). Let $S_2 = S_1 \cap \bigcap\{S^\dagger : S^\dagger \in D \text{ and } S^\dagger \in N\}$, so clearly $S_2 = \{\alpha_i : i < \kappa\} \subseteq S_1$ is an indiscernible sequence over $N \bigcup \omega_1$ in the model $(H(\lambda), \in, <^*)$ (but does not belong to N). Note that for a formulae $\varphi = \varphi(x_1, \ldots, x_\kappa; y_1, \ldots, y_n)$ with n parameters $y_1, \ldots, y_n$ from ω_1, and k parameters $x_1 < \ldots < x_k$, from κ, the corresponding function $f : [\kappa]^k \to \{true, \quad false\}^{\aleph_1^n}$ is in N and it is constant on S_2. (The function f is: for $\alpha_1 < \ldots < \alpha_\kappa < \kappa$ let $f(\alpha_1, \ldots, \alpha_\kappa) = \{(\beta_1, \ldots, \beta_n, \mathbf{t}) : \beta_1, \ldots, \beta_n < \omega_1$ the $\mathbf{t}$ is the truth value of $\varphi(\alpha_1, \ldots, \alpha_\kappa, \beta_1, \ldots, \beta_n)\})$. Clearly $p \leq p_0$.

Let $N \cap P_\kappa \subseteq P_\mu$, $S_3 = S_2 \setminus (\mu + 1)$ (with $\mu < \kappa$, of course).

Clearly $S_3 \in D$ hence $S_3 \neq \emptyset$. Let $\chi \in S_3$ be such that $\chi = \sup(S_3 \cap \chi)$, and N^* be the Skolem Hull (in $(H(\lambda), \in, <^*)$) of $N \bigcup \{\chi\}$, by the choice of S_2 (and Rowbotton theorem), clearly $\delta \stackrel{\text{def}}{=} N^* \cap \omega_1 = N \cap \omega_1$ and $\kappa \cap$(Skolem hull of $N \cup \chi) = \chi$ (or you can choose such χ).

Also $P_\chi \in N^*$ (as $\langle P_i, \underset{\sim}{Q}_i : i < \kappa \rangle \in N^*, \chi \in N^*$) and clearly $P_\chi \lessdot P_\kappa$. Now P_κ is $\{\aleph_1\}$-semiproper and $\langle p_{\lambda,0} : \lambda \in S_1 \rangle \in N^*$ and $\chi \in N^*$, hence $p_{\chi,0} \in N^*$ and $p_0 \leq p_{\chi,0}$ and there is $p_1 \in P_\kappa, p_1 \geq p_{\chi,0}$, which is $\{\aleph_1\}$-semi (N^*, P_κ)-generic. As $N^* \cap \omega_1 = N \cap \omega_1$, p_1 is also $\{\aleph_1\}$-semi (N, P_κ)-generic.

Let $G \subseteq P_\kappa$ be generic, $p_1 \in G$, and we shall find $f \in \underset{\sim}{Q}_\kappa[G]$ which is $\{\aleph_1\}$-semi $(N[G], \underset{\sim}{Q}_\kappa[G])$-generic, this obviously suffices. We have $\delta = N^*[G] \cap \omega_1$. In $V[G]$, χ has cofinality $\aleph_0$, (as $p_{\chi,0} \Vdash$ “$\chi \in \underset{\sim}{S}$” and $\underset{\sim}{S} \subseteq (S_0^\chi)^{V^{P_\chi}}$). So there are $\alpha_0 < \ldots < \alpha_n < \alpha_{n+1} < \ldots, \bigcup_{n<\omega} \alpha_n = \chi$, $\alpha_n \in N^*[G]$, and let $\{\underset{\sim}{\gamma}_n : n < \omega\}$ be a list of all $\underset{\sim}{Q}_\kappa[G]$-names of countable ordinals which belong to $N[G]$ (not $N^*[G]$!). We let $f_0 = \underset{\sim}{q}[G] \in N[G]$, N_n be the Skolem Hull of $N \bigcup \{\alpha_0, \ldots, \alpha_n\}$ in $(H(\lambda), \in, <^*)$ and define by induction $f_n \in N_n[G]$ such that $\underset{\sim}{Q}_\kappa[G] \models$ "$f_{n+1} \geq f_n$", $\chi > \operatorname{Sup}\operatorname{Rang}(f_n) \geq \alpha_n$ and $f_n \Vdash_{\underset{\sim}{Q}_\kappa[G]}$ "$\underset{\sim}{\gamma}_n = \beta_n$" for some β_n. This will suffice because $\bigcup_{n<\omega} f_n \bigcup \{\langle \delta, \chi \rangle\} \in \underset{\sim}{Q}_\kappa[G]$ and is $\{\aleph_1\}$-semi $(N[G], \underset{\sim}{Q}_\kappa[G])$-generic because $N_n \cap \omega_1 \subseteq N^* \cap \omega_1 = N \cap \omega_1$. Defining f_{n+1}, the only nontrivial point is $\chi > \operatorname{Sup}\operatorname{Rang}(f_{n+1})$, but $f_{n+1} \in N_{n+1}[G]$, and $N_{n+1}[G] \cap \kappa \subseteq \chi$, (as by a version of Rowbottom's partition theorem on normal ultrafilters $S_3 \setminus \chi$ is indiscernible, in $(H(\lambda), \in, <^*)$, over $N \bigcup \chi$). Now, for every P_κ-name $\underset{\sim}{\beta} \in N_n$ of an ordinal $< \kappa$, for some $\beta \in N_n \cap \kappa, \Vdash_{P_\kappa}$ "$\underset{\sim}{\beta} < \beta$" (as P_κ satisfies the κ-c.c., see 5.3) hence $\operatorname{Sup}(N_n[G] \cap \kappa) = \operatorname{Sup}(N_n \cap \kappa) \leq \chi$. So we can define f_{n+1}, hence all the f_n's hence, as said above, we finish.

(2) By 3.8(2) the complete Boolean algebra $P = RO(P_\lambda)$ is (S, ω)-complete, where S will be $\{2, \aleph_0, \aleph_1\}$; let $\underset{\sim}{Q} = RO(\underset{\sim}{Q}_\kappa)$.

Let $(p, \underset{\sim}{q}) \in P * \underset{\sim}{Q}$. Clearly it is enough to describe the winning strategy of player II in $\mathrm{E}\supset^\omega_S\Big((p,\ \underset{\sim}{q}),\ P * \underset{\sim}{Q}\Big)$.

Suppose in the n-th move, player I chooses the $P * \underset{\sim}{Q}$-name $\underset{\sim}{\beta}_n$ of an ordinal $< \aleph_1$, and player II will choose β_n. Player II will do the following: after the n-th move he will have $(p, \underset{\sim}{q}_\eta) \in P * \underset{\sim}{Q}$ and $\underset{\sim}{\beta}'_\eta, \beta_\eta$ for every increasing sequence η of ordinals $< \kappa$ of length $\leq n$ such that:

(1) $(p, q_{<>}) = (p, \underset{\sim}{q})$,

(2) $(p, q_{\eta \restriction \ell}) \leq (p, \underset{\sim}{q}_\eta)$,

(3) $(p, \underset{\sim}{q}_\eta) \Vdash$ "$\underset{\sim}{\beta}_{\ell g(\eta)} = \underset{\sim}{\beta}'_\eta$", $\underset{\sim}{\beta}'_\eta$ a P-name (of an ordinal $< \aleph_1$).

(4) for some $A_n \in D$, for every increasing $\eta \subseteq A_n$ of length $\leq n$ we have $\beta_\eta = \beta_{\ell g(\eta)}$,

(5) $p \Vdash_P$ "$\operatorname{Sup}\operatorname{Rang}(\underset{\sim}{q}_\eta) > \operatorname{Max}\operatorname{Rang}(\eta)$",

(6) For $\eta \in {}^n\kappa$ increasing, $\langle \underset{\sim}{\beta}'_{\eta\restriction 0}, \beta_{\eta \restriction 0}, \underset{\sim}{\beta}'_{\eta \restriction 1}, \beta_{\eta\restriction 1}, \ldots, \underset{\sim}{\beta}'_{\eta\restriction n}, \beta_{\eta \restriction n}\rangle$ is an initial segment of a play of $\mathrm{E}\supset^\omega_S(p, P)$ in which player II uses his winning strategy.

Clearly Player II can do the above and it gives him a strategy. (i.e. the zero-th move is easy. In the $(n+1)$'th move; first for every increasing $\nu \in {}^n\kappa$ for every $\eta = \nu\hat{\ }\langle\alpha\rangle$ first choose q_η to satisfy (2), (5) and force $\underset{\sim}{\beta}_{\ell g(\eta)}$ to be equal to a P-name $\underset{\sim}{\beta}'_\eta$; then choose β_η to satisfy (6). Finally he chooses A_n and β_n to satisfy (4)). We have to prove that he wins by this strategy. So let $A = \bigcap\limits_n A_n$, and for $\eta \in {}^\omega A$ increasing, we know that for some (by clause 6) $p_\eta \in P_\kappa$, $p_\eta \Vdash$ "$\underset{\sim}{\beta}'_{\eta\restriction\ell} = \beta_{\eta\restriction\ell}$" for $\ell < \omega$.

Let $K = \{T : T$ a tree of finite increasing sequences from A, closed under initial segments, $\langle\rangle \in T$ and for every $\eta \in T$, $\{i \in A : \eta\hat{\ }\langle i\rangle \in T\} \in D\}$ (we can replace D by $\mathcal{D}_\kappa + A$ or $\mathcal{D}^{cb}_\kappa + A$ in this context since we only need κ-completeness). Remember $\lim T = \{\eta : \ell g(\eta) = \omega,\ \eta\restriction k \in T$ for every $k < \omega\}$. So K is closed under intersection of $< \kappa$ elements. For each $T \in K, \eta \in T$, let x^T_η be, in $RO(P)$, $\mathrm{Sup}_P\{p_\nu : \nu \in \lim T$ and $\nu \trianglerighteq \eta\}$ (Remember, we replaced P_κ by a complete Boolean algebra B^P or see 1.4(9)). Clearly x^T_η decreases with T, so as P satisfies the κ-chain condition, for some $T, x^T_{\langle\rangle}$ is minimal (i.e., $T^\dagger \subseteq T, T^\dagger \in K$ implies $x^{T^\dagger}_{\langle\rangle} = x^T_{\langle\rangle}$), and similarly for every $\eta \in T$.

Obviously,

(1) $x^T_\eta = \mathrm{Sup}\{x^T_{\eta\hat{\ }\langle i\rangle} : \eta\hat{\ } < i > \in T\}$: (this holds for any tree),

(2) $RO(P) \models 0 < b \leq x^T_\eta$ implies $\{i : RO(P) \models b \cap x^T_{\eta\hat{\ }<i>} \neq 0\} \neq \emptyset \operatorname{mod} D$ (by T's minimality).

Let $\underset{\sim}{T}^* = \{\eta : x^T_\eta$ belongs to the generic set of $P\}$. Hence $x^T_{\langle\rangle} \Vdash_P$ "$T^* \neq \emptyset$, in fact $\langle\rangle \in T^*$", and

(3) $x^T_{\langle\rangle} \Vdash_P$ " for any $\eta \in \underset{\sim}{T}^*$ for κ many i's we have $\eta\hat{\ }\langle i\rangle \in T^*$".

Now if $G \subseteq P$ is generic, $x^T_{\langle\rangle} \in G$ then $\underset{\sim}{S}[G]$ is a stationary subset of S^2_0, and $C = \{\delta$: if $\eta \in {}^{\omega>}\delta$, then $\mathrm{Rang}(\underset{\sim}{q}_\eta[G]) \subseteq \delta\}$ is closed unbounded. Hence for some η, δ with $\eta \in {}^\omega\delta$ the following holds: $\delta \in \underset{\sim}{S}[G] \cap C$, $(\forall k)\eta\restriction k \in \underset{\sim}{T}^*[G]$, and $\bigcup_{\ell<\omega} \eta(\ell) = \delta$. Let $q^* = \bigcup_{\ell<\omega} q_{\eta\restriction\ell} \bigcup\{\langle \mathrm{Sup} \bigcup_{\ell<\omega} \mathrm{Dom}(q_{\eta\restriction\ell}), \delta\rangle\} \in Q$. Let $\underset{\sim}{q}^*$ be the P-name of such a q^*. It is easy to check $(x^T_{\langle\rangle}, \underset{\sim}{q}^*)$ is as required. $\square_{7.2}$

7.2A Remark. In 7.2, we can weaken the assumption allowing $S = \emptyset \bmod D$ when e.g. $P_j, P_j/P_{i+1}$ are semiproper. (A complete proof of a better theorem appeared in XI.)

7.3 Theorem. If "ZFC + G.C.H. + there is a measurable of order 1" is consistent, then so is "ZFC + G.C.H.+ for every subset of S^2_0, either it or its complement, contains a closed copy of ω_1".

Remark. We do not try to get the weakest hypothesis. It will be interesting to find an equi-consistency result. (See XI.)

Proof. So let V satisfy G.C.H., $B \subseteq \kappa$ the set of measurables of order 0, not 1, and for every $\mu \in B$, let D_μ be a normal ultrafilter on μ; we know (see below why) that $\diamondsuit_B$ holds, and let $\bar{S} = \langle S_\mu : S_\mu \subseteq H(\mu), \mu \in B \rangle$, exemplify it. Moreover, if $S \subseteq H(\kappa)$, φ a Π^1_1 sentence, $(H(\kappa), \in, S) \vDash \varphi$ then $\{\mu \in B : S \cap H(\mu) = S_\mu, (H(\mu), \in, S_\mu) \vDash \varphi\}$ is a stationary subset of κ. It is well known that there are such S_μ. [Why $\bar{S}$ exists? Choose inductively $S_\mu \subseteq H(\mu)$ for $\mu \in B$ such that if possible $\{\mu' : \mu' \in \mu \cap B$ and $S_\mu \cap H(\mu') \neq S_{\mu'}\}$ is not a stationary subset of μ].

We define an RCS iterated forcing $\langle P_i, \underset{\sim}{Q}_i : i < \kappa \rangle$ by induction on i, such that $|P_i| < \kappa$, and for every measurable $\mu < \kappa$, $i < \mu \Rightarrow |P_i| < \mu$.

When we have defined $\underset{\sim}{Q}_j$ for $j < i$ then $P_j (j \leq i)$ are defined. If $i \in \kappa \setminus B$, $\underset{\sim}{Q}_i$ is $\{f : f$ a countable function from $\aleph_1$ to $|P_i|^+ + 2^{\aleph_2}\}$ ($2^{\aleph_2}$ of V^{P_i}).

If $i \in B$, $S_i = \langle p, \underset{\sim}{S} \rangle, p \in P_i$, $\underset{\sim}{S}$ a P_i-name, $p \Vdash_{P_i}$ "$\underset{\sim}{S}$ is a subset of S^i_0 and $\underset{\sim}{S} \neq \emptyset \bmod D$ for some normal ultrafilter $D \in V$ on i", then we let $\underset{\sim}{Q}_i$ be as in 7.2 if p is in the generic set, and trivial otherwise. We can finish as in previous proofs. $\square_{7.3}$

7.3A Remark. 1) We leave the checking that the forcing works, to the reader. For the normal ultrafilters D', D'' on κ if, for $B' \subseteq \kappa$ we have $B' \in D' \Leftrightarrow \{\lambda \in B : B' \cap \lambda \in D_\lambda\} \in D''$, then we can get in V^{P_κ} every $A' \in (D')^+$ contains a closed copy of ω_1.

2) Note that P_κ will be $\{\aleph_1\}$-semiproper and even $\{2, \aleph_0, \aleph_1\}$-complete (see 3.5).

3) In fact we could have gotten that every stationary $S \subseteq S_0^2$ contains a closed copy of ω_1, i.e. $Fr^+(\aleph_2)$ if we use 7.2A.

4) The forcing in the proof of 7.3 preserve "$\mathrm{cf}(\delta) > \aleph_0$". [Why? Use 2.7 and simple properties of the $\underset{\sim}{Q}_i$'s.]

7.4 Theorem. Suppose "ZFC + there are two supercompact cardinals" is consistent. Then so is ZFC + G.C.H. + "$Fr^+(\aleph_\alpha)$ for every regular $\aleph_\alpha (\alpha > 1)$".

Remark. Slightly better is XI 7.6. We can also get result like XI 7.2(c) i.e.

(*) for every $\theta = \mathrm{cf}(\theta) > \aleph_4$ and stationary $W_i \subseteq \{\delta < \theta : \mathrm{cf}(\delta) = \aleph_0\}$ we can find an increasing continuous $h : \omega_1 \to S$ such that $h(i) \in W_i$

Proof. Let $V \models$ "$2^\mu = \mu^+$ for $\mu \geq \lambda$" and $\kappa < \lambda$ and κ, λ are supercompact.

By a theorem of Laver [L] we can assume no κ-complete forcing will destroy the supercompactness of κ. The following is known:

$(*)_0$ If $\aleph_\alpha \geq \lambda$ is regular, $S \subseteq S_0^\alpha$ is stationary, then for some $\mu, \kappa < \mu < \lambda$ and $\delta < \aleph_\alpha$, we have $\mathrm{cf}(\delta) = \mu$ and $S \cap \delta$ is stationary.

Let P be the Levy collapse of λ to κ^+. By Baumgartner [B2], in V^P,

$(*)_1$ for every stationary $S \subseteq \lambda \cap S_0^\infty$, for some $\delta < \lambda$, $\mathrm{cf}(\delta) = \kappa$ (in V^P), $S \cap \delta$ is stationary.

Moreover,

$(*)_2$ If in V^P, $\aleph_\beta > \lambda$, $\aleph_\beta$ regular, $S \subseteq S_0^\beta$ stationary, then for some $\delta < \aleph_\beta$, $\mathrm{cf}(\delta) = \kappa$, and $S \cap \delta$ is stationary in δ.

(why? as $|P| = \lambda < \aleph_\beta = \mathrm{cf}(\aleph_\beta)$, S is the union of λ sets from V, so at least one of them is stationary (subset of λ in V), so without loss of generality $S \in V$. Now by $(*)_0$ above we can find δ as there. But P is κ-complete and collapses δ to size κ, so $\mathrm{cf}(\delta)^{V^P} = \kappa$, and $S \cap \delta$ is stationary in V^P. We want to deduce $\Vdash_P$ "$S \cap \delta$ is stationary in δ", as $\delta \in S \Rightarrow \mathrm{cf}(\delta) = \aleph_0$ this is easy).

We can conclude

(∗) in V^P, if $\mu > \kappa$ is regular, $S \subseteq \mu \cap S_0^\infty$ is stationary then for some $\delta < \mu, \mathrm{cf}(\delta) = \kappa$ and $S \cap \delta$ is stationary.

Let Q be the forcing from 7.3 A(3) (or the proof of XI 7.1), we shall show that $V^{P * \underset{\sim}{Q}}$ is as required.

Note: $\aleph_1^{V^{P*\underset{\sim}{Q}}} = \aleph_1^V$, $\aleph_2^{V^{P*\underset{\sim}{Q}}} = \kappa$, $\aleph_3^{V^{P*\underset{\sim}{Q}}} = \lambda$ and every cardinal $\mu > \lambda$ of V remains a cardinal in $V^{P*\underset{\sim}{Q}}$ and the properties "δ a limit ordinal" "$\mathrm{cf}(\delta) = \aleph_0$", $\mathrm{cf}(\delta) > \aleph_0$ are preserved by P (being $\aleph_1$-complete) and Q (by 7.3A(4)) so S_0^∞ has the same interpretation in V, V^P and $V^{P*\underset{\sim}{Q}}$.

Let, in $V^{P*\underset{\sim}{Q}}$, μ be a regular cardinal $> \aleph_1$ and $S \subseteq \mu \cap S_0^\infty$ be stationary. If $\mu \leq \kappa$, apply the proof of 7.3, 7.3A(3). If $\mu > \kappa$ then, as $V^P \models$ "$|Q| = \kappa$", S is the union of κ subsets which belong to V^P, so at least one is stationary, so w.l.o.g. $S \in V^P$.

So in V^P, for some $\delta, \mathrm{cf}(\delta) = \kappa, \delta \cap S$ is stationary; as $\underset{\sim}{Q}$ satisfies the κ-chain condition, $S \cap \delta$ is still stationary in $V^{P*\underset{\sim}{Q}}$, as required. $\square_{7.4}$