Perspectives in Mathematical Logic

Keith J. Devlin

Constructibility

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Preface to the Series

Perspectives in Mathematical Logic

(Edited by the Ω-group for "Mathematische Logik" of the Heidelberger Akademie der Wissenschaften)

On Perspectives. *Mathematical logic arose from a concern with the nature and the limits of rational or mathematical thought, and from a desire to systematise the modes of its expression. The pioneering investigations were diverse and largely autonomous. As time passed, and more particularly since the mid-fifties, interconnections between different lines of research and links with other branches of mathematics proliferated. The subject is now both rich and varied. It is the aim of the series to provide, as it were, maps or guides to this complex terrain. We shall not aim at encyclopaedic coverage; nor do we wish to prescribe, like Euclid, a definitive version of the elements of the subject. We are not committed to any particular philosophical programme. Nevertheless we have tried by critical discussion to ensure that each book represents a coherent line of thought; and that, by developing certain* themes, *it will be of greater interest than a mere assemblage of results and techniques.*

The books in the series differ in level: some are introductory, some highly specialised. They also differ in scope: some offer a wide view of an area, others present a single line of thought. Each book is, at its own level, reasonably self-contained. Although no book depends on another as prerequisite, we have encouraged authors to fit their book in with other planned volumes, sometimes deliberately seeking coverage of the same material from different points of view. We have tried to attain a reasonable degree of uniformity of notation and arrangement. However, the books in the series are written by individual authors, not by the group. Plans for books are discussed and argued about at length. Later, encouragement is given and revisions suggested. But it is the authors who do the work; if, as we hope, the series proves of value, the credit will be theirs.

History of the Ω-Group. *During 1968 the idea of an integrated series of monographs on mathematical logic was first mooted. Various discussions led to a meeting at Oberwolfach in the spring of 1969. Here the founding members of the group (R. O. Gandy, A. Levy, G. H. Müller, G. E. Sacks, D. S. Scott) discussed the project in earnest and decided to go ahead with it. Professor F. K. Schmidt and Professor Hans Hermes gave us encouragement and support. Later Hans Hermes joined the group. To begin with all was fluid. How ambitious should we be? Should we write the books ourselves? How long would it take? Plans for authorless books were promoted, savaged and scrapped. Gradually there emerged a form and a method. At the end of an infinite discussion we found our name, and that of the series. We established our centre in Heidelberg. We agreed to meet twice a year together with authors, consultants and* *assistants, generally in Oberwolfach. We soon found the value of collaboration: on the one hand the permanence of the founding group gave coherence to the over-all plans; on the other hand the stimulus of new contributors kept the project alive and flexible. Above all, we found how intensive discussion could modify the authors' ideas and our own. Often the battle ended with a detailed plan for a better book which the author was keen to write and which would indeed contribute a perspective.*

Oberwolfach, September 1975

Acknowledgements. *In starting our enterprise we essentially were relying on the personal confidence and understanding of Professor Martin Barner of the Mathematisches Forschungsinstitut Oberwolfach, Dr. Klaus Peters of Springer-Verlag and Dipl.-Ing. Penschuck of the Stiftung Volkswagenwerk. Through the Stiftung Volkswagenwerk we received a generous grant (1970 – 1973) as an initial help which made our existence as a working group possible.*

Since 1974 the Heidelberger Akademie der Wissenschaften (Mathematisch-Naturwissenschaftliche Klasse) has incorporated our enterprise into its general scientific program. The initiative for this step was taken by the late Professor F. K. Schmidt, and the former President of the Academy, Professor W. Doerr.

Through all the years, the Academy has supported our research project, especially our meetings and the continuous work on the Logic Bibliography, in an outstandingly generous way. We could always rely on their readiness to provide help wherever it was needed.

Assistance in many various respects was provided by Drs. U. Felgner and K. Gloede (till 1975) and Drs. D. Schmidt and H. Zeitler (till 1979). Last but not least, our indefatigable secretary Elfriede Ihrig was and is essential in running our enterprise.

We thank all those concerned.

Heidelberg, September 1982

R. O. Gandy *H. Hermes*
A. Levy *G. H. Müller*
G. E. Sacks *D. S. Scott*

Author's Preface

This book is intended to give a fairly comprehensive account of the theory of constructible sets at an advanced level. The intended reader is a graduate mathematician with some knowledge of mathematical logic. In particular, we assume familiarity with the notions of formal languages, axiomatic theories in formal languages, logical deductions in such theories, and the interpretation of languages in structures. Practically any introductory text on mathematical logic will supply the necessary material. We also assume some familiarity with Zermelo-Fraenkel set theory up to the development or ordinal and cardinal numbers. Any number of texts would suffice here, for instance *Devlin* (1979) or *Levy* (1979).

The book is not intended to provide a complete coverage of the many and diverse applications of the methods of constructibility theory, rather the theory itself. Such applications as are given are there to motivate and to exemplify the theory.

The book is divided into two parts. Part A ("Elementary Theory") deals with the classical definition of the L_α-hierarchy of constructible sets. With some pruning, this part could be used as the basis of a graduate course on constructibility theory. Part B ("Advanced Theory") deals with the J_α-hierarchy and the Jensen "fine-structure theory".

Chapter I is basic to the entire book. The first seven or eight sections of this chapter should be familiar to the reader, and they are included primarily for completeness, and to fix the notation for the rest of the book. Sections 9 through 11 may well be new to the reader, and are fundamental to the entire development. Thus a typical lecture course based on the book would essentially commence with section 9 of Chapter I. After Chapter II, where the basic development of constructibility theory is given, the remaining chapters of Part A are largely independent, though it would be most unnatural to cover Chapter IV without first looking at Chapter III. Likewise, in Part B, after the initial chapter (Chapter VI) there is a large degree of independence between the chapters. (Indeed, given suitable introduction by an instructor, Chapter IX could be read directly after Chapter IV.)

Constructibility theory is plagued with a large number of extremely detailed and potentially tedious arguments, involving such matters as investigating the exact logical complexity of various notions of set theory. In order to try to strike a balance between the need to have a readable book of reasonable length, and the requirements of a beginning student of the field, as our development proceeds we give progressively less detailed arguments, relying instead upon the developing ability of the reader to fill in any necessary details. Thus the experienced reader may well find that it is necessary to skip over some of the earlier proofs, whilst the novice will increasingly need to spend time supplying various details. This is particularly true of Chapter II and the latter parts of Chapter I upon which Chapter II depends.

As this is intended as an advanced reference text, we have not provided an extensive selection of exercises. Those that are given consist largely of extensions or enlargements of the main development. Together with filling in various details in our account, these should suffice for a full understanding of the main material, which is their only purpose. The exercises occur at the end of each chapter (except for Chapter I), with an indication of the stage in the text which must be reached in order to attempt them.

Chapters are numbered by Roman numerals and results by normal numerals. A reference to "II.5" means section 5 of Chapter II, whilst "V.3.7" would refer to result 7 in section 3 of Chapter V. The mention of the chapter number would be supressed within that chapter. The end of a proof is indicated by the symbol $\square$. If this occurs directly after the statement of a result, it should be understood that either the proof of the result is obvious (possibly in view of earlier remarks) or else (according to context) that the proof is a long one that will stretch over several pages and involve various lemmas. During the course of some of the longer proofs, many different symbols are introduced. In order to help the reader to keep track of them, at the points where new symbols are defined the symbol concerned appears in the outer margin of the book.

Finally, I would like to express my gratitude to all of those who have helped me in the preparation of this book. There are the members of the Ω-Group, who gave me the benefit of their views during the early stages of planning. Gert Müller kept a watchful eye on matters managerial, and Azriel Levy took on the task of editor, reading through various versions of the manuscript and making countless suggestions for improvements. Others who read through all or parts of the final manuscript are (in order of the number of errors picked up) Stevo Todorčević, Klaus Gloede, Jakub Jasinski, Włodek Bzyl, Martin Lewis, and Dieter Donder. Not to forget Ronald Jensen. Although he played no part in the writing of this book, it is clear (or will be if you get far enough into the book) that without his work there would have been practically nothing to write about!

Financial support during the preparation of the manuscript was provided by the Heidelberger Akademie der Wissenschaften.

Keith J. Devlin

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