Introduction

Even though the title of this book is related to the old Lecture Notes "Proper Forcing", it is a new book: not only did it have six new chapters and some sections moved in (and the thirteenth chapter moved out to the author's book on cardinal arithmetic), but the old material and also the new have been revised for clarification and corrected several times.

Now, twenty years after its discovery, I feel that perhaps "proper forcing" is a household concept in set theory, and the reader probably knows the basic facts about forcing and proper forcing. However we demand no prerequisites except some knowledge of naive set theory (including stationary sets, Fodor Lemma, strongly inaccessible, Mahlo, weakly compact etc.; occasionally we mention some large cardinals in their combinatorial definitions (measurable, supercompact), things like $0^{\#}$ and complementary theorems showing some large cardinals are necessary, but ignorance in those directions will not hamper the reader), and the book aims at giving a complete presentation of the theory of proper and improper forcing from its beginning avoiding the metamathematical considerations; in particular no previous knowledge of forcing is demanded (though the forcing theorem is stated and explained, not proved). This is the main reason for not just publishing the additional material in a shorter book. Another reason is the complaints about shortcomings of the Proper Forcing Lecture Notes.

Forcing was founded by Cohen's proof of the independence of the continuum hypothesis; Solovay and many others developed the theory (works prior to 1977). Particularly relevant to this book are Solovay and Tennenbaum [ST] and Martin and Solovay [MS], Jensen (see [DeJo]), Silver [Si67], Mitchell [Mi1], Baumgartner [B1], Laver [L1], Abraham, Devlin and Shelah [ADS:81].

We do not elaborate here on the history of the subject (but of course there are credits and references to it in each chapter or in its sections, e.g. on Baumgartner's axiom A see VII §4). And in the first two chapters we review classical material.

Our aim is to try to develop a theory of iterated forcing for the continuum. In addition to particular consistency results that are hopefully of some interest, I try to give to the reader methods which he could use for such independence results. Many of the results are presented in an "axiomatic" framework for this reason.

The main aim of the book is thus to enable a researcher interested in an independence result of the appropriate kind, to have much of the work done for him, thus allowing him to quote general results.

We know how for any partial order P (which we call here a forcing notion) we can find an extension V^P of the universe V of set theory (e.g. thinking of V as being countable); this is explained in Chapter I. So for $G \subseteq P$ generic over V (i.e. not disjoint from any dense subset of P from V), $V[G]$ is an extension of V, a model of set theory with the same ordinals, consisting of sets constructible from V and G. By fine tuning P we can get universes of set theory with various desired properties, we are particularly interested in those of the form "for every x there is y such that ...".

So, e.g., for every Souslin tree there is a c.c.c. forcing notion P changing the universe so that it is no longer a Souslin tree, but to prove the consistency of "there is no Souslin tree" we need to repeat it till we "catch our tail". For this an iteration $P * \underset{\sim}{Q}$ is defined, and $(V^P)^{\underset{\sim}{Q}} = V^{P*\underset{\sim}{Q}}$ is proved (i.e. two successive generic extensions can be conceived as one). More delicate is the limit case, and for this case "finite support iteration" $\langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha \rangle$ works; by natural bookkeeping we can consider all Souslin trees and even all Souslin trees in V^{P_i} for $i < \alpha$, and as the c.c.c. is preserved if $\mathrm{cf}(\alpha) > \aleph_1$, since a Souslin tree can be coded as $A \subseteq \omega_1$, we "catch our tail".

In other cases countable support iteration can serve for "catching our tail"; some chain condition is then needed.

The main issue is thus *preservation*, i.e. if each $\underset{\sim}{Q}_i$ has (in V^{P_i}) a property, does P_α, the limit of an iteration, have it (this depends of course on the kind of the iteration, that is on the support). The most basic property here is not collapsing $\aleph_1$, which is treated for several kinds of iteration, i.e. supports.

Classically, a natural condition guaranteeing that $\aleph_1$ is not collapsed is (the countable chain condition) c.c.c. under finite support iterations; it is preserved (see Chapter II). It is natural to ask that the stationarity of subsets of ω_1 be preserved as well (as if $\langle S_n : n < \omega \rangle$ is a partition of ω_1 to stationary sets, we can force closed unbounded subsets E_n of ω_1 without collapsing $\aleph_1$ (see Baumgartner, Harrington and Kleinberg [BHK]) but in the limit necessarily $\aleph_1$ is collapsed). So it is natural to strengthen this somewhat (to preserve the stationarity of subsets of $\mathcal{S}_{\leq \aleph_0}(\lambda)$ for every λ), and we get the notion of proper forcing, which is preserved under the countable support iteration (see Chapter III, see alternative proof in XII §1 - and for $\aleph_1$ - free iteration in IX). This is a major notion here and the proof of its preservation serve as paradigm here.

However some forcing notions preserving $\aleph_1$ are not proper: Prikry forcing, Namba forcing. We have to use such forcing (i.e. non-proper) when during the iteration, in some intermediate stage we have to change the cofinality of some uncountable regular cardinal (e.g. $\aleph_2$) to $\aleph_1$. To deal with them we prove the preservation of semiproperness under revised countable support (see Chapter X) and show that Prikry forcing is (always) semiproper and Namba forcing is "often" semiproper. But there may be not semiproper non-proper forcing notions, in particular Namba forcing may be not semiproper (see XII §2), so we introduced properties like the S-condition (guaranteeing that no real is added (see Chapter XI)) and UP($\mathbb{I}$) (see Chapter XV), and another one (for continuum $> \aleph_2$) see Chapter XIV.

But of course we really would like to have a general framework for preserving other properties. This is dealt with in Chapter VI §1, §2; a prototypical property is preserving "every new $f \in {}^\omega\omega$ is dominated by an old $g \in {}^\omega\omega$" which is done in V §4 (but this is done only under the assumption that the forcing does not collapse $\aleph_1$ by some of the earlier versions), but includes many examples like "(f,g)-bounding for a closed enough family of pairs (f,g)". Later we deal with more general theorems (XIII §3, with a forerunner VI §3, which deals e.g. with "no new $f \in {}^\omega\omega$ dominates F").

However the case of "no new reals" is somewhat harder and it is treated in several places: in Chapter V §3 (for S-complete forcing, $S \subseteq \mathcal{S}_{\leq\aleph_0}(\lambda)$ stationary), in Chapter V §6, §7 (for $(<\omega_1)$-proper $\mathbb{D}$-complete (e.g. $\mathbb{D}$ is a simple $\aleph_1$-completeness system)), Chapter VIII §4 (for a generalization, in particular $\mathbb{D}$ is a simple 2-completeness system) and Chapter XVIII §2 (for essentially strongly proper forcing notions). Some other properties do not fall (or are not presented) under any of those cases: strongly proper (see IX, VI §6, IX §4), not collapsing $\aleph_2$ for $\aleph_1$-complete forcing when we use mixed support (VIII §1).

In all iterations somewhere we need to prove a suitable chain condition in order to catch our tails; for finite support iteration this is done directly (II), for countable support iteration see III §4, and more VII §1, VIII §2, and for revised countable support later in XVII §4 this plays a central role.

Every iteration theorem + chain condition gives the consistency of an axiom, see VII §3, VIII §4, XVII §1, §2, §3. This stresses the problem of having a general iteration theorem for "continuum $\geq \aleph_3$".

Of course much of the book deals with specific problems which serve both as an illustration of the methods and for their interest per se (see the annotated content and the separate introductions to the chapters).

The mathematical work was done between 1977 and 1989; the author approaches other aspects of our subject in Judah and Shelah [JdSh:292], [Sh:630] (on nicely defined forcing, e.g. Borel), Rosłanowski and Shelah [RoSh:470] (on even more nicely defined forcing, quite explicitly in fact), [Sh:176] §7, §8, Goldstern and Shelah [GoSh:295] (on amalgamation and projective sets in the final model) and [Sh:311] (continuing Chapter XV), [Sh:587], [Sh:F259], [Sh:655] (on replacing $\aleph_0, \aleph_1$ by λ, λ^+) and [Sh:592] (more on FS iteration of c.c.c. forcing notions).

Prerequisites: we assume that the reader has some knowledge of naive set theory (including stationary sets, Fodor lemma, etc.). The metamathematical side is avoided by stating the forcing theorem without proof in Chapter I. If you have read Jech [J] or Kunen [Ku83] you should be well prepared. Several places present some preservation theorems with less generality and they may be of some help to the reader (and they can be treated as good preparation for this book too). Let us list some of them: Baumgartner [B3], Abraham [Ab], Bartoszyński and Judah [BaJu95], Goldstern [Go], Jech [J86], Judah and Repicky [JuRe].

* * *

There are many people who gave indispensable help for the book in various stages.

Concerning the old "Proper Forcing" I heard from Baumgartner the idea of avoiding the metamathematical side, Azriel Levy, who has a much better name than the author in such matters, made notes from the lectures in the Hebrew University, rewrote them, and they appear as Chapters I, II, and part of III. These chapters were somewhat corrected and expanded by Rami Grossberg and the author. Most of XI §1–5 were lectured on and (the first version) written up by Shai Ben David. And most of all Rami Grossberg has taken care of it in all respects and Danit Sharon typed it.

Concerning the present work, Azriel Levy has helped in transforming the book from Troff to TEX. Menachem Kojman in addition wrote up first version of section I §7 from my lectures in the eighties.

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† This footnote was omitted by mistake in [Sh:b].

Notation

Natural numbers are denoted by k, ℓ, m, n.

Ordinals are denoted by $i, j, \alpha, \beta, \gamma, \delta, \xi, \zeta$, θ where δ is reserved for limit ordinals.

Cardinals (usually infinite) are denoted by $\lambda, \mu, \kappa, \chi$. Let $\aleph_\alpha$ be the α'th infinite cardinal, $\omega_\alpha = \aleph_\alpha$, $\omega = \omega_0$. Let ${}^\beta\alpha = \{f : f$ a function from β to $\alpha\}$, ${}^{\beta>}\alpha = \bigcup_{\gamma<\beta} {}^\gamma\alpha$. For sequences of ordinals (i.e., members of some ${}^\beta\alpha$), $\ell g(\eta) = \mathrm{Dom}\eta$, $\eta(i)$ the i'th ordinal so $\eta = \langle \eta(i) : i < \ell g(\eta) \rangle$. We write also $\langle \eta(0), \ldots, \eta(n) \rangle$ or $\eta(0), \ldots, \eta(n)$ as seems fit. We denote sequences, usually of ordinals, by η, ν, ρ and also τ. The concatanation of η, ν is denoted by $\eta\char`\^\nu$ Let c.l.u.b. or club mean closed unbounded.

Let $|A|$ denote the cardinality of the set A, $\mathcal{P}(A)$ denote the power set of A, and $\mathrm{cf}(\alpha)$ the cofinality of α.

"A real" means here a subset of ω, or its characteristic functions. The quantifiers "$\exists^* n$", "$\forall^* n$" (sometimes written $\exists^\infty$ and $\forall^\infty$, respectively) are abbreviations for "for infinitely many $n \in \omega$" and "for all but finitely many $n \in \omega$", respectively.

Let φ, ψ denote first order formulas. Let $\varphi(x_0, \ldots, x_{n-1})$ means every free variable of φ appears in $\{x_0, \ldots, x_{n-1}\}$.

Let P (and also Q and R) denote a partially ordered set or even a quasi ordered set (i.e., $p \le q \le p$ does not necessarily imply $p = q$). We call such P a forcing notion, and assume $\emptyset$ ot $\emptyset_P$is a minimal member of P. We use G for a generic subset of P, (usually G_α or G_{P_α} for a generic subset of P_α), (for a definition of generic see I §1). Let p, q, r denote members of forcing notions, we say p, q are incompatible in P if they have no common upper bound in P.

We do not distinguish strictly between a model M, or a forcing notion P, and their universe. $\bigtimes_{i \in I} A_i$ is the cartesian product sometimes also denoted by $\prod_{i\in I} A_i$. Distinguish (in principle) it from $\prod_{i \in I} \mu(i)$ (multiplication of cardinals). We shall not distinguish notationally between multiplication of cardinals or ordinals. For an uncountable cardinal λ of uncountable cofinality, $\mathcal{D}_\lambda$ stands for the filter generated by the closed unbounded subsets of λ; when $S \subseteq \lambda$ is stationary (also denoted by $S \not\equiv 0 \bmod \mathcal{D}_\lambda$) $\mathcal{D}_\lambda + S$ is the filter generated by $\mathcal{D}_\lambda \cup \{S\}$.

For ordinals α, β such that $\beta < \alpha$ we let $S^\alpha_\beta = \{\gamma < \aleph_\alpha : \mathrm{cf}\gamma = \aleph_\beta\}$ but when μ, λ are (infinite) cardinals such that $\mu < \lambda$ then $S^\lambda_\mu = \{\gamma < \lambda : \mathrm{cf}\lambda = \mu\}$ when no confusion arise.

We made a special effort to uniformize the notation we use but still there may be some exceptions in the chapters, for example in Chapter III, $\mathcal{S}_{\aleph_0}(A)$ is used to denote the family of countable subsets of A (see Definition III 1.2) but the same family in Chapter V is denoted by $\mathcal{S}_{<\aleph_1}(A)$ (see Definition V 3.3). So since some of the notions are redefined the reader is advised to check the nearest definition rather than the first.

Models are denoted by letters M, N perhaps with an index. We shall not always distinguish between the model and its universe, but always $|M|$ will denote the universe of M and $\|M\|$ its cardinality.

Content by Subject

1. Results Outside Set Theory

In the Appendix, Sect. 2 there are results on the power of Ext $(G, \mathbb{Z})$ assuming various weak diamonds.

2. Results in Naive Set Theory

In the Appendix §1 we investigate weak variants of the diamond (continuing Devlin and Shelah [DvSh:65].)

In the Appendix §3 we prove that CH implies some kind of weak diamond to S^2_1 (and generalizations).

In IX §3 we discuss various specializations of Aronszajn trees.

In XIII §4 a sufficient condition for the existence of Ulam filters is given.

3. Basics of Forcing

In Chapter I we explain how to use forcing and discuss some basic forcing. In Chapter II we explain iteration with $< \kappa$-support, (in particular finite support) deal for example with Martin Axiom, and more examples.

4. Specific Independence Results on Trees

In III 5.4 we present a proof that every $\aleph_1$-Aronszajn tree can be specialized by a c.c.c. forcing notion (and generally in III §5 deal with κ-trees).

In III §6 we present a proof of the consistency of " ZFC+$2^{\aleph_0} = \aleph_2$+ there is no $\aleph_2$-Aronszajn trees".

In V §6, §7, we present a proof of CON (ZFC+G.C.H.+ every Aronszajn tree is special) which seems to us more adaptable to further needs (e.g. $2^{\aleph_1}$ large).

In V §8 we present a proof of the consistency of the Kurepa hypothesis.

In VII §3 we prove consistency results on Aronszajn trees strengthening the previous ones, motivated by general topology. This implies the consistency of G.C.H. + there is a countably paracompact regular space which is not normal (see VII 3.25).

In VII §3 we present a proof of the consistency of a strengthening of CON (ZFC+G.C.H. + there is no $\aleph_1$-Kurepa tree.)

In VIII §3 we prove the consistency of ZFC+CH+SH+$2^{\aleph_1} > \aleph_2$.

In IX§4 we prove that SH$\not\Rightarrow$ " every Aronszajn tree is special" and variations.

5. Theorems on $\aleph_1$-Complete Forcing

In VIII §1 we prove that we can iterate $\aleph_2$-complee forcing and $\aleph_1$-complete forcing satisfying a strong $\aleph_2$-chain condition, without collapsing $\aleph_1$ and $\aleph_2$.

In VIII 2.7A we remark on another strong $\aleph_2$-chain condition preserved by CS iteration.

6. Chain Conditions

The c.c.c. and κ-c.c. are presented in Chapter II and the preservation of the c.c.c. by FS is proved there.

In III 4.1 we prove that a CS iteration of proper forcing notions of power $< \kappa$, κ regular $(\forall \mu < \kappa)\ \mu^{\aleph_0} < \kappa$, of length κ satisfies the κ-c.c. (by proving that if the length is $< \kappa$, it has a dense subset $< \kappa$).

Of course IV is dedicated to oracle-c.c.

Lemma V 1.5 proves the $\aleph_2$-c.c. of a CS iteration of E-complete forcing notions each of power $\aleph_1$, (assuming CH).

VII §1 deals with a strong κ-chain (e.c.c.), such that if we have a CS iteration of length κ with the condition we use for not adding reals (in V§7, VIII§4) then the forcing satisfies the κ-c.c. So this helps to get consistency results with ZFC+CH.

In VII §2 we deal with κ-pic (= κ-properness isomorphism conditions). If P satisfies it, then P satisfies the κ-c.c., and adds $< \kappa$ reals, and an iteration of length κ still satisfies the κ-c.c. This helps to get consistency results with ZFC+$2^{\aleph_0} = \aleph_2 + 2^{\aleph_1} = \lambda$ (so we start with $V \models$ "$CH + 2^{\aleph_1} = \kappa$" and this holds for the intermediate stages). Application is starting with $V \models$ "CH+$2^{\aleph_1} = \kappa$" and use a CS iteration $\bar{Q} = \langle P_i, Q_i : i < \omega_2 \rangle$ to specialize all Aronszajn trees without adding reals.

On the κ-c.c. for RCS iteration of length κ see X §5 (5.3, 5.4) and XI 6.3(2).

On λ-c.c. for κ-RS iterations see XIV.

7. Preservation of Properness and Variants

As the book was written in the generic way, i.e. as the author advance, there are several such proofs. In III §3 the preservation of properness by CS iteration is proved. In IX §2 the preservation of properness under $\aleph_1$-free limit is proved. In X §2 the preservation of properness and semi-properness under RCS is proved. (remember that semi-proper forcing may change the cofinality of some regular $\lambda > \aleph_1$ to ω). In XII §2 the preservation of properness under CS iteration is reproved, using the definition of properness by games (similarly for semiproperness) .

We deal with preservation of α-properness and $(\omega, 1)$- properness in V §2, §3, X §7.

8. Consistency Results on the Uniformization Properties and Variants

In II §4 we prove the consistency of "some family $\mathcal{P}$ of $\aleph_1$ subsets of ω has k-uniformization property" i.e. if $f_A : A \to k$ for $A \in \mathcal{P}$ then for some $f : \omega \to k$ we have

$$A \in \mathcal{P} \Rightarrow (\forall^* n \in A)(f(n) = f_A(n)).$$

In V§1, we prove the consistency with G.C.H. of "for some stationary $S \subseteq \omega_1$, $\langle A_\delta : \delta < \omega_1 \rangle$ have the $\aleph_0$-uniformization property if each A_δ is a set of order type ω, $\sup(A_\delta) = \delta$".

In VIII §4 we prove the consistency with G.C.H. of $\neg\Phi^3_{\aleph_2}$. Moreover, e.g.: for $\langle A_\delta : \delta < \omega_1 \text{ limit}\rangle$ as above, $n_\delta < 3$ then for some $f : \omega_1 \to 3$, for every $\delta < \omega_1$ limit, for some $m_\delta \in \{0, 1, 2\} \setminus \{n_\delta\}$, we have $(\forall^* \alpha \in A_\delta)(f(\alpha 0 = m_\delta)$.

9. Consistency of "Large Ideals"

In XIII we get consistency results on ω_1, in XVI use smaller ones, in XIV with larger continuum.

In XVII §4 we deal with properties consistent with $\neg 0^{\#}$ (on ω_1 being an α-th function from ω_1 to ω_1 even $\alpha = 2^{\aleph_1}$).

10. Other Consistency Results

In III §7 we deal with " for a family of $\aleph_1$ countable subsets A_i of $\omega_1 (i < \omega_1)$, order type $(A_i) < \mathrm{Sup} A_i$, there is a club $C \subseteq \omega_1$ $\bigwedge_i \mathrm{Sup}(C \cap A_i) < \mathrm{Sup} A_i$.

In VI §4 we prove the consistency of " there is no P-point".

In VI §5 we prove the consistency of " there is a unique Ramsey ultrafilter (on ω)".

In X, XI we prove various independence results on ω_2, read XI §1, X 8.4, and XI §7's theorem.

In VII §3, §4, many applications are listed.

In XVIII §4 we prove the consistency of "there is a unique P-point (on ω)" (which necessarily is a Ramsey ultrafilter).

11. Other Preservations of Generalizations of Properness

In V §3 we prove the preservation of ω-properness+ the ω^ω-bounding property (properness suffices - see VI).

In VI §1, §2, §3, XVIII §3 we deal with general preservation theorems, e.g. of covering models, hence of various specific properties which can be formulated this way (F-bounding properties, Sacks property, Laver property, PP-property, D generates a Ramsey ultrafilter, D generates a P-point (ultrafilter)). Now VI §2 relies on VI §1 to give specific results (using covering models); a different approach, guaranteeing only preservation continue to hold in limit stages, is VI §3, which apply e.g. to "$F \subseteq {}^\omega\omega$ is not dominated by $\cdot$"; this is further developed in XVIII §3.

In IX §4 we deal with preservation of "(T^*, S)-preserving" which means T^* looks like Souslin trees at levels $\delta \notin S$, and in the end we comment on possible generalizations.

Not adding reals is dealt with in V §1, §2 and mainly V §7, VIII §4 and X §7 and in XVIII §2.

In V §1 and X §3 we deal with preservation of generalizations of $\aleph_1$-completeness.

In XI §5, §6 we deal with the preservation of the S-condition (always satisfied by Nm which change the cofinality of $\aleph_2$ to $\aleph_0$ but guarantees reals are not added).

In XV §3 we deal with a generalization, i.e. proving preserving of a condition implying $\aleph_1$ not collapsed, which is satisfied by all proper forcing notions and all forcing satisfying the S-condition.

12. Forcing Axioms

On consistency, see III §4, VII §2, VIII §3, X 2.6, XIII, XIV, XVI §2. Also XVII §3 (SPFA$\nvdash$ weak Chang conjecture), XVII §1 (SPFA$\equiv$ MM), XVII §2 (SPFA$\nvdash$ PFA$^+$)

13. Counterexamples

In VII §5 we build an iteration $\langle P_n, \underset{\sim}{Q}_n : n < \omega\rangle$ such that each $\underset{\sim}{Q}_n$ does not collapse any stationary subset of ω_1; but any limit we take collapses $\aleph_1$.

In III §4 we build (in ZFC) a forcing notion of power $\aleph_1$, not collapsing $\aleph_1$ but also not preserving the stationarity of some $S \subseteq \omega_1$.

Examples of iteration $\langle P_n, \underset{\sim}{Q}_n : n < \omega\rangle$ each $\underset{\sim}{Q}_n$ is α-proper for each countable α not adding reals, but any limit we take that collapses $\aleph_1$ is presented in V 5.1 using Appendix §1 (really the previous result of Devlin and Shelah [DvSh:65]).

In XII §2 we show that many forcing notions satisfy the conditions from XI but are not semiproper. In fact if there is an $\{\aleph_1\}$-semiproper forcing notion changing cofinaltity of $\aleph_2$ to $\aleph_0$ (e.g. if Nm is $\{\aleph_1\}$-semiproper then Chang conjecture holds.

Examples of iteration $\langle P_n, \underset{\sim}{Q}_n : n < \omega\rangle$, each Q_n is a α-proper and is a $\mathbb{D}$-complete for some simple $\aleph_1$-complete completeness system but any limit adds reals is presented in XVIII §1.

Annotated Content

I. FORCING, BASIC FACTS

§0. Introduction

§1. Introducing Forcing

We define generic sets, names for a forcing notion, and formulate the forcing theorems.

§2. The Consistency of CH

Our aim is to construct by forcing a model of ZFC were CH holds. First we explain the problem of not collapsing cardinals, and second prove that $\aleph_1$-complete forcing notion does not add reals.

§3. On the Consistency of the Failure of CH

We construct a model of ZFC in which the Continuum Hypothesis fails; define the c.c.c., prove that forcing with c.c.c. forcing preserves cardinalities and cofinalities, and prove also the Δ-system lemma for finite sets.

§4. More on the Cardinality $2^{\aleph_0}$ and Cohen Reals

We construct for every cardinal λ in V which satisfies $\lambda^{\aleph_0} = \lambda$ a model $V[G]$ such that $V[G] \models 2^{\aleph_0} = \lambda$. Also Cohen reals are defined.

§5. Equivalence of Forcing Notions, and Canonical Names

We define when two forcing notions are equivalent, introduce canonical names and prove that for every P-name $\underset{\sim}{\tau}$ there is a canonical P-name $\underset{\sim}{\sigma}$ such that $\Vdash_P$ "$\underset{\sim}{\tau} = \underset{\sim}{\sigma}$".

§6. Random Reals, Collapsing Cardinals and Diamonds

We introduce random reals and the Levy collapse, and prove that for regular λ Levy$(\aleph_0, < \lambda)$ satisfies the λ-c.c. For every uncountable regular λ and a stationary $S \subseteq \lambda$ define a forcing notion P which preserve the regularity of λ and stationarity of S add no bounded subsets to λ and such that $V^P \models \Diamond_S$.

§7. ♣ Does Not Imply Diamond

♣ is a weak relative of diamond, for $S \subseteq \lambda$ stationary $\clubsuit(S)$ says that we can find $\langle A_\delta : \delta \in S\rangle$ $A_\delta \subseteq \delta = \sup(A_\delta)$, and for every unbounded $A \subseteq \lambda$ for some ($\equiv$ stationarily many) $\delta \in S$, $A_\delta \subseteq A$. If CH, $\clubsuit(\aleph_1) \equiv \diamondsuit_{\aleph_1}$, so we prove the consistency of $\clubsuit(\aleph_1) + \neg CH$, but forcing three times.

Start with $V \models GCH$, for $\diamondsuit_s$ for $S = \{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_0\}$, using it construct a "very good" $\clubsuit(S)$ sequence $\langle A_\delta : \delta \in S\rangle$. Then force by adding $> \aleph_2$ subsets of $\aleph_1$ by countable conditions, $\bar{A}$ was constructed to withstood it, lastly collapse $\aleph_1$ to $\aleph_0$ by Levy$(\aleph_0, \aleph_1)$. As any new unbounded subset of $\aleph_2$ contains an old one, $\bar{A}$ still witness $\clubsuit(S)$ but now S is a stationary subset of $\aleph_1$ of the last universe.

II. ITERATION OF FORCING

§0. Introduction

§1. The Composition of Two Forcing Notions

Composition of two notions and state the associativity lemma are defined.

§2. Iterated Forcing

We define iterated forcing, and prove that the c.c.c is preserved by FS (finite support) iteration.

§3. Martin's Axiom and Few Applications

We prove that ZFC $+2^{\aleph_0} > \aleph_1$+MA is consistent. Use MA to prove many simple uniformization properties.

§4. The Uniformization Property

Here we deal with more general uniformization properties some of which contradict MA. We strenghten the demand of almost disjointness to being a kind of tree and prove the consistency of a version contradicting MA.

§5. Maximal Almost Disjoint Families of Subsets of ω

A maximal almost disjoint (MAD) subset of $\mathcal{P}(\omega)$ is a family of infinite subsets of ω such that the intersection of any two members is finite and maximal with this property. We prove using MA that every MAD set has cardinality $2^{\aleph_0}$. Also the other direction: for every $\aleph_1 \leq \lambda < 2^{\aleph_0}$ there exists a generic extension of V by c.c.c. forcing such that in it there exist mad set of power λ.

III. PROPER FORCING

§0. Introduction

§1. Introducing Properness

We define "P is a proper forcing notion ", prove some definitions are equivalent (and deal with the closed unbounded filter $\mathcal{D}_{\aleph_0}(\lambda)$).

§2. More on Properness

We define "p is (N, P)-generic " and deal more with equivalent definitions of properness.

§3. Preservation of Properness Under CS Iteration

We prove the theorem mentioned in the title, CS is countable support.

§4. Martin's Axiom Revisited

We discuss the popularity of the c.c.c., whether we can replace it by a more natural and weaker condition. We give a sufficient condition for a countable support iteration of length κ to satisfy the κ-c.c. We prove the consistency (assuming existence of an inaccessible cardinal) of "ZFC + $2^{\aleph_0} = \aleph_1+$ Ax[for forcing notions not destroying stationary subsets of ω_1 of cardinality $\aleph_1$]". We show that the last demand cannot be replaced by "not collapsing cardinalities or cofinalities".

§5. On Aronszajn Trees

We define κ-Aronszajn and κ-Souslin trees. We then present existence theorems (for λ^+ when $\lambda = \lambda^{<\lambda}$) and prove that under MA every Aronszajn tree is special.

§6. Maybe There Is No $\aleph_2$-Aronszajn Tree

We prove the consistency of ZFC $+2^{\aleph_0} = \aleph_2+$ there is no $\aleph_2$-Aronszajn tree, the method being collapsing successively all $\lambda, \aleph_1 < \lambda < \kappa$ (κ a weakly compact cardinal) and treating every potential initial segment of an $\aleph_2$-Aronszajn tree to ensure it will not actually become an initial segment of such a tree.

§7. Closed Unbounded Subsets of ω_1 Can Run Away from Many Sets

We prove the consistency of ZFC $+2^{\aleph_0} = \aleph_2$ with if for $i < \omega_1$, $A_i \subseteq \omega_1$ has order type $<$ SupA_i, then for some closed unbounded $C \subseteq \omega_1, (\forall i)[\mathrm{Sup}(C \cap A_i) < \mathrm{Sup}A_i]$.

IV. ON ORACLE-C.C., THE LIFTING PROBLEM OF THE MEASURE ALGEBRA, AND "$\mathcal{P}(\omega)$/FINITE HAS NO NON-TRIVIAL AUTOMORPHISM"

§0. Introduction

The oracle-c.c. method enables us to start with $V \models \diamondsuit_{\aleph_1}$ and extend the set of reals ω_2-times (by iterated forcing), in the intermediate stages $\diamondsuit_{\aleph_1}$ holds, and we omit types of power $\aleph_1$ along the way, i.e. promise that some intersections of $\aleph_1$ Borel sets remain empty.

§1. On Oracle Chain Condition

One way to build forcing notions satisfying the $\aleph_1$-c.c., is by successive countable approximations including promises to maintain the predensity of countably many subset, many times using the diamond. We formalize a corresponding property ($\bar{M}$-c.c., $\bar{M}$ an oracle) and prove the equivalence of some variants of the definition.

§2. The Omitting Type Theorem

We prove that if the intersection of $\aleph_1$ Borel sets is empty and even if we add a Cohen real it remains empty, (and $\diamondsuit_{\aleph_1}$) *then* for some oracle $\bar{M}$, for every forcing notion P satisfying the $\bar{M}$-c.c., in V^P the intersection of the Borel sets (reinterpreted) is still empty.

§3. Iterations of $\bar{M}$-c.c. Forcings

We show that for Finite Support iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \leq \omega_2 \rangle$, if $\bar{M}_i \in V^{P_i}$ is an $\aleph_1$-oracle large enough for $\langle \langle \bar{M}_j, P_j, \bar{Q}_j \rangle : j < i \rangle$, and Q_i satisfies the $\bar{M}_i$-c.c. *then* $P_\alpha = \mathrm{Lim}\bar{Q}$ satisfies the $\bar{M}_0$-c.c. The first three sections give the exact formulation of the aim stated in the introduction and prove that it works.

§4. The Lifting Problem of the Measure Algebra

We show how to apply the method described in §1 - §3 in order to get a model in which the natural homomorphism $\mathcal{B} \to \mathcal{B}/$ (measure zero sets) does not lift. Where $\mathcal{B}$ is the Boolean algebra of Borel sets of reals.

§5. Automorphisms of $\mathcal{P}(\omega)$/finite

We use our method to prove that the Boolean algebra "power set of ω divided by the ideal of finite sets" can have only trivial automorphisms, where those are defined as the ones induced by permutations of ω or "almost" permutation of ω. This is equivalent to having the topological space $\beta(\mathbb{N}) \setminus \mathbb{N}$ having only trivial autohomeomorphisms. However a main lemma is delayed to the next section.

§6. Proof of Main Lemma 5.6

The point missing in §5 is: if F is an automorphism of $\mathcal{P}(\omega)$/finite, $\bar{M}$ an $\aleph_1$-oracle, then there is forcing notion P satisfying the $\bar{M}$-c.c., and a P-name $\underset{\sim}{X}$ of a real such that in V^P, for no $Y \subseteq \omega$, $(\forall A, B \in \mathcal{P}(\omega)^V)[X \cap A =_{ae} B \Rightarrow \underset{\sim}{Y} \cap F(A) =_{ae} F(B)]$, moreover even a Cohen forcing does not introduce such a Y, where $=_{ae}$ means equal modulo finite. We try to build such P, $\underset{\sim}{X}$ and prove that if we always fail F is trivial.

V. α-PROPERNESS AND NOT ADDING REALS

§0. Introduction

§1. $\mathcal{E}$-Completeness – a Sufficient Condition for Not Adding Reals

We define what it means to be $\mathcal{E}$-complete e.g., if $P \subseteq ({}^{\omega_1>}2, \lhd)$, $\mathcal{E} \subseteq \omega_1$ stationary and $f_n \subseteq f_{n+1} \in P$, $\mathrm{Sup}(\mathrm{Dom} f_n) \in \mathcal{E}$. We show that properness + $\mathcal{E}$-completeness are preserved by CS iteration and get corresponding Axiom. We also introduce a forcing axiom which is consistent with CH use it to prove a uniformization property which implies existence of a non free Whitehead group. In this way we get universe in which ω_1 is "schizophrenic": for two disjoint stationary subsets S_1, S_2 we have $\diamondsuit$ on S_1 but on S_2 the situation is as with MA.

§2. Generalizations of Properness

We introduce various variants of properness. We find it interesting and beneficial to have properties like properness dealing with sequences of countable models defining in particular α-properness.

§3. α-Properness and $(\mathcal{E}, \alpha)$-Properness Revisited

We repeat the previous section in more detail.

§4. Preservation of "ω-Properness + the ${}^\omega\omega$ Bounding Property"

P satisfies the ${}^\omega\omega$-bounding property if $[\forall f \in ({}^\omega\omega)^{V^P}][\exists g \in ({}^\omega)^V]\ (\bigwedge_n f(n) < g(n))$ i.e.: every new function $f : \omega \to \omega$ is dominated by an old one. We prove in great detail the theorem stated in the title as it serve a prototype of having preservation of "properness + X", dealt with later and is a case of a central theme to this book.

§5. Which Forcings Can We Iterate Without Adding Reals

We explain why "not adding reals" is not preserved by any kind of iteration, and suggest a remedy - $\mathbb{D}$-completeness. More elaborately the weak diamond tell us we have to exclude some forcing notion and $\mathbb{D}$-completeness seems a simple way to exclude then.

§6. Specializing an Aronszajn Tree Without Adding Reals

We prove that every Aronszajn tree can be specialized by a "nice" forcing: α-proper for every $\alpha < \omega_1$ and $\mathbb{D}$-complete for some $\aleph_1$-completeness system $\mathbb{D}$. Together with the next section this gives a proof of Con(ZFC + $\exists\kappa[\kappa$ inaccessible]) $\to$ Con(ZFC + GCH + SH) and with Chapter VIII a new proof of Jensen's Con(ZFC + GCH + SH) where SH is the Souslin Hypothesis.

§7. Iteration of $(E, \mathbb{D})$-Complete Forcing Notions

We prove that the limit of a CS iteration of Q_i each is α-proper for every $\alpha < \omega_1$, and $\mathbb{D}$-complete for some simple $\aleph_1$-completeness from V does not add reals.

§8. The Consistency of SH + CH + There Are No Kurepa Trees

VI. PRESERVATION OF ADDITIONAL PROPERTIES AND APPLICATIONS

§0. Introduction

§1. A General Preservation Theorem

We present a way to prove preservation of "properness $+\varphi$" for properties φ restricting our set of reals. Our hope is that this framework is easy to be applied to many properties. In the end we present a version, more general in one respect, less general in most respects (for readability).

§2. Examples

We specify more the framework in §1 to capture more of the common properties. Then we prove that the ${}^{\omega}\omega$-bounding property, the Sacks property, the Laver property, the PP-property, some (f, g)-bounding properties and some others come under the framework of §1; the "Sacks" and "Laver" properties appear first and most characteristically in the forcing notions bearing the respective names.

§3. Preservation of Unboundedness

We prove a presentation theorem suitable to prove the preservation of: no $g \in {}^{\omega}\omega$ satisfies $\bigwedge_{f\in F} f \leq^* g$ for a fixed $F \subseteq {}^{\omega}\omega$. We then look at other examples and prove the consistency of "ZFC + $\mathfrak{s} > \mathfrak{b}$" i.e. there is a non-dominated $F \subseteq {}^{\omega}\omega$, $|F| = \aleph_1$, but for every $\mathcal{B} \subseteq \mathcal{P}(\omega)/\text{finite}$ of cardinality $\aleph_1$, some infinite $A \subseteq \omega$ induce an ultrafilter $\{B \in \mathcal{B} : A \subseteq_{ae} B\}$, but relaying on the existence of a forcing notion presented in §6.

§4. There May Be No P-Point

We present a proof of this theorem, using the preservation of the PP-property. This may serve as a preliminary test, whether our general machinery simplifies and clarifies proofs.

§5. There May Exist a Unique Ramsey Ultrafilter

The main result is the consistency of ZFC + $2^{\aleph_0} = \aleph_2+$ "there is a unique Ramsey ultrafilter on ω up to permutations of ω". For this we have to prove that "D generates a Ramsey ultrafilter " is preserved - by another application of §1, and of course mainly to work on each iterand.

§6. On the Splitting Number $\mathfrak{s}$ and Domination Number $\mathfrak{b}$ and on $\mathfrak{a}$

In §3 we have proved the consistency of $\mathfrak{s} > \mathfrak{b}$, modulo the existence of a suitable forcing notion: the one that adds an infinite $A \subseteq \omega$ including an ultrafilter on the "old" $\mathcal{P}(\omega)$ (helping to prove $\mathfrak{s}$ large), is proper of course and is almost ${}^{\omega}\omega$-bounding (helping to prove $\mathfrak{b}$ small). We define suitable creatures, and then the forcing notion build from them, and prove that it has the desired properties. We then prove parallel lemmas on a similar forcing notion "respecting" an ideal (usually one coming from a MAD family enough indistructible). This helps to get a universe with $\mathfrak{a}$ large.

§7. On $\mathfrak{s} > \mathfrak{b} = \mathfrak{a}$

We prove the consistency of the statement in the title. The main point is that if we force as for "CON(ZFC + $\mathfrak{s} > \mathfrak{b}$)" and we build in ground model a "bad enough MAD family", a suitable preservation theorem shows that it remains MAD.

§8. On $\mathfrak{h} < \mathfrak{s} = \mathfrak{b}$

We prove the consistency of the statement in the title; for this we use an iteration in which we add many Cohen reals making $\mathfrak{s}$ large (as in §6) and dominating real. We use the Cohen reals to construct a family witnessing $\mathfrak{h}$ is small (i.e. $\aleph_1$). To show that this works use our iterating only "nicely definable" forcing notions.

VII. AXIOMS AND THEIR APPLICATION

§0. Introduction

§1. On the κ-Chain Condition, When Reals Are Not Added

When we iterate $\aleph_2$ times forcings not adding real, (but not necessarily $\aleph_1$-complete) we suggest a condition called $\aleph_2$-e.c.c. so that if each Q_i satisfies the $\aleph_2$-e.c.c., then P_{ω_2} satisfies the $\aleph_2$-c.c.

§2. The Axioms

We suggest some axioms whose consistency follows from the theorems on preservation under iteration of various properties.

§3. Applications of Axiom II

We prove several applications of an axiom consistent with G.C.H.

§4. Applications of Axiom I

We prove some applications and mention others of an axiom consistent with $2^{\aleph_0} = \aleph_2$.

§5. A Counterexample Connected to Preservation

An example is given of a countable support iteration of length ω of forcing not collapsing stationary subsets of ω_1, but the limit collapse $\aleph_1$.

VIII. κ-PIC AND NOT ADDING REALS

§0. Introduction

§1. Mixed Iteration

We prove that we can iterate $\aleph_2$-complete forcings and $\aleph_1$-complete forcings satisfying a strong $\aleph_2$-chain condition, without collapsing $\aleph_1$ and $\aleph_2$.

§2. Chain Conditions Revisited

We suggest another condition, κ - pic, to ensure the limit of the iteration P_κ satisfies the κ-c.c. The aim is e.g., to start with $V \models$ "$2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} > \aleph_2$", and use CS iteration $\bar{Q}$ of length ω_2 , each time dealing with "all problems" (there are $2^{\aleph_1}$) at once.

§3. The Axioms Revisited

We discuss what axioms we can get according to the four possibilities of the truth of $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$ but assuming always $2^{\aleph_0} \le \aleph_2$.

§4. More on Forcings Not Adding ω-Sequences and on the Diagonal Argument

We prove e.g., that CH does not imply $\Phi^3_{\aleph_1}$ (a kind of uniformization) by dealing with completeness systems which are 2-complete. Our main results are of the form: Suitable CS iteration does not add reals. So we continued the proof in Ch.V on CS iterations not adding reals, weakening the demand on the comleteness system from getting $\aleph_1$-complete filter to "the intersection of any two members is non empty, and we try to get properties preserved by the iterations.

IX. SOUSLIN HYPOTHESIS DOES NOT IMPLY "EVERY ARONSZAJN TREE IS SPECIAL"

§0. Introduction

§1. Free Limits

We look at Boolean algebras generated by a set of sentences in infinitary propositional calculus (mainly $L_{\omega_1,\omega}$). This enables us to define free limit.

§2. Preservation by Free Limit

We prove that an iteration in which we use $L_{\omega_1,\omega}$-free limit at limit stages, preserve properness, in some sense this gives a more natural proof of the preservation of properness.

§3. Aronszajn Trees: Various Ways to Specialize

We introduce some new ways to specialize Aronszajn trees, and present the old ones, as well as some connection between those properties.

§4. Independence Results

Here are the main results of the chapter. We use an iterated forcing S – st-specializing any Aronszajn tree. The problem is to make sure that some fixed tree T^* will remain not special. In fact we get that on one stationary S all Aronszajn trees are special but some trees are "Souslin" on $\omega_1 \setminus S$, a strong way to be not special. We introduce such a property of forcing "(T^*, S)-preserving" and show that is is preserved in iteration. There is a discussion of the problem and our strategy in the beginning of the section and a discussion of open problems and how the preservation theorem can be generalized.

X. ON SEMI-PROPER FORCING

§0. Introduction

We would like to deal with forcing notions not preserving e.g. "δ has uncountable cofinality".

§1. Iterated Forcing with RCS (Revised Countable Support)

The standard countable support iteration cannot be applied when some uncountable cofinalities are changed to ω, we introduce the revised version suitable for this case. Though harder to define, this iteration conforms better with our intuition concerning iterations.

§2. Proper Forcing Revisited

We define semi-properness, and prove that it is strongly preserved by RCS iteration.

§3. Pseudo-Completeness

We prove that a weakening of $\aleph_1$-completeness (compatible with changing some cofinalities to $\aleph_0$) is strongly preserved by RCS iteration.

§4. Specific Forcings

We deal with Prikry forcing, Namba forcing and generalizations which are semi-proper when we use "large" filters which may exist even on small cardinals.

§5. Chain Conditions and Abraham's Problem

We prove that under reasonable conditions when we iterate up to κ the κ-c.c. holds and get the first application: a universe V in which for every $A \subseteq \omega_1$ there is a countable subset of ω_2^V which does not belong to $L(A)$.

§6. Reflection Properties of S_0^2. Refining Abraham's Problem and Precipitous Ideals

For some large cardinal κ, by iteration we find a forcing notion P, such that $V^P \models$ "$\kappa = \aleph_2$ and $A = \{\delta < \kappa : \mathrm{cf}\delta = \aleph_0, \delta$ regular in $V\}$ is stationary ". So we may make A large in some sense, as mentioned in the title.

§7. Friedman's Problem

We collapse some large κ, by iterated forcing, which sometimes collapses $(2^{\aleph_2})^+$ to $\aleph_1$, sometimes changes the cofinality of $\aleph_2$ to $\aleph_0$, and sometimes adds a closed unbounded $C \subseteq S$ of order type ω_1, where $S \subseteq S_0^2$ is stationary. We get a model V in which every stationary $S \subseteq S_0^2 = \{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_0\}$ contains a closed copy of ω_1. By stronger hypothesis we get it for every stationary $S \subseteq S_0^\alpha$, $\mathrm{cf}(\aleph_\alpha) > \aleph_0$.

XI. CHANGING COFINALITIES: EQUI-CONSISTENCY RESULTS

§0. Introduction

We try here to weaken semiproperness.

§1. The Theorems

Here we describe what kind of a condition on forcing notions we want (i.e. satisfied by some specific notions and preserved by suitable iterations). Then we proceed to get consistency results. The proof uses RCS-iteration of length κ, κ a strongly inaccessible cardinal. In each step, we allow Namba forcing. The consistency results are mostly from Chapter X but here we use the minimal large cardinals required.

§2. The Condition

We describe here the condition, called the S-condition or $\mathbb{I}$-condition for S a set of regular cardinals $> \aleph_1$, $\mathbb{I}$ a set of $\aleph_2$-complete ideals, and some helping definitions and conventions.

§3. The Preservation Properties Guaranteed by the S-Condition

We prove that, assuming a forcing notion P satisfies our condition, forcing with P implies $\aleph_1$ is not collapsed, and (assuming CH) no real is added; and for this we need partitions theorems on trees.

§4. Forcing Notions Satisfying the S-Condition

We show that Namba forcing, Nm satisfies the $\{\aleph_2\}$-condition that Nm and Nm$'$ are really different forcing notions; that Nm, Nm$'$ may satisfy the $\aleph_4$-c.c. (while $2^{\aleph_0} = \aleph_1, 2^{\aleph_1}$ is large) We also prove $\aleph_1$-complete forcing and a forcing notion shooting a closed unbounded subset of order type ω_1 through a stationary $S \subseteq S_0^2$ satisfies our condition.

§5. Finite Composition

We prove that under suitable hypothesis, a composition of forcing satisfying an S-condition satisfies it. For this we prove a combinatorial theorem on trees.

§6. Preservation of the $\mathbb{I}$-Condition by Iteration

Here we prove that if we iterate forcing notions satisfying our conditions, but enough times collapse the present $2^{|P|}$ to $\aleph_1$, the composite forcing satisfies the condition. So usually we have large segments of cardinals which we have to collapse by $\aleph_1$-complete forcings, but for strongly inaccessible we can use Nm straight away (by 6.5).

§7. Further Independence Results

We prove the equiconsistency of "ZFC + κ is Mahlo" and " ZFC $+\aleph_2$ has the Friedman property", and a further result using weakly compact cardinal. We also prove the equiconsistency of "ZFC + κ is 2-Mahlo " and of "ZFC+ there is the club of $\aleph_2$ consisting of regular cardinals of L".

§8. Relativising to a Stationary Set

XII. IMPROPER FORCING

§0. Introduction

§1. Games and Properness

Equivalent definitions of variants of properness by games are given, and it is exemplified how the proofs of the preservation theorems in this context look like.

§2. When Is Namba Forcing Semiproper, Chang's Conjecture and Games

We prove e.g., that if some $\{\aleph_1\}$-semi proper forcing changes the cofinality of $\aleph_2$ to ω then Namba forcing is semi-proper, and Chang's Conjecture holds hence $0^{\#} \in V$. So without some fairly large cardinals (in some inner model) all semi proper forcing notions are proper, so actually Chapter X cannot do what Chapter XI does.

XIII. LARGE IDEALS ON ω_1

§0. Introduction

§1. Semi-Stationarity

We define and prove the basic properties of semi-stationarity of subsets of $\mathcal{S}_{\leq\aleph_0}(\lambda)$, and the connection with semiproper iterations up to measurables.

§2. S-Suitable Iterations and Sealing Forcing

S-suitable iterations, for $S \subseteq \omega_1$ stationary, are semi-proper iterations in which we "promise" for some subalgebras of $\mathfrak{B}{\restriction}S = \mathcal{P}(\omega_1)/(\mathcal{D}_{\omega_1} + S)$ in some V^{P_i}, that they remain $\lessdot\ \mathfrak{B}^{V^{P_j}}$ for $j > i$ ($\lessdot$ means a complete subalgebra - see 0.1(4)); usually the subalgebras are $\mathfrak{B}{\restriction}S$ in V^{P_i}. They are reasonable if we try to get a universe in which not only does $\mathcal{P}(\omega_1)/(\mathcal{D}_{\omega_1} + S)$ satisfies the $\aleph_2$-c.c. but also has more specific structure. We prove various lemmas on how we can continue such iterations, mainly using sealing forcing, which "seal" various antichains.

§3. On $\mathfrak{B} = \mathcal{P}(\omega_1)/\mathcal{D}_{\omega_1}$ Being Layered or the Levy Algebra

We prove, starting from enough supercompacts the consistency of G.C.H. + $(\mathfrak{B}\restriction S) \overset{\text{def}}{=} \mathcal{P}(\omega_1)/(\mathcal{D}_{\omega_1} + S)$ is layered, which means that ($S \subseteq \omega_1$ is stationary and) "almost" every $\mathcal{A} \subseteq \mathfrak{B}\restriction S$ of power $\aleph_1$, satisfies $\mathcal{A} \lessdot \mathfrak{B}$ (almost means a club of cofinality $\aleph_1$); we can have $S = \omega_1$, if we omit CH. Moreover we can do this without adding reals and have some forcing axiom. We then get a stronger condition of this form: $\mathfrak{B}\restriction S$ is (the completion of) the Levy($\aleph_0, < \aleph_2$) algebra; in fact gives two proofs of this with extra things as above.

§4. On $\mathcal{P}(\omega_1)/(\mathcal{D}_{\omega_1} + S)$ is Reflective or Ulam

We prove, again starting from supercompacts, that in some generic extensions not adding reals, a positive answer to the Ulam question holds: there are $\aleph_1$ measures on ω_1 (really $\{0,1\}$-measures) with *countable* additivity such that every set $\subseteq \omega_1$ is measurable with respect to at least one of them. For this we define when a filter is Ulam, and give a combinatorial sufficient condition for it (which take some space to show). Then we have forcing doing it, with extra conclusions as in the solution for layerness. Note that here we have, for $\delta < \kappa$ of cofinality $\aleph_0$, a demand dual to the one when we want to get the Levy algebra: decreasing sequences of members of $\bigcup_{i<\delta} \mathfrak{B}^{P_i}$ has a positive intersection if this is reasonable. We also prove the consistency of a weaker statement - among any $\aleph_2$ stationary subsets of S, there are $\aleph_2$, the intersection of any countably many is stationary. Before all this we deal with reflective Boolean algebra.

XIV. ITERATED FORCING WITH UNCOUNTABLE SUPPORT

§0. Introduction

We try to deal with "P does not change the cofinality of every $\mu \leq \kappa$".

§1. κ-Revised Support Iteration

We define and investigate κ-RS (κ-revised support), intended for iterating forcing not collapsing cardinals $\leq \kappa$, but possibly changing some cardinals with cofinality $> \kappa$ to ordinals with cofinality $< \kappa$.

§2. Pseudo-Completeness

We do not know to generalize the theorem on proper forcing (III) and semi proper forcing (Chapter X) but we generalize properties like pseudo completeness (see X §3). We deal with such properties, some do not add bounded subsets to κ some may add even reals. For this we use a forcing notion which has also a notion of pure extensions $[\leq_0]$. We define properties like (S, γ)-Pr_1^+ saying we have pure decidability and $\leq_0$ is γ-complete; this is less restrictive than it may look by 2.4. For such forcing notions we define an iteration with some kind of mixed support and prove the appropriate preservation theorems and theorems on chain conditions.

§3. Axioms

We deal with the relevant forcing axioms, and prove that it applies to some forcing notions.

§4. On Sacks Forcing

We prove that we can apply our condition to Sacks forcing.

§5. Abraham's Second Problem – Iterating Changing Cofinality to ω

We here apply the previous theorem to solve Abraham's second problem (solved for $\aleph_2$ in Chapter X). For this we use a forcing notion based on an initial segment of a play of a game (using a fixed winning strategy) rather than positive sets for a fixed ideal.

XV. A MORE GENERAL ITERABLE CONDITION ENSURING $\aleph_1$ IS NOT COLLAPSED

§0. Introduction

A drawback of Chapter IX is that it implies "the forcing notion does not add real"; our aim is to overcome it.

§1. Preliminaries

§2. Trees of Models and UP

We define a property of a forcing notion, $UP(\mathbb{I}, \mathbf{S}, \mathbf{W})$ where $\mathbb{I}$ is a set of ideals, $\mathbf{S}$ a set of regular cardinals (most popular value $\{\aleph_1\}$), $\mathbf{W} \subseteq \omega_1$ stationary (e.g. ω_1); this condition generalizes the one in chapter XI, gives an equivalent definition for the interesting cases there. We prove some basic properties: it does not destroy the stationarity of subsets of $\mathbf{W}$, and in suitable cases it is preserved by composition. For this we again have to deal with the partition theorems on trees.

§3. Preservation of the $UP(\mathbb{I}, \mathbf{S}, \mathbf{W})$ by Iteration

We prove that if we iterate forcing notions with such a property, each time collapsing soon enough, then the limit satisfies a similar condition. We first define (3.1), deal with limit of length ω (3.2), ω_1 (3.3), note continuity and chain condition (3.4), iterates up to an inaccessible (3.5), do one more step (3.6), prove that we have enough freedom in reorganizing the forcings such that the previous cases suffice (3.7, see 1.7) and conclude enough such iterations exists (3.8).

§4. Families of Ideals and Families of Partial Orders

A bothersome point in §2, §3 is that for $\underset{\sim}{Q}$, a P-name of a forcing notion satisfying some $UP(\underset{\sim}{\mathbb{I}}_1)$, we demand that the members of $\underset{\sim}{\mathbb{I}}_1$ were ideals in V, not V^P. We show here that we can replace a general $\underset{\sim}{\mathbb{I}}$, a P-name of set of ideals in V^P by a family of ideals $\mathbb{I}^1$ in V (not changing much the relevant properties). For this we find it better to replace the ideals by partial orders, revising the definition of UP. We then restate our iteration theorem, and have a theorem on not adding reals parallel to $\mathbf{W}$-complete.

XVI. LARGE IDEALS ON ω_1 FROM SMALLER CARDINALS

§0. Introduction

§1. Bigness of Stationary $T \subseteq \mathcal{S}_{\leq\aleph_0}(\lambda)$

We consider various bigness properties of subsets of $\mathcal{S}_{\leq\aleph_0}(\lambda)$, connected reflection properties and large cardinal properties and interrelation.

§2. Getting Large Ideal on ω_1

Here we deal with the appropriate lemmas on iterated forcing and sealing maximal antichains of $\mathcal{P}(\omega_1)/\mathcal{D}_{\omega_1}$, and then get consistency results of the form: (CH)+$\mathcal{P}(\omega_1)/(\mathcal{D}_{\omega_1}+S)$ is $\aleph_2$-saturated or even $\mathcal{P}(\omega_1/\mathcal{D}_{\omega_1}+S)$ is layered (hence we can get non regular ultrafilter) or even in the Levy algebra.

XVII. FORCING AXIOMS

§0. Introduction

§1. Semiproper Forcing Axiom Implies Martin's Maximum

We prove the SPFA=MM. Also we prove that SPFA$\vee Ax_1[\aleph_1$- complete] implies that a forcing notion is semi proper iff it does not destroy the stationarity of subsets of ω_1. We then show that SPFA implies $P(\omega_1/\mathcal{D}_{\omega_1})$ is $\aleph_2$-saturated.

§2. SPFA Does Not Imply PFA$^+$

We prove that SPFA does not imply Ax_1 [semi proper]=SPFA$^+$, and even AX_1[proper]=PFA$^+$. We first prove this consistency starting with a supercompact limit of a supercompact and then show that one supercompact suffices. We then show that properness is (provably in ZFC) not productive, and that SPFA implies Ax_1 [$\aleph_1$-complete] or even Ax_1[c.c.c. $*\ \aleph_1$-complete].

§3. Canonical Functions for ω_1

We start with $g : \omega_1 \to \omega_1$, such that for no stationary $A \subseteq \omega_2$ and $\alpha < \omega_2$ is $g{\restriction}A$ equal to an α-th function, and κ supercompact and find a κ-c.c. semi proper forcing notion forcing $\kappa = \aleph_2$, PFA$^+$, and g retains its property hence Chang conjecture fails. For this we define "Q is a g-small proper forcing notion". We then prove for $\alpha < \omega_1$ additively indecomposable, Ax[α-proper] is consistent with the existence of $\langle c_\delta : \delta < \omega_1,\ \alpha \text{ divides } \delta\rangle$, $c_\delta \subseteq \delta$ closed unbounded of order type α and for every club E of ω_1 for stationarily many $\delta < \omega_1$, $c_\delta \subseteq E$. So for $\alpha' < \alpha'' < \omega_1$ additively indecomposable $\mathrm{Ax}_{\omega_1}[\alpha'$-proper] $\not\Rightarrow$Ax[α'-proper]. Note that CS iteration of $\aleph_1$-complete and $\aleph_1$-c.c. forcing notion, is α-proper for every $\alpha < \omega_1$.

§4. A Largeness of $\mathcal{D}_{\omega_1}$ in Forcing Extensions of L and Canonical Functions

We define a property $(*)^1_\lambda$ of a cardinal λ, such that if it holds in V it holds in L (and appropriate Erdös cardinals satisfies it, and assuming $0^{\#}$ we show that $L^{\mathrm{Levy}(\aleph_0,<\kappa)} \models (*)^1_\kappa$ holds when $\lambda > \kappa > \aleph_0$).

Now we can iterate (CS) first adding λ functions from ω_1 to ω_1 and then kill all stationary subsets of ω_1 which contradict "f_α is an α-th function for $\alpha < \lambda$ and for every $g \in {}^{\omega_1}\omega_1$, $\{\mathrm{eq}(g, f_\alpha) : \alpha < \lambda\}$ is predense in $P(\omega_1)/\mathcal{D}_{\omega_1}$ (we get a more explicit version: for every $x \in H(\chi)$ there is a countable N, $x \in N \prec (H(\chi), \in, <^*_\chi)$ such that $[g \in {}^{\omega_1}\omega_1 \cap N \Rightarrow \bigvee_{\alpha \in \lambda \cap N} g(N \cap \omega_1) = f_\alpha(N \cap \omega_1)$. We then play with cardinal arithmetic.

XVIII. MORE ON PROPER FORCING

§0. Introduction

§1. No New Reals: A Counterexample and New Questions

We prove that the weak diamond (see AP§1) is not the only obstacle to "the limit of a CS iteration of proper forcing not adding real does not add reals." This points to some quite simple forcing notions, such that if we iterate them with CS the theorem in Ch V (and the weak diamond) does not tell us whether reals may be added in limits or not.

§2. Not Adding Reals

We give here a quite weak condition guaranteeing that CS iterations of forcing notions satisfying those conditions do not add reals. Those conditions are particularly strong for forcing notions of cardinality $\aleph_1$ and answer the questions raised by the first section.

§3. Other Preservations

We deal with preservations of the properties (by CS iteration of proper forcing). This was done in VI, here our framework is weaker so more general. We then carry various examples, including preservation of the P-point.

§4. There May Be a Unique P-Point

We prove the consistency of " there is a unique P-point"; it is (necessarily) a Ramsey ultrafilter but there is no other P-point. In VI§5 we have proved " there is a unique Ramsey ultrafilter". The missing part proved here is: given $D_0 \leq_{\mathrm{RK}} D_1$, D_0 a Ramsey ultrafilter, D_1 a P-point, $D_1 \not\leq_{\mathrm{RK}} D_0$, find a forcing notion Q, $\Vdash_Q$ "D_0 is not a P-point nor can it be completed to a P-point by further forcing notions with all the required properties".

APPENDIX: ON WEAK DIAMONDS AND THE POWER OF EXT

§0. Introduction

§1. Unif: a Strong Negation of the Weak Diamond

Introduce a generalization of the negation of the weak diamond (i.e., $\Phi^2_{\aleph_1}$) and prove cases of this principle from an appropriate replacement of $2^{\aleph_0} < 2^{\aleph_1}$.

§2. On the Power of Ext and Whitehead's Problem

We show that for "every non free abelian group G is not Whitehead, moreover $\mathrm{EXT}(G, \mathbb{Z})$ large", much less than $V = L$ is needed; this exemplifies the use of §1.

§3. Weak Diamond for $\aleph_2$ Assuming CH

We prove that every ladder system $\bar{\eta} = \langle \eta_\delta : \delta \in S^2_1 \rangle$ when η_δ is continuous cannot be uniformized assuming $2^{\aleph_0} = \aleph_1$. This shows some hopeful theorem on cases of "an CS iteration of $\aleph_2$-c.c. forcing notion not adding reals gives an $\aleph_2$-c.c. forcing notion not adding reals", similarly for "($\leq \aleph_1$)-support iteration of $\aleph_1$-complete forcing notions not collapsing $\aleph_2$ gives a forcing notion not collapsing $\aleph_2$.

REFERENCES