Appendix B

Lattice Tables and Representation Theorems

Lattice tables and usl tables of various kinds were an important part of the proofs of Parts B and C which characterized various initial segments of $\mathscr{D}$. We now indicate how to construct such tables. These tables are related to representations of lattices as lattices of equivalence relations.

1. Finite Distributive Lattices

We construct lattice tables for finite distributive lattices. These tables are the ones needed to obtain the results of Chap. VI.

1.1 Definition. A lattice $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ is *distributive* if the following conditions are satisfied for all a, b, $c \in L$:

(i) $\quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

(ii) $\quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Given a finite distributive lattice $\mathscr{L}$ with $k + 1$ elements, we wish to construct a *homogeneous lattice table* for $\mathscr{L}$. This table will consist of a set of $k + 1$-tuples of integers $< n$. We recall some definitions from Chap. VI.

1.2 Definition. Let Θ be a set of $k + 1$-tuples and let $\mathscr{L}$ be a lattice with $k + 1$ elements, $\{p_0, p_1, \ldots, p_k\}$. Let $\alpha, \beta \in \Theta$ and $i, j, m \leqslant k$ be given. We say that $\alpha \equiv_i \beta$ if $\alpha^{[i]} = \beta^{[i]}$, i.e., if α and β *agree on coordinate i*. If $p_i \vee p_j = p_m$, then we say that $\alpha \equiv_{i \vee j} \beta$ if $\alpha \equiv_i \beta$ and $\alpha \equiv_j \beta$. If $p_i \wedge p_j = p_m$, then we say that $\alpha \equiv_{i \wedge j} \beta$ if there is a finite sequence $\gamma_0, \ldots, \gamma_r$ of elements of Θ such that $\alpha = \gamma_0 \equiv_i \gamma_1 \equiv_j \gamma_2 \equiv_i \cdots \equiv_j \gamma_r = \beta$.

1.3 Definition. Let $n, k \in N$ and $\Theta \subseteq [0, n)^{k+1}$ be given. Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ be a lattice with elements $\{p_0, \ldots, p_k\}$ such that p_0 is the least element of $\mathscr{L}$ and p_k is the greatest element of $\mathscr{L}$. Then Θ is said to be a *finite homogeneous lattice table* for $\mathscr{L}$ if the following conditions are satisfied:

(i) $\quad \forall \alpha, \beta \in \Theta (\alpha \equiv_0 \beta)$.

(ii) $\forall \alpha, \beta \in \Theta(\alpha \equiv_k \beta \rightarrow \alpha = \beta)$.

(iii) $\forall i,j \leqslant k(p_i \leqslant p_j \leftrightarrow \forall \alpha, \beta \in \Theta(\alpha \equiv_j \beta \rightarrow \alpha \equiv_i \beta))$.

(iv) $\forall i,j,m \leqslant k(p_i \vee p_j = p_m \leftrightarrow \forall \alpha, \beta \in \Theta(\alpha \equiv_{i \vee j} \beta \leftrightarrow \alpha \equiv_m \beta))$.

(v) $\forall i,j,m \leqslant k(p_i \wedge p_j = p_m \leftrightarrow \forall \alpha, \beta \in \Theta(\alpha \equiv_{i \wedge j} \beta \leftrightarrow \alpha \equiv_m \beta))$.

(vi) For all $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \Theta$, if

(a) $\forall i \leqslant k(\alpha_0 \equiv_i \alpha_1 \rightarrow \beta_0 \equiv_i \beta_1)$

then there is a function $f\colon \Theta \rightarrow \Theta$ such that for all $j \leqslant 1$ and α, $\beta \in \Theta$

(b) $f(\alpha_j) = \beta_j$

and

(c) $\forall i \leqslant k(\alpha \equiv_i \beta \rightarrow f(\alpha) \equiv_i f(\beta))$.

The existence of suitable lattice tables for finite distributive lattices will follow from the existence of such tables for finite boolean algebras and a canonical embedding of finite distributive lattices into finite boolean algebras.

1.4 Definition. A *boolean algebra* $\mathscr{B} = \langle B, \leqslant, \vee, \wedge, ', 0, 1\rangle$ is a distributive lattice $\langle B, \leqslant, \vee, \wedge\rangle$ with least element 0 and greatest element 1 together with a total function $'\colon B \rightarrow B$ which satisfies:

$$\forall x \in B(x \vee x' = 1 \,\&\, x \wedge x' = 0).$$

A finite boolean algebra is completely characterized by its *atoms* and a finite distributive lattice is completely characterized by its *join irreducible* elements. We define these types of elements, and indicate how they are related.

1.5 Definition. Let $\mathscr{B} = \langle B, \leqslant, \vee, \wedge, ', 0, 1\rangle$ be a finite boolean algebra. An *atom* of $\mathscr{B}$ is an element $a \in B$ such that $a \neq 0$ and for all $b \in B$, if $b < a$ then $b = 0$.

1.6 Definition. Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge\rangle$ be a finite distributive lattice. A *join irreducible element* of $\mathscr{L}$ is an element $a \in L$ such that $a \neq 0$ and for all b, $c \in L$, if $b \vee c = a$ then either $b = a$ or $c = a$.

We refer the reader to Birkhoff [1940] for the following facts about distributive lattices and boolean algebras.

1.7 Theorem. *Let $\mathscr{B} = \langle B, \leqslant, \vee, \wedge, ', 0, 1\rangle$ be a finite boolean algebra. Then every non-zero element of B can be expressed in a unique way as a join of atoms of B.*

1.8 Theorem. *Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge\rangle$ be a finite distributive lattice. Then there is a finite boolean algebra $\mathscr{B} = \langle B, \leqslant, \vee, \wedge, ', 0, 1\rangle$ and a lattice embedding $g\colon L \rightarrow B$ preserving least and greatest elements.*

The idea behind the proof of Theorem 1.8 is the following. A boolean algebra with n atoms is isomorphic to $\mathscr{B}_n$, the set of all subsets of $\{0, 1, \ldots, n-1\}$ ordered by inclusion. $\vee$ and $\wedge$ are interpreted, respectively as $\cup$ and $\cap$, and $'$ is interpreted as complementation. The isomorphism maps the ith atom of the boolean algebra to

$\{i - 1\}$. A distributive lattice $\mathscr{L}$ with n join irreducibles is isomorphic to a sublattice $\mathscr{L}^*$ of $\mathscr{B}_n$ which contains the least and greatest elements of $\mathscr{B}_n$. This isomorphism establishes a one-one correspondence between the atoms of $\mathscr{B}_n$ and the join irreducibles of $\mathscr{L}^*$ with the property that the join irreducible of $\mathscr{L}^*$ is the least element of $\mathscr{L}^*$ which contains the atom to which it corresponds.

Theorems 1.7 and 1.8 can be used to relate lattice tables for boolean algebras to lattice tables for distributive lattices.

1.9 Lemma. *Let $\mathscr{L}$ be a finite distributive lattice with universe L and n join irreducible elements, and assume that $\mathscr{L}$ is a sublattice of the boolean algebra $\mathscr{B}$ with universe B and n atoms. Let Θ be a finite homogeneous lattice table for $\mathscr{B}$ and let $\alpha, \beta \in \Theta$ and $i \in N$ be given such that $\alpha \not\equiv_i \beta$. Then there is a $p_j \in L$ such that $\alpha \not\equiv_j \beta$.*

Proof. By 1.3(iv) and Theorem 1.7, we may assume that p_i is an atom of $\mathscr{B}$. Let $p_j \in L$ be the join irreducible corresponding to p_i as described in the paragraph following Theorem 1.8. Then $p_j \geqslant p_i$ so by 1.3(iii), $\alpha \not\equiv_j \beta$. ▯

We now construct finite homogeneous lattice tables for finite boolean algebras.

1.10 Theorem. *Let $\mathscr{B} = \langle B, \leqslant, \vee, \wedge, ', 0, 1\rangle$ be a finite boolean algebra. Then $\mathscr{B}$ has a finite homogeneous lattice table Θ.*

Proof. Let $a_1, \ldots, a_n$ be the atoms of $\mathscr{B}$. Let $a_0 = 0$ and let $a_{n+1}, \ldots, a_{2^n - 1}$ be the remaining elements of B, with $a_{2^n-1} = 1$. Let Θ be the set of all 2^n-tuples α which satisfy the following conditions:

(1) $\alpha^{[0]} = 0$.

(2) $\forall i \leqslant n(\alpha^{[i]} \in \{0, 1\})$.

(3) If $n < j < 2^n$ and $A = \{a_{i_1}, \ldots, a_{i_{r(j)}}\}$ is the set of all atoms a of $\mathscr{B}$ such that $a < a_j$, and A is ordered so that $i_1 < i_2 < \cdots < i_{r(j)}$, then $\alpha^{[j]} = \Sigma\{2^{r(j)-m}\colon \alpha^{[i_m]} = 1\}$.

It follows from (1) that 1.3(i) is satisfied. It follows from Theorem 1.7 and (3) that 1.3(ii) is satisfied.

Let $i, j < 2^n$ be given. First assume that $a_i \leqslant a_j$. Let $\alpha, \beta \in \Theta$ be given such that $\alpha \equiv_j \beta$. For each $\sigma \in \Theta$ and $u < 2^n$, define $I(\sigma, u) = \{r\colon 1 \leqslant r \leqslant n \,\&\, \sigma^{[r]} = 1 \,\&\, a_r \leqslant a_u\}$. By (3), $I(\alpha, j) = I(\beta, j)$ so since $a_i \leqslant a_j$, we must have $I(\alpha, i) = I(\beta, i)$. Hence $\alpha \equiv_i \beta$. Conversely, assume that $a_i \nleqslant a_j$. Then there is an atom a_u of $\mathscr{B}$ such that $a_u \leqslant a_i$ but $a_u \nleqslant a_j$. Let α be the unique 2^n-tuple in Θ such that $\alpha^{[u]} = 1$ and $\alpha^{[r]} = 0$ for all $r \leqslant n$ such that $r \neq u$, and let β be the unique 2^n-tuple in Θ such that $\beta^{[r]} = 0$ for all $r \leqslant n$. Then $\alpha^{[j]} = \beta^{[j]} = \beta^{[i]} = 0$ but $\alpha^{[i]} \neq 0$. Hence $\alpha \equiv_j \beta$ but $\alpha \not\equiv_i \beta$, so 1.3(iii) holds.

Next let $i, j, m < 2^n$ be given. First assume that $a_i \vee a_j = a_m$. By Theorem 1.7, $\{r \leqslant n\colon a_r \leqslant a_m\} = \{r \leqslant n\colon a_r \leqslant a_i\} \cup \{r \leqslant n\colon a_r \leqslant a_j\}$. Let $\alpha, \beta \in \Theta$ be given such that $\alpha \equiv_{i \vee j} \beta$. By Definition 1.2, $\alpha \equiv_i \beta$ and $\alpha \equiv_j \beta$. Hence $I(\alpha, i) = I(\beta, i)$ and $I(\alpha, j) = I(\beta, j)$. Furthermore, $I(\alpha, i) \cup I(\alpha, j) = I(\alpha, m)$ and $I(\beta, i) \cup I(\beta, j) = I(\beta, m)$. Hence $I(\alpha, m) = I(\beta, m)$, so $\alpha \equiv_m \beta$. Next let $\alpha, \beta \in \Theta$ be given such that $\alpha \equiv_m \beta$. Since $a_i \vee a_j = a_m$, $a_i \leqslant a_m$ and $a_j \leqslant a_m$, so by 1.3(iii), $\alpha \equiv_i \beta$ and $\alpha \equiv_j \beta$. By Definition

1.2, $\alpha \equiv_{i \vee j} \beta$. Conversely, assume that $a_p = a_i \vee a_j \neq a_m$. Then there is an atom a_u of $\mathscr{B}$ such that $a_u \leqslant a_m \Leftrightarrow a_u \nleqslant a_p$. Let α be the unique 2^n-tuple in Θ such that $\alpha^{[r]} = 0$ for all $r \leqslant n$, and let β be the unique 2^n-tuple of Θ such that $\beta^{[u]} = 1$ and $\beta^{[r]} = 0$ for all $r \leqslant n$ such that $r \neq u$. Then u is an element of exactly one of $\{I(\beta, p), I(\beta, m)\}$. If $u \in I(\beta, p)$ then $\alpha \equiv_m \beta$ but $\alpha \not\equiv_p \beta$, and if $u \in I(\beta, m)$ then $\alpha \equiv_p \beta$ but $\alpha \not\equiv_m \beta$. Furthermore, $\alpha \equiv_p \beta \Leftrightarrow \alpha \equiv_{i \vee j} \beta$. Hence $\alpha \not\equiv_m \beta \Leftrightarrow \alpha \equiv_{i \vee j} \beta$, so 1.3(iv) holds.

Let $i, j, m < 2^n$ be given. First assume that $a_i \wedge a_j = a_m$. Then $\{r \leqslant n: a_r \leqslant a_m\} = \{r \leqslant n: a_r \leqslant a_i\} \cap \{r \leqslant n: a_r \leqslant a_j\}$. Let $\alpha, \beta \in \Theta$ be given such that $\alpha \equiv_{i \wedge j} \beta$. By Definition 1.2, there are $\gamma_0, \ldots, \gamma_s \in \Theta$ such that $\alpha = \gamma_0 \equiv_i \gamma_1 \equiv_j \gamma_2 \equiv_i \cdots \equiv_j \gamma_s = \beta$. Since $a_i \wedge a_j = a_m$, $a_m \leqslant a_i$ and $a_m \leqslant a_j$. Hence by 1.3(iii), $\alpha = \gamma_0 \equiv_m \gamma_1 \equiv_m \gamma_2 \equiv_m \cdots \equiv_m \gamma_s = \beta$. Next let α, $\beta \in \Theta$ be given such that $\alpha \equiv_m \beta$. Let γ be the unique element of Θ such that for all $r \leqslant n$

$$\gamma^{[r]} = \begin{cases} 1 & \text{if} \quad \alpha^{[r]} = 1 \,\&\, a_r \leqslant a_i \quad \text{or if} \quad \beta^{[r]} = 1 \,\&\, a_r \leqslant a_j \\ 0 & \text{otherwise.} \end{cases}$$

Since $a_i \wedge a_j = a_m$, $\gamma \equiv_m \alpha \equiv_m \beta$. Furthermore, $\gamma \equiv_i \alpha$ and $\gamma \equiv_j \beta$ so by Definition 1.2, $\alpha \equiv_{i \wedge j} \beta$. Conversely, suppose that $a_p = a_i \wedge a_j \neq a_m$. Then there is an atom a_u of $\mathscr{B}$ such that $a_u \leqslant a_m \Leftrightarrow a_u \nleqslant a_p$. Let α be the unique element of Θ such that $\alpha^{[r]} = 0$ for all $r \leqslant n$ and let β be the unique element of Θ such that $\beta^{[u]} = 1$ and $\beta^{[r]} = 0$ for all $r \leqslant n$ such that $r \neq u$. Then $\alpha \equiv_m \beta \Leftrightarrow \alpha \not\equiv_p \beta$. Since $a_p = a_i \wedge a_j$, we have already shown that $\alpha \equiv_p \beta \Leftrightarrow \alpha \equiv_{i \wedge j} \beta$. Hence $\alpha \equiv_m \beta \Leftrightarrow \alpha \not\equiv_{i \wedge j} \beta$, so 1.3(v) holds.

Finally, let $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \Theta$ be given satisfying 1.3(via). For every $\alpha \in \Theta$ and $r \leqslant n$, define $f(\alpha)^{[r]}$ as follows:

$$f(\alpha)^{[r]} = \begin{cases} \beta_0^{[r]} & \text{if} \quad \beta_0 \equiv_r \beta_1 \\ \alpha^{[r]} & \text{if} \quad \alpha_1 \not\equiv_r \alpha_0 \equiv_r \beta_0 \not\equiv_r \beta_1 \\ 1 - \alpha^{[r]} & \text{if} \quad \alpha_0 \not\equiv_r \alpha_1 \,\&\, \beta_0 \not\equiv_r \beta_1 \,\&\, \alpha_0 \not\equiv_r \beta_0. \end{cases}$$

Then for all $\alpha \in \Theta$, it follows from (3) that the $n + 1$-tuple $f(\alpha)$ defined above has a unique extension to an element of Θ. Hence without loss of generality, we can treat f as a map from Θ into Θ. It is easily verified that 1.3(vib) and 1.3(vic) are satisfied. ▯

1.11 Corollary. *Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ be a finite distributive lattice. Then $\mathscr{L}$ has a finite homogeneous lattice table.*

Proof. By Theorem 1.8, there is a finite boolean algebra $\mathscr{B}$ with universe B and a lattice embedding $g: L \to B$ preserving least and greatest elements. Let $B = \{a_0, \ldots, a_{2^n - 1}\}$ be ordered as in the proof of Theorem 1.10, and let $J = \{i < 2^n: \exists c \in L(g(c) = a_i)\} = \{i_0 < i_1 < \cdots < i_r\}$. By Theorem 1.10, there is a finite homogeneous lattice table Θ for $\mathscr{B}$. For each $\alpha \in \Theta$, define the $r + 1$-tuple α^* by $\alpha^{*[j]} = \alpha^{[i_j]}$ for all $j \leqslant r$, and let $\Theta^* = \{\alpha^*: \alpha \in \Theta\}$. It follows from the correspondence between atoms and join irreducibles in the proofs of Lemmas 1.9 and 1.10 that Θ^* is a finite homogeneous lattice table for $\mathscr{L}$. ▯

The lattice tables we have been discussing are closely related to representations of lattices as lattices of equivalence relations over sets as introduced by Whitman [1946]. The finite set on which the equivalence relations are defined is the lattice table Θ. The equivalence relations are just the relations $\equiv_i$ corresponding to the elements p_i of the lattice being represented, and meets and joins are defined as in Definition 1.2. Whitman defined meets and joins dually to the way they were defined in Definition 1.2, but since the dual of a lattice is a lattice, the two definitions give rise to the same class of theorems. Jonsson [1953] proved the representation theorem for distributive lattices without the homogeneity property. The proof presented here is along the lines of that given by Thomason [1970a].

***1.12 Exercise.** Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ be a countable distributive lattice with least element a_0 and greatest element a_1. For each $i \in N$, let $\mathscr{L}_i = \langle L_i, \leqslant, \vee, \wedge \rangle$ be a finite sublattice of $\mathscr{L}$ containing a_0 and a_1 such that $\mathscr{L} = \bigcup\{\mathscr{L}_i \colon i \in N\}$. Given a lattice table Θ for $\mathscr{L}_{i+1}$, we let $\Theta \upharpoonright i = \{\langle \alpha^{[0]}, \ldots, \alpha^{[n(i)]} \rangle : \alpha \in \Theta\}$, where $L_i = \{a_0, \ldots, a_{n(i)}\}$. A *sequential table* for $\mathscr{L}$ has the form $\{\Theta_{i,j} \colon i,j \in N\}$ where

(i) $\forall i,j \in N(\Theta_{i,j}$ is a finite homogeneous lattice table for $\mathscr{L}_i)$.

(ii) $\forall i \in N \exists j_0 \in N \forall j \geqslant j_0(\Theta_{i+1,j} \upharpoonright i \subseteq \Theta_{i,j})$.

(iii) $\forall i \in N(\{\langle \alpha, j \rangle \colon \alpha \in \Theta_{i,j}\}$ is recursive).

Show that $\mathscr{L}$ has a sequential table. (*Hint*: First show that given any finite boolean algebra $\mathscr{B}$, there is a finite boolean algebra $\mathscr{B}^*$ such that for all finite lattices $\mathscr{L}$, all embeddings $f \colon \mathscr{L} \hookrightarrow \mathscr{B}$ and all extensions $\mathscr{L}^*$ of $\mathscr{L}$ generated by one element, there is an embedding $f^* \colon \mathscr{L}^* \hookrightarrow \mathscr{B}^*$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{L} & \xrightarrow{\subseteq} & \mathscr{L}^* \\ f \downarrow \hookrightarrow & & \downarrow \hookrightarrow f^* \\ \mathscr{B} & \xrightarrow{\subseteq} & \mathscr{B}^* \end{array}$$

Fig. 1.1

($\mathscr{B}^*$ is obtained from $\mathscr{B}$ by splitting all the atoms of $\mathscr{B}$ in half, a recursive process.) Use this sequence of boolean algebras and the proofs of Theorem 1.10 and Corollary 1.11 to obtain the desired lattice table.)

2. Finite Lattices

We construct lattice tables for finite lattices. These tables are the ones needed to obtain the results of Chap. VII.

2.1 Definition. Let $\Theta \subseteq N^{k+1}$ be given, and let $\mathscr{L} = \langle L, \leqslant, \vee \rangle$ be a finite usl. Θ is said to be a *usl table* for $\mathscr{L}$ if there is an enumeration $p_0, \ldots, p_k$ of the elements of L such that:

(i) $\forall \alpha, \beta \in \Theta(\alpha \equiv_0 \beta)$.

(ii) $\forall \alpha, \beta \in \Theta(\alpha \equiv_k \beta \rightarrow \alpha = \beta)$.

(iii) $\forall i,j \leqslant k(p_i \leqslant p_j \leftrightarrow \forall \alpha, \beta \in \Theta(\alpha \equiv_j \beta \rightarrow \alpha \equiv_i \beta))$.

(iv) $\forall i,j,m \leqslant k(p_i \vee p_j = p_m \leftrightarrow \forall \alpha, \beta \in \Theta(\alpha \equiv_{i \vee j} \beta \leftrightarrow \alpha \equiv_m \beta))$.

We first show that every finite usl has a finite usl table.

2.2 Lemma. *Let $\mathscr{L} = \langle L, \leqslant, \vee\rangle$ be a finite usl with universe $L = \{p_0, \ldots, p_k\}$ where p_0 is the smallest element of L and p_k is the greatest element of L. Then $\mathscr{L}$ has a finite usl table $\Theta \subseteq [0, 2k)^{k+1}$.*

Proof. Θ is constructed through a process consisting of $k + 1$ steps.

Step 0. Place $\alpha_0 \in \Theta$ where $\alpha_0^{[i]} = 0$ for all $i \leqslant k$.

Step s; $0 < s < k$. Place α_{2s-1} and $\alpha_{2s} \in \Theta$ where

$$\alpha_{2s-1}^{[i]} = \begin{cases} 0 & \text{if } i = 0 \\ 2s - 1 & \text{if } i \neq 0 \end{cases}$$

and

$$\alpha_{2s}^{[i]} = \begin{cases} 0 & \text{if } i = 0 \\ 2s - 1 & \text{if } p_i \leqslant p_s \\ 2s & \text{otherwise.} \end{cases}$$

Step k. Place $\alpha_{2k-1} \in \Theta$ where

$$\alpha_{2k-1}^{[i]} = \begin{cases} 0 & \text{if } i = 0 \\ 2k - 1 & \text{if } i \neq 0. \end{cases}$$

Note that if $i \neq 0$, $\alpha \neq \beta \in \Theta$, and $\alpha \equiv_i \beta$, then $\{\alpha, \beta\} = \{\alpha_{2s-1}, \alpha_{2s}\}$ for some s such that $0 < s < k$. The lemma now follows routinely. ▯

The tables which are needed are lattice tables rather than usl tables. However, it is not known whether every finite lattice $\mathscr{L}$ has a finite homogeneous lattice table. Hence we construct an infinite homogeneous lattice table for $\mathscr{L}$. In order to be able to use this table in tree constructions of initial segments of $\mathscr{D}$, we require that the lattice tables be nicely approximated to by a sequence of usl tables.

2.3 Definition. Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge\rangle$ be a finite lattice with universe $L = \{p_0, \ldots, p_k\}$ where p_0 is the smallest element of L and p_k is the greatest element of L. Let $\{\Theta_i\colon i \in N\}$ be given such that for all $i \in N$, $\Theta_i \subseteq N^{k+1}$. Then $\{\Theta_i\colon i \in N\}$ is said to be a *sequential lattice table* for $\mathscr{L}$ if the following conditions hold:

(i) $\forall i \in N(\Theta_i$ is a finite usl table for $\mathscr{L})$.

(ii) $\forall i \in N(\Theta_i \subseteq \Theta_{i+1})$.

(iii) $\forall i, j, m \leqslant k(p_i \wedge p_j = p_m \leftrightarrow \forall r \in N\, \forall \alpha, \beta \in \Theta_r(\alpha \equiv_m \beta \leftrightarrow$

$\exists \gamma_0, \ldots, \gamma_s \in \Theta_{r+1}(\alpha = \gamma_0 \equiv_i \gamma_1 \equiv_j \gamma_2 \equiv_i \cdots \equiv_j \gamma_s = \beta)))$.

As in the previous section, the tables we need must satisfy a homogeneity condition.

2.4 Definition. Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge\rangle$ be a finite lattice with universe $L = \{p_0, \ldots, p_k\}$ where p_0 is the smallest element of L and p_k is the greatest element of L Let $\{\Theta_i\colon i \in N\}$ be a sequential lattice table for $\mathscr{L}$. Then $\{\Theta_i\colon i \in N\}$ is *weakly* homogeneous if for all $r \in N$ and all $\alpha_0, \alpha_1, \beta_0, \beta_2 \in \Theta_r$, if

$$\text{(i)} \qquad \forall i \leqslant k(\alpha_0 \equiv_i \alpha_1 \rightarrow \beta_0 \equiv_i \beta_2)$$

then there is a $\beta_1 \in \Theta_{r+1}$ and functions $f_s\colon \Theta_r \rightarrow \Theta_{r+1}$ for $s = 0, 1$ such that for all $s \leqslant 1$ and α, $\beta \in \Theta_r$,

$$\text{(ii)} \qquad f_s(\alpha_0) = \beta_s, \qquad f_s(\alpha_1) = \beta_{s+1}$$

and

$$\text{(iii)} \qquad \forall i \leqslant k(\alpha \equiv_i \beta \rightarrow f_s(\alpha) \equiv_i f_s(\beta)).$$

The sequential lattice table for $\mathscr{L}$ is built by iterating a process described in some lemmas below, which enable us to start with a usl table in the sequence and place all necessary interpolants into the next usl table in the sequence.

2.5 Lemma. *Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ be a finite lattice with universe $L = \{p_0, \ldots, p_k\}$ where p_0 is the least element of L and p_k is the greatest element of L. Let $\Theta \subseteq [0, u]^{k+1}$ be a usl table for $\mathscr{L}$. Let α, $\beta \in \Theta$ and $i, j, m \leqslant k$ be given such that $p_i \wedge p_j = p_m$ and $\alpha \equiv_m \beta$. Then there is a usl table $\Theta^* \subseteq [0, u + 4]^{k+1}$ for $\mathscr{L}$ which extends Θ and has interpolants $\gamma_1, \gamma_2, \gamma_3 \in \Theta^*$ such that $\alpha \equiv_i \gamma_1 \equiv_j \gamma_2 \equiv_i \gamma_3 \equiv_j \beta$.*

Proof. If $p_i \leqslant p_j$ or $p_j \leqslant p_i$, then we can set $\Theta^* = \Theta$ and $\gamma_1 = \gamma_2 = \gamma_3 = \beta$ to prove the lemma. Otherwise, let $\Theta^* = \Theta \cup \{\gamma_1, \gamma_2, \gamma_3\}$ where for all $n \leqslant k$

$$\gamma_1^{[n]} = \begin{cases} \alpha^{[n]} & \text{if} \quad p_n \leqslant p_i \\ u + 1 & \text{otherwise,} \end{cases}$$

$$\gamma_2^{[n]} = \begin{cases} \gamma_1^{[n]} & \text{if} \quad p_n \leqslant p_j \\ u + 2 & \text{if} \quad p_n \leqslant p_i \,\&\, p_n \nleqslant p_j \\ u + 3 & \text{otherwise,} \end{cases}$$

$$\gamma_3^{[n]} = \begin{cases} \beta^{[n]} & \text{if} \quad p_n \leqslant p_j \\ u + 2 & \text{if} \quad p_n \leqslant p_i \,\&\, p_n \nleqslant p_j \\ u + 4 & \text{otherwise.} \end{cases}$$

It is easily checked that $\alpha \equiv_i \gamma_1 \equiv_j \gamma_2 \equiv_i \gamma_3 \equiv_j \beta$ once we note that since $p_i \nleqslant p_j$, $\gamma_2^{[i]} = \gamma_3^{[i]} = u + 2$. It is routine to check that Θ^* is a usl table for $\mathscr{L}$. ∎

A finite iteration of the process described in Lemma 2.5 will yield the following corollary.

2.6 Corollary. *Let $\mathscr{L}$ be a finite lattice with universe $L = \{p_0, \ldots, p_k\}$, and let Θ_r be a finite usl table for $\mathscr{L}$. Then there is a finite usl table $\Theta_{r+1} \supseteq \Theta_r$ for $\mathscr{L}$ such that*

$$\text{(i)} \qquad \forall i, j, m \leqslant k(p_i \wedge p_j = p_m \leftrightarrow \forall \alpha, \beta \in \Theta_r(\alpha \equiv_{i \wedge j} \beta \leftrightarrow \alpha \equiv_m \beta))$$

where all congruence relations in (i) *are considered relative to Θ_{r+1}.*

The next lemma provides the interpolants required for the weak homogeneity property.

2.7 Lemma. *Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ be a finite lattice with universe $L = \{p_0, \ldots, p_k\}$ where p_0 is the least element of L and p_k is the greatest element of L, and let $\Theta \subseteq [0, u]^{k+1}$ be a finite usl table for $\mathscr{L}$. Let $\alpha_0, \alpha_1, \beta_0, \beta_2 \in \Theta$ be given such that for all $i \leqslant k$, if $\alpha_0 \equiv_i \alpha_1$ then $\beta_0 \equiv_i \beta_2$. Then there is a finite usl table $\Theta^* \supseteq \Theta$ for $\mathscr{L}$, an element $\beta_1 \in \Theta^*$ and maps $f_s\colon \Theta \to \Theta^*$ for $s \leqslant 1$ such that $f_0(\alpha_0) = \beta_0$, $f_0(\alpha_1) = f_1(\alpha_0) = \beta_1, f_1(\alpha_1) = \beta_2$, and for all $i \leqslant k$, $s \leqslant 1$ and $\alpha, \beta \in \Theta$, if $\alpha \equiv_i \beta$ then $f_s(\alpha) \equiv_i f_s(\beta)$.*

Proof. Let $p_m = \bigvee\{p_i\colon \alpha_0 \equiv_i \alpha_1\}$. By Definition 2.1(iv), $\alpha_0 \equiv_m \alpha_1$. We define β_1 by specifying $\beta_1^{[n]}$ for all $n \leqslant k$ as follows:

$$\beta_1^{[n]} = \begin{cases} \beta_0^{[n]} & \text{if } \; p_n \leqslant p_m \\ u+1 & \text{otherwise.} \end{cases}$$

The construction of Θ^* is accomplished in two steps.

Step $j \leqslant 1$. Let $\Theta = \{\alpha_0, \ldots, \alpha_r\}$. We define $\gamma_v^j = f_j(\alpha_v)$ in a sequence of $r+1$ substeps. At the first two substeps, we set $\gamma_0^0 = \beta_0$, $\gamma_1^0 = \gamma_0^1 = \beta_1$ and $\gamma_1^1 = \beta_2$.

Substep $v \geqslant 2$. Let c be the least integer not yet used as a coordinate of an element of Θ. For each $n \leqslant k$, let $i(n)$ be the least i such that $\alpha_i \equiv_n \alpha_v$. We define γ_v^j by specifying its nth coordinate for each $n \leqslant k$ as follows:

$$\gamma_v^{j[n]} = \begin{cases} \gamma_{i(n)}^{j[n]} & \text{if } \; i(n) < v \\ c & \text{otherwise.} \end{cases}$$

This completes the construction. Let $\Theta^* = \Theta \cup \{\gamma_v^j\colon j \leqslant 1 \;\&\; v \leqslant r\}$. It follows easily from the construction that for all $i \leqslant k$, $s \leqslant 1$ and $\alpha, \beta \in \Theta$, if $\alpha \equiv_i \beta$ then $f_s(\alpha) \equiv_i f_s(\beta)$. It is routine to check that Θ^* is a usl table for $\mathscr{L}$. ▯

A finite iteration of the process described in Lemma 2.7 will yield the following corollary.

2.8 Corollary. *Let $\mathscr{L}$ be a finite lattice with universe $L = \{p_0, \ldots, p_k\}$, and let Θ_r be a finite usl table for $\mathscr{L}$. Then there is a finite usl table $\Theta_{r+1} \supseteq \Theta_r$ for $\mathscr{L}$ such that for all $\alpha_0, \alpha_1, \beta_0, \beta_2 \in \Theta_r$, if* 2.4(i) *holds then there is a $\beta_1 \in \Theta_{r+1}$ and functions $f_s\colon \Theta_r \to \Theta_{r+1}$ for $s \leqslant 1$ such that* 2.4(ii) *and* (iii) *hold for all $s \leqslant 1$ and $\alpha, \beta \in \Theta_r$ (the congruence relations in* 2.4(ii) *and* (iii) *are considered relative to Θ_{r+1}).*

Corollaries 2.6 and 2.8 contain the information needed to build weakly homogeneous sequential lattice tables.

2.9 Theorem. *Let $\mathscr{L}$ be a finite lattice. Then $\mathscr{L}$ has a weakly homogeneous sequential lattice table.*

Proof. By Lemma 2.2, there is a finite usl table Θ_0 for $\mathscr{L}$. Assume by induction that Θ_r has been defined and possesses all the required properties. Let Θ_{r+1}^* be the Θ_{r+1} of Corollary 2.6, and let Θ_{r+1} be obtained from Corollary 2.8 applied to Θ_{r+1}^* in place of Θ_r. Then Θ_{r+1} has all the desired properties. ▯

2.10 Remark. The constructions of this section can be carried out in a uniformly effective manner, so that the sequence $\{\Theta_r\colon r \in N\}$ can be chosen to be a recursive sequence of canonically finite sets.

2.11 Remarks. Lemma 2.5 was proved by Jonsson [1953]. The remaining results of this section were proved by Lerman [1971]. The proofs we present are based on ideas of Jonsson [1953] and Thomason [1970a]. The formulation of Lemma 2.7 and its proof are due to Lachlan and Lebeuf [1976] who simplified the original proof.

3. Countable Uppersemilattices

We construct tables for countable usls. These are the tables needed to prove the results of Chap. VIII.

We begin with some facts about countable usls. Given a countable usl $\mathscr{U}$, we will describe an approximation to $\mathscr{U}$ by finite usls. The approximation which we describe can also be used to construct a recursive countable universal usl (a countable usl $\mathscr{U}$ is *universal* if every countable usl can be embedded into $\mathscr{U}$).

3.1 Definition. Let $\{\mathscr{L}_i\colon i\in N\}$ be a sequence of usls such that for each $i\in N$, $\mathscr{L}_i=\langle L_i, \leqslant_i, \vee_i\rangle$ and $\mathscr{L}_0\subseteq\mathscr{L}_1\subseteq\cdots$. We define the usl $\mathscr{L}=\bigcup\{\mathscr{L}_i\colon i\in N\}=\langle L, \leqslant, \vee\rangle$ by letting $L=\bigcup\{L_i\colon i\in N\}$, defining $a\leqslant b$ for $a, b\in L$ if for some $i\in N$, $a, b\in L_i$ and $a\leqslant_i b$, and defining $a\vee b$ to be the element $c\in L$ such that $a\vee_i b=c$ where i is the least element of N such that $a, b\in L_i$.

3.2 Remark. Let $\mathscr{L}$ be as in Definition 3.1 and assume that each L_i is finite and has a least element. Then each $\mathscr{L}_i$ is a lattice since every finite usl with least element is a lattice; define $a\wedge b$ to be $\bigvee\{c\colon c\leqslant a\,\&\,c\leqslant b\}$.

3.3 Remark. Let $\mathscr{L}=\langle L, \leqslant, \vee\rangle$ be a usl, and let $\mathscr{L}^*=\langle L^*, \leqslant, \vee\rangle$ be a finite subusl of $\mathscr{L}$. Fix $a\in L-L^*$ and let $\mathscr{L}^+=\langle L^+, \leqslant, \vee\rangle$ be the smallest subusl of $\mathscr{L}$ such that $L^*\cup\{a\}\subseteq L^+$. Then L^+ is finite since each element b of L^+ can be obtained as a finite join, $b=\bigvee\{d\colon d\in M\}$ where $M\subseteq L^*\cup\{a\}$.

The preceding remarks can be used to obtain nice approximations to infinite usls.

3.4 Remark. Let $\mathscr{L}=\langle L, \leqslant, \vee\rangle$ be a countable usl with least element. Then there is a sequence $\{\mathscr{L}_i\colon i\in N\}$ of finite lattices such that for each $i\in N$, $\mathscr{L}_i$ is a subusl of $\mathscr{L}_{i+1}$ and $\mathscr{L}=\bigcup\{\mathscr{L}_i\colon i\in N\}$.

In order to construct suitable lattice tables for countable usls, we must express the countable usl $\mathscr{L}=\langle L, \leqslant, \vee\rangle$ as $\mathscr{L}=\bigcup\{\mathscr{L}_i\colon i\in N\}$ as in Remark 3.4, where for each $i\in N$, $\mathscr{L}_i=\langle L_i, \leqslant_i, \vee_i, \wedge_i\rangle$. We must then find a nice way to extend a lattice table for $\mathscr{L}_i$ to a lattice table for $\mathscr{L}_{i+1}$ for each $i\in N$. This will be the aim of the lemmas of this section.

A sequential lattice table for $\mathscr{L}_i$ will be expandable to one for $\mathscr{L}_{i+1}$ only if all tables in the sequence are admissible extensions of the previous tables.

3.5 Definition. Let $\Theta\subseteq[0,n]^{k+1}$ and $\Theta^*\subseteq[0,m]^{k+1}$ be finite usl tables for the finite lattice $\mathscr{L}$. We say that Θ^* *extends* Θ if $\Theta\subseteq\Theta^*$. If Θ^* extends Θ, then Θ^* is an *admissible extension* of Θ (write $\Theta \subseteq_a \Theta^*$) if

(i) $\forall \alpha \in \Theta^* \, \exists \beta \in \Theta \, \forall \gamma \in \Theta \, \forall i \leqslant k(\alpha \equiv_i \gamma \rightarrow \alpha \equiv_i \beta)$.

Condition 3.5(i) says that any new $k+1$-tuple in Θ^* is associated with an n-tuple β in Θ which it duplicates on some coordinates, and no other duplication of coordinates of elements of Θ is possible except the coordinates duplicated from β.

The next definition is needed to relate usl tables for $\mathscr{L}_i$ to usl tables for $\mathscr{L}_{i+1}$.

3.6 Definition. Let $\mathscr{L}_0$ and $\mathscr{L}_1$ be finite lattices such that $\mathscr{L}_0 \subseteq \mathscr{L}_1$, and let $\Theta \subseteq [0,n]^{k+1}$ and $\Theta^* \subseteq [0,m]^{j+1}$ be finite usl tables for $\mathscr{L}_0$ and $\mathscr{L}_1$ respectively. We assume that the table Θ^* is organized in such a way that if $\mathscr{L}_i$ has universe L_i and $L_1 = \{p_0, \ldots, p_j\}$, then $L_0 = \{p_0, \ldots, p_k\}$. Θ is said to be the *restriction* of Θ^* to $\mathscr{L}_0$ (write $\Theta = \Theta^* \upharpoonright \mathscr{L}_0$) if $\Theta = \{\alpha \in [0,n]^{k+1} \colon \exists \beta \in \Theta^* \, \forall i \leqslant k(\alpha \equiv_i \beta)\}$. Θ^* is an *expansion* of Θ if Θ is the restriction of Θ^* to $\mathscr{L}_0$. Given $\beta \in \Theta^*$, we write $\beta \upharpoonright \mathscr{L}_0$ for the $k+1$-tuple $\alpha \in [0,n]^{k+1}$ such that $\alpha \equiv_i \beta$ for all $i \leqslant k$.

Admissible extensions will play an important role in our construction. We first check that the extensions of Sect. 2 are admissible.

3.7 Lemma. *The extension of Lemma 2.5 is admissible.*

Proof. In the notation of Lemma 2.5, we must find a β as in Definition 3.5(i) for each $\alpha \in \{\gamma_1, \gamma_2, \gamma_3\}$. For γ_1 and γ_2, we can choose β to be the α of Lemma 2.5. And for γ_3, we can choose β to be the β of Lemma 2.5. ▯

3.8 Lemma. *The extension of Lemma 2.7 is admissible.*

Proof. In the notation of Lemma 2.7, we must find a β as in Definition 3.5(i) for each $\alpha \in \{\gamma_v^j \colon j \leqslant 1 \,\&\, v < r\}$. If $\gamma_v^j = \beta_1$, then we can choose β to be the β_0 of Lemma 2.7. Otherwise, we proceed by induction as in Lemma 2.7. Fix j and v. If $j = 0$, then we can choose β to be the β_0 of Lemma 2.7. By Lemma 2.7, we note that $\beta_0 \equiv_m \beta_2$ so $\beta_1 \equiv_i \beta_2 \Rightarrow p_i \leqslant p_m$. Hence if $j = 1$, we can choose β to be the β_2 of Lemma 2.7. ▯

The next four lemmas deal with properties of admissible extensions and the existence of admissible extensions under various hypotheses.

3.9 Transitivity Lemma. *Let $\mathscr{L}$ be a finite lattice and let Θ_0, Θ_1, and Θ_2 be finite usl tables for $\mathscr{L}$ such that Θ_{i+1} is an admissible extension of Θ_i for $i \leqslant 1$. Then Θ_2 is an admissible extension of Θ_0.*

Proof. Fix $\alpha \in \Theta_2 - \Theta_0$. If $\alpha \in \Theta_1$, then a β as in 3.5(i) exists since Θ_1 is an admissible extension of Θ_0. Otherwise, $\alpha \in \Theta_2 - \Theta_1$. Hence there is a $\beta^* \in \Theta_1$ which satisfies 3.5(i) for Θ_1 in place of Θ and Θ_2 in place of Θ^*. If $\beta^* \in \Theta_0$ then the proof is complete. Otherwise, since Θ_1 is an admissible extension of Θ_0, it follows from Definition 3.5 for Θ_0 in place of Θ and Θ_1 in place of Θ^* that there is a β corresponding to $\alpha = \beta^*$ which satisfies 3.5(i). This β satisfies 3.5(i) for the original α. ▯

3.10 Corollary. *The extensions of Corollary 2.6 and Corollary 2.8 are admissible.*

Proof. Immediate from Lemmas 3.7, 3.8 and 3.9. ▯

In order to build sequential lattice tables, we must satisfy both the weak homogeneity and infemum preserving conditions. We name extensions which satisfy these properties.

3.11 Definition. Let $\mathscr{L}$ be a finite lattice and let Θ and Θ^* be finite usl tables for $\mathscr{L}$ such that Θ^* extends Θ. Θ^* is said to be a *type 1* extension of Θ if Θ^* is an admissible extension of Θ and the conclusions of Corollaries 2.6 and 2.8 hold for Θ in place of Θ_r and Θ^* in place of Θ_{r+1}. (In other words, Θ^* is a suitable successor for Θ as an element of a weakly homogeneous sequential lattice table for $\mathscr{L}$.)

3.12 Corollary. *Let $\mathscr{L}$ be a finite lattice and let Θ be a finite usl table for $\mathscr{L}$. Then Θ has a type 1 extension.*

Proof. Starting with Θ, extend it to Θ^+ using Corollary 2.6 and then extend Θ^+ to Θ^* using Corollary 2.8. By Corollary 3.10 and the Transitivity Lemma, Θ^* is a type 1 extension of Θ. ▯

We now investigate the interaction between restrictions and admissibility.

3.13 Restriction Lemma. *Let $\mathscr{L}$ be a finite lattice and let Θ and Θ^* be finite usl tables for $\mathscr{L}$ such that Θ^* is an admissible extension of Θ. Let $\mathscr{L}^*$ be a subusl of $\mathscr{L}$. Then $\Theta^* \restriction \mathscr{L}^*$ is an admissible extension of $\Theta \restriction \mathscr{L}^*$, i.e., the following diagram commutes.*

$$\begin{array}{ccc} \Theta & \xrightarrow{\subseteq_a} & \Theta^* \\ \restriction\downarrow & & \downarrow\restriction \\ \Theta \restriction \mathscr{L}^* & \xrightarrow[\subseteq_a]{} & \Theta^* \restriction \mathscr{L}^* \end{array}$$

Fig. 3.1

Proof. Given $\alpha \in \Theta^* \restriction \mathscr{L}^* - \Theta \restriction \mathscr{L}^*$, find $\alpha^+ \in \Theta$ such that $\alpha = \alpha^+ \restriction \mathscr{L}^*$. Let $\beta^+ \in \Theta$ correspond to α^+ as in 3.5(i). Then $\beta = \beta^+ \restriction \mathscr{L}^*$ corresponds to α as in 3.5(i). ▯

The remaining lemmas which allow us to combine tables, allow us to do so only if the tables are sufficiently disjoint. We introduce notation for various disjointness conditions.

3.14 Definition. Let $j, k \in N$ be given such that $j \leqslant k$, and let $\Psi, \Psi^* \subseteq N^{k+1}$ and $\Theta \subseteq N^{j+1}$ be finite usl tables. We say that $\Psi \equiv_0 \Theta$ if

(i) $\quad \forall \alpha \in \Psi\, \forall \beta \in \Theta\, \forall i \leqslant j(\alpha \equiv_i \beta \leftrightarrow i = 0).$

We say that Ψ^* and Θ *are disjoint above* Ψ if

(ii) $\quad \forall \alpha \in \Psi^*\, \forall \beta \in \Theta\, \forall i \leqslant j(\alpha \equiv_i \beta \rightarrow \exists \gamma \in \Psi(\alpha \equiv_i \gamma \equiv_i \beta)).$

Different tables for the same lattice can be combined as follows.

3.15 Joint Embedding Lemma. *Let $\mathscr{L}$ be a finite lattice, and let Θ and Θ^* be finite usl tables for $\mathscr{L}$ such that $\Theta \equiv_0 \Theta^*$. Then $\Theta \cup \Theta^*$ is a usl table for $\mathscr{L}$ which is an* *admissible extension of both Θ and Θ^*, i.e., the following diagram commutes*:

$$\begin{array}{ccc} & & \Theta \\ & & \downarrow{\scriptstyle \subseteq_a} \\ \Theta^* & \xrightarrow[\subseteq_a]{} & \Theta \cup \Theta^* \end{array}$$

Fig. 3.2

Proof. Straightforward. ▯

It will sometimes be necessary to amalgamate two extensions of a given usl table. The following lemma will enable us to carry out such amalgamations.

3.16 Amalgamation Lemma. *Let $\mathscr{L}$ be a finite lattice and let Θ, Θ^* and Θ^+ be finite usl tables for $\mathscr{L}$ such that Θ^+ extends Θ and Θ^* is an admissible extension of Θ. Assume that Θ^+ and Θ^* are disjoint above Θ. Then $\Theta^* \cup \Theta^+$ is a usl table for $\mathscr{L}$ and the following diagram commutes*:

$$\begin{array}{ccccc} & & \Theta & & \\ & {\scriptstyle \subseteq_a}\swarrow & & \searrow{\scriptstyle \subseteq} & \\ \Theta^* & & & & \Theta^+ \\ & {\scriptstyle \subseteq}\searrow & & \swarrow{\scriptstyle \subseteq_a} & \\ & & \Theta^* \cup \Theta^+ & & \end{array}$$

Fig. 3.3

Furthermore, if Θ^+ is an admissible extension of Θ then $\Theta^* \cup \Theta^+$ is an admissible extension of Θ^*.

Proof. We first show that $\Theta^* \cup \Theta^+$ is a usl table for $\mathscr{L}$. 2.1(i) is easily verified and 2.1(ii) follows immediately from 3.14(ii). Let $\mathscr{L} = \langle L, \leqslant, \vee, \wedge \rangle$ and let $p_i, p_j \in L$ be given such that $p_i \leqslant p_j$. Let $\alpha, \beta \in \Theta^* \cup \Theta^+$ be given such that $\alpha \equiv_j \beta$. Since $\Theta \subseteq \Theta^*$ and $\Theta \subseteq \Theta^+$, we may assume without loss of generality that $\alpha \in \Theta^*$ and $\beta \in \Theta^+$, else 2.1(iii) will follow immediately. By 3.14(ii), there is a $\gamma \in \Theta$ such that $\alpha \equiv_j \gamma \equiv_j \beta$. Since $\gamma \in \Theta^* \cap \Theta^+$ and Θ^* and Θ^+ are usl tables for $\mathscr{L}$, $\alpha \equiv_i \gamma \equiv_i \beta$. Conversely, let $p_i, p_j \in \Theta^* \cup \Theta^+$ be given such that $p_i \nleqslant p_j$. Since Θ^* is a usl table for $\mathscr{L}$, 2.1(iii) implies that there are $\alpha, \beta \in \Theta^*$ such that $\alpha \equiv_j \beta$ but $\alpha \not\equiv_i \beta$. Since $\Theta^* \subseteq \Theta^* \cup \Theta^+$, 2.1(iii) holds for $\Theta^* \cup \Theta^+$.

Let $p_i, p_j, p_m \in L$ be given such that $p_i \vee p_j = p_m$. Let $\alpha, \beta \in \Theta^* \cup \Theta^+$ be given such that $\alpha \equiv_m \beta$. By the previous paragraph, $\alpha \equiv_i \beta$ and $\alpha \equiv_j \beta$ so $\alpha \equiv_{i \vee j} \beta$. Next let $\alpha, \gamma \in \Theta^* \cup \Theta^+$ be given such that $\alpha \equiv_i \gamma$ and $\alpha \equiv_j \gamma$. If $\alpha, \gamma \in \Theta^*$ or $\alpha, \gamma \in \Theta^+$ then $\alpha \equiv_m \gamma$ since Θ^* and Θ^+ are usl tables for $\mathscr{L}$. Hence we may assume that $\alpha \in \Theta^* - \Theta$ and $\gamma \in \Theta^+ - \Theta$. Since Θ^* is an admissible extension of Θ, there is a $\beta \in \Theta$ satisfying 3.5(i). Hence $\alpha \equiv_i \beta$ and $\alpha \equiv_j \beta$. Since $\beta \in \Theta^*$ and Θ^* is a usl table for $\mathscr{L}$, $\alpha \equiv_m \beta$. But $\gamma \equiv_i \alpha \equiv_i \beta$ and $\gamma \equiv_j \alpha \equiv_j \beta$ and $\beta \in \Theta \subseteq \Theta^+$ and Θ^+ is a usl table for $\mathscr{L}$, so $\gamma \equiv_m \beta$. Hence $\alpha \equiv_m \beta \equiv_m \gamma$. Conversely, if $p_i, p_j, p_m \in L$ are given such that $p_i \vee p_j \neq p_m$, then there are $\alpha, \beta \in \Theta^*$ such that $\alpha \equiv_{i \vee j} \beta \Leftrightarrow \alpha \not\equiv_m \beta$. Since $\Theta^* \subseteq \Theta^* \cup \Theta^+$ we see that 2.1(iv) holds, so $\Theta^* \cup \Theta^+$ is a usl table for $\mathscr{L}$.

We now show that $\Theta^* \cup \Theta^+$ is an admissible extension of Θ^+. Let $\alpha \in \Theta^* \cup \Theta^+$ be given. If $\alpha \in \Theta^+$, then we can choose $\beta = \alpha$ to verify 3.5(i). So we may assume that $\alpha \in \Theta^*$. Since Θ^* is an admissible extension of Θ, there is a $\beta \in \Theta$ such that for all $\gamma \in \Theta$ and $i \leqslant k$, $\alpha \equiv_i \gamma \Leftrightarrow \alpha \equiv_i \beta$. Let $\delta \in \Theta^+$ and $i \leqslant k$ be given such that $\alpha \equiv_i \delta$. By 3.14(ii), there is a $\gamma \in \Theta$ such that $\alpha \equiv_i \gamma$, hence $\beta \equiv_i \alpha \equiv_i \gamma$. Thus $\beta \in \Theta \subseteq \Theta^+$ will witness the fact that $\Theta^* \cup \Theta^+$ is an admissible extension of Θ^+.

The last sentence of the lemma follows as in the above paragraph by interchanging Θ^* and Θ^+. ▯

The preceding lemmas provide some of the building blocks for obtaining usl tables. We will introduce other building blocks which require that we modify certain tables *isomorphically*. These tables are needed to extend certain diagrams to commuting diagrams.

3.17 Definition. Let Θ, Ψ, N^{k+1} be finite usl tables. We say that Θ and Ψ are *isomorphic* (write $\Theta \simeq \Psi$) if there are permutations $\{p_i \colon i \leqslant k\}$ of N such that $\Psi = \{\langle p_0(m_0), \ldots, p_k(m_k)\rangle \colon \langle m_0, \ldots, m_k\rangle \in \Theta\}$. We then write the isomorphism map as $p = \langle p_0, \ldots, p_k\rangle$, and write $\Psi = p(\Theta)$.

We now show that isomorphisms preserve admissible extensions.

3.18 Admissibility Preservation Lemma. *Let Ψ, Ψ^+ and Θ^* be finite usl tables such that the diagram of Fig. 3.4 commutes, where $p = \langle p_0, \ldots, p_k\rangle$. Then $\Psi \subseteq_a \Psi^*$.*

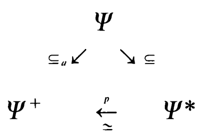

Fig. 3.4

Proof. Let $\beta \in \Psi^*$ be given. Since Ψ^+ is an admissible extension of Ψ, there is an $\alpha \in \Psi$ such that for all $\gamma \in \Psi$ and $i \leqslant k$, if $p(\beta) \equiv_i \gamma$ then $p(\beta) \equiv_i \alpha$. Since Fig. 3.4 is a commuting diagram, it must be the case that for all $\gamma \in \Psi$, $\gamma \equiv_i p(\beta)$ if and only if $\gamma \equiv_i \beta$. Hence for all $\gamma \in \Psi$ and $i \leqslant k$, if $\beta \equiv_i \gamma$ then $\beta \equiv_i \alpha$, i.e., $\Psi \subseteq_a \Psi^*$. ▯

In order to apply the Joint Embedding Lemma to Θ and $\Theta^\#$, we require that $\Theta \equiv_0 \Theta^\#$. If this is not the case, we use the next lemma to find $\Theta^* \simeq \Theta^\#$ such that $\Theta^* \equiv_0 \Theta$ and apply the Joint Embedding Lemma to Θ and Θ^*.

3.19 $\equiv_0$ Modification Lemma. *Let $\mathscr{L}$ and $\mathscr{M}$ be finite lattices such that $\mathscr{L} \subseteq \mathscr{M}$. Let Θ and $\Theta^\#$ be finite usl tables for $\mathscr{L}$ and $\mathscr{M}$ respectively. Then there is a finite usl table Θ^* for $\mathscr{M}$ such that $\Theta^\# \simeq \Theta^* \equiv_0 \Theta$.*

Proof. Since Θ and $\Theta^\#$ are finite, there are $j, k \in N$ and finite sets, $S, T \subseteq N$ such that $\Theta \subseteq S^{k+1}$ and $\Theta^\# \subseteq T^{j+1}$. Hence there is a permutation q of N such that $q(T) \cap S = \emptyset$. For $1 \leqslant i \leqslant k$, let $p_i = q$. Fix $\alpha \in \Theta$ and $\beta \in \Theta^\#$. Then there is a permutation p_0 of N such that $p_0(\beta^{[0]}) = \alpha^{[0]}$. Let $p = \langle p_0, \ldots, p_k\rangle$ and let $\Theta^* = p(\Theta^\#)$. It is easily verified that $\Theta \equiv_0 \Theta^* \simeq \Theta^\#$. ▯

In order to apply the Amalgamation Lemma to $\Psi^+ \upharpoonright \mathscr{L} \supseteq \Psi \upharpoonright \mathscr{L}$ and Θ^* above Θ, we require that $\Psi^+ \upharpoonright \mathscr{L}$ and Θ^* are disjoint above Θ. If this is not the case, we use the next lemma to define $\Psi^* \simeq \Psi^+$ such that $\Psi^* \restriction \mathscr{L}$ and Θ^* are disjoint above Θ, and apply the Amalgamation Lemma to $\Psi^* \restriction \mathscr{L}$ and Θ^*.

3.20 Disjointness Modification Lemma. *Let $\mathscr{L}$ and $\mathscr{M}$ be finite lattices such that $\mathscr{L} \subseteq \mathscr{M}$. Let Ψ and Ψ^* be finite usl tables for $\mathscr{M}$ and let Θ and Θ^* be finite usl tables for $\mathscr{L}$ as in the solid part of the diagram of Fig. 3.5. Then there is a finite usl table $\Psi^+ \simeq \Psi^*$ for $\mathscr{M}$ such that $\Psi \subseteq \Psi^+$ and Θ^* and Ψ^+ are disjoint above Ψ. Furthermore, if Ψ^* is an admissible extension of Ψ, then Ψ^+ can be chosen to be an admissible extension of Ψ.*

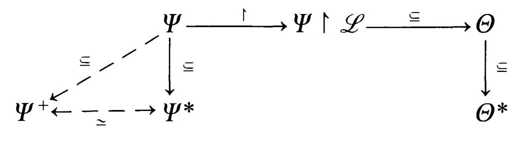

Fig. 3.5

Proof. The lemma asserts that we can extend the solid part of Fig. 3.5 to the full diagram so that the diagram commutes and disjointness and admissibility properties are established.

Since Ψ, Ψ^* and Θ^* are finite, there are finite sets $R, S, T \subseteq N$ and $j, k \in N$ such that $R \subseteq S \cap T$, $\Theta^* \subseteq S^{j+1}$, $\Psi \subseteq R^{k+1}$ and $\Psi^* \subseteq T^{k+1}$. For each $i \leqslant k$, fix a permutation p_i of N which satisfies the following conditions: If $\beta \in \Psi^*$ and there is an $\alpha \in \Psi$ such that $\alpha \equiv_i \beta$, then $p_i(\beta^{[i]}) = \beta^{[i]}$; and if no such α exists, $p_i(\beta^{[i]}) \in N - S$. If $k < i \leqslant j$, let p_i be the identity permutation of N. Let $p = \langle p_0, \ldots, p_j \rangle$ and let $\Psi^+ = p(\Psi^*)$. Then $\Psi^+ \simeq \Psi^*$, Ψ^+ and Θ^* are disjoint above Ψ, and $\Psi \subseteq \Psi^+$. Also, if $\Psi \subseteq_a \Psi^*$, then by the Admissibility Preservation Lemma (3.18), $\Psi \subseteq_a \Psi^+$. ▯

The next two lemmas are applied to build a sequential lattice table for $\mathscr{L}_i$ which can be extended to a sequential lattice table for any finite $\mathscr{M} \supseteq \mathscr{L}_i$.

3.21 Lemma. *Let $\mathscr{L}$ and $\mathscr{M}$ be finite lattices such that $\mathscr{L} \subseteq \mathscr{M}$. Let $\Theta \subseteq N^{k+1}$ be a finite usl table for $\mathscr{L}$, and let $\Theta^\#$ be a finite usl table for $\mathscr{M}$. Then there are finite usl tables Θ^+ and Θ^* for $\mathscr{L}$ and $\mathscr{M}$ respectively such that the following diagram commutes:*

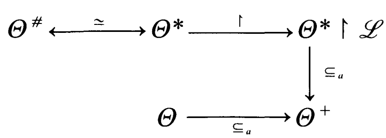

Fig. 3.6

Proof. Fix $\mathscr{L}$, $\mathscr{M}$, Θ and $\Theta^\#$ as in the hypothesis of the lemma. By the $\equiv_0$ Modification Lemma (3.19), there is a finite usl table Θ^* for $\mathscr{M}$ such that $\Theta^\# \simeq \Theta^* \equiv_0 \Theta$. By the Restriction Lemma (3.13), $\Theta^* \restriction \mathscr{L}$ is a finite usl table for $\mathscr{L}$. Note that $\Theta^* \restriction \mathscr{L} \equiv_0 \Theta$. Hence by the Joint Embedding Lemma (3.15), we can set $\Theta^+ = \Theta \cup \Theta^* \restriction \mathscr{L}$ to make Fig. 3.6 a commuting diagram. ▯

3.22 Lemma. *Let $\mathscr{L}$ and $\mathscr{M}$ be finite lattices such that $\mathscr{L} \subseteq \mathscr{M}$. Let Θ and Θ^* be finite usl tables for $\mathscr{L}$, and let Ψ and Ψ^* be finite usl tables for $\mathscr{M}$ as in the solid part of the diagram of Fig. 3.7. Then there are usl tables Θ^+ for $\mathscr{L}$ and Ψ^+ for $\mathscr{M}$ such that Fig. 3.7 is a commuting diagram.*

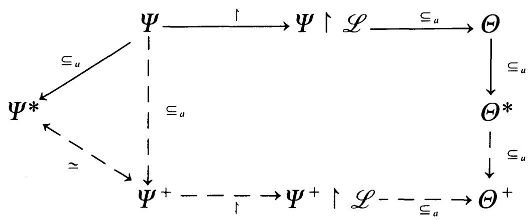

Fig. 3.7

Proof. By the Disjointness Modification Lemma (3.20), there is a finite usl table Ψ^+ for $\mathscr{M}$ such that $\Psi^+ \simeq \Psi^*$, Θ^* and Ψ^+ are disjoint above Ψ, and Ψ^+ is an admissible extension of Ψ. By the Restriction Lemma (3.13) and the Transitivity Lemma (3.9), the following diagram commutes:

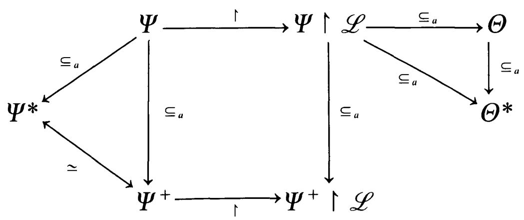

Fig. 3.8

The lemma now follows from the Amalgamation Lemma (3.16), with $\Theta^+ = \Theta^* \cup \Psi^+ \upharpoonright \mathscr{L}$. ▯

The final two lemmas of this section are isomorphism pullback lemmas which allow us to show that certain induction hypotheses specified in the construction of a sequential lattice table are satisfied.

3.23 First Pullback Lemma. *Let $\mathscr{L}$ and $\mathscr{M}$ be finite lattices such that $\mathscr{L} \subseteq \mathscr{M}$. Let $\Psi \subseteq N^{j+1}$ be a finite usl table for $\mathscr{M}$, and let $\Theta, \Theta^* \subseteq N^{k+1}$ be finite usl tables for $\mathscr{L}$ as in the solid part of Fig. 3.9. Then there is a finite usl table Ψ^* for $\mathscr{M}$ such that Fig. 3.9 is a commuting diagram.*

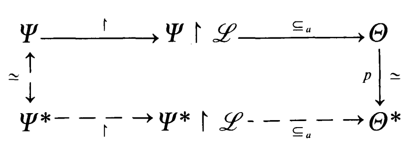

Fig. 3.9

Proof. Let $p = \langle p_0, \ldots, p_k \rangle$. Define $q = \langle q_0, \ldots, q_j \rangle$ by $q_i = p_i$ if $i \leqslant k$, and let q_i be the identity permutation of N if $k < i \leqslant j$. Then q is an isomorphism between Ψ and

$q(\Psi) = \Psi^*$, and for all $\alpha \in \Psi$, $p(\alpha \restriction \mathscr{L}) = q(\alpha) \restriction \mathscr{L}$. It will therefore follow that Fig. 3.9 is a commuting diagram once we show that $\Psi^* \restriction \mathscr{L} \subseteq_a \Theta^*$.

Let $\beta^* \in \Theta^*$ be given. Then there is a $\beta^+ \in \Theta$ such that $p(\beta^+) = \beta^*$. Since $\Psi \restriction \mathscr{L} \subseteq_a \Theta$, there is an $\alpha^+ \in \Psi \restriction \mathscr{L}$ such that for all $\gamma^+ \in \Psi \restriction \mathscr{L}$ and $i \leqslant k$, if $\beta^+ \equiv_i \gamma^+$ then $\beta^+ \equiv_i \alpha^+$. Fix $\alpha \in \Psi$ such that $\alpha^+ = \alpha \restriction \mathscr{L}$. Suppose that $\gamma^* \in \Psi^* \restriction \mathscr{L}$ and $i \leqslant k$ are given such that $\gamma^* \equiv_i \beta^*$. Then there is a $\gamma \in \Psi$ such that $q(\gamma) \restriction \mathscr{L} = \gamma^*$. Since each p_i is a permutation and Fig. 3.9 is a commuting diagram, $\gamma \restriction \mathscr{L} \equiv_i \beta^+$. By choice of α, $\gamma \equiv_i \alpha$. Hence $q(\gamma) \restriction \mathscr{L} \equiv_i q(\alpha) \restriction \mathscr{L}$. We thus see that for all $\gamma^* \in \Psi^* \restriction \mathscr{L}$ and $i \leqslant k$, if $\gamma^* \equiv_i \beta^*$ then $q(\alpha) \restriction \mathscr{L} \equiv_i \beta^*$, so $\Psi^* \restriction \mathscr{L} \subseteq_a \Theta^*$. ▯

3.24 Second Pullback Lemma. *Let $\mathscr{L}$ and $\mathscr{M}$ be finite lattices such that $\mathscr{L} \subseteq \mathscr{M}$. Let Γ and $\Gamma^\#$ be finite usl tables for $\mathscr{M}$ and let Θ, Θ' and Θ'' be finite usl tables for $\mathscr{L}$ such that the solid part of Fig. 3.10 commutes.*

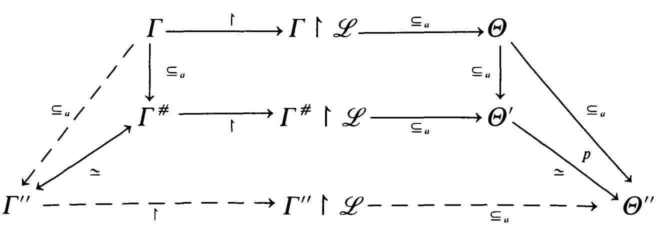

Fig. 3.10

Then there is a finite usl table Γ'' for $\mathscr{M}$ such that Fig. 3.10 is a commuting diagram.

Proof. Let $p = \langle p_0, \ldots, p_k \rangle$ be the isomorphism map such that $p(\Theta') = \Theta''$. Fix j, $k \in N$ such that $\Theta \subseteq N^{k+1}$ and $\Gamma \subseteq N^{j+1}$. Let $q_i = p_i$ for $i \leqslant k$, and let q_i be the identity permutation of N if $k < i \leqslant j$. Set $q = \langle q_0, \ldots, q_j \rangle$, and let $\Gamma'' = q(\Gamma^\#)$. By the Transitivity Lemma (3.9), $\Gamma \restriction \mathscr{L} \subseteq_a \Theta''$ and $\Gamma \restriction \mathscr{L} \subseteq_a \Theta'$. Hence $p_i(\alpha^{[i]}) = \alpha^{[i]}$ for all $\alpha \in \Gamma$ and $i \leqslant k$. Since q_i is a permutation of N for all $i \leqslant j$ and $q(\alpha^{[i]}) = \alpha^{[i]}$ for all i such that $k < i \leqslant j$, it follows from the Admissibility Preservation Lemma (3.18) that $\Gamma \subseteq_a \Gamma''$. It now follows easily from the definition of q and the proof of Theorem 3.23 that Fig. 3.10 is a commuting diagram and $\Gamma'' \restriction \mathscr{L} \subseteq_a \Theta''$. ▯

We now define the type of usl table which is needed to obtain the results of Chap. VIII. Fix a countable usl $\mathscr{L}$ together with a sequence $\{\mathscr{L}_i\colon i \in N\}$ of finite lattices which approximates to $\mathscr{L}$ as in Remark 3.4.

3.25 Definition. Let $\mathscr{M}$ be a finite lattice. A sequence $\{\Theta_i\colon i \in N\}$ is a *spasmodic weakly homogeneous sequential lattice table for $\mathscr{M}$* if there is a subsequence $\{\Theta_{f(i)}\colon i \in N\}$ of the above sequence such that $\{\Theta_{f(i)}\colon i \in N\}$ is a weakly homogeneous sequential lattice table for $\mathscr{M}$ and

(i) $\forall i,j \in N(f(i) \leqslant j < f(i+1) \rightarrow \Theta_j = \Theta_{f(i)})$.

3.26 Definition. An array $\{\Theta_{i,j}\colon i,j \geqslant 0\}$ is a *uniform sequential lattice table* for $\{\mathscr{L}_i\colon i \in N\}$ if there is a function $h\colon N \rightarrow N$ such that the following two conditions hold:

(i) $\forall i \in N(\{\Theta_{i,j}: j \geqslant h(i)\}$ is a recursive spasmodic weakly homogeneous sequential lattice table for $\mathscr{L}_i$).

(ii) $\forall i \in N \forall j \geqslant h(i+1)(\Theta_{i+1,j} \upharpoonright \mathscr{L}_i \subseteq \Theta_{i,j})$.

We now construct a uniform sequential lattice table for $\{\mathscr{L}_i: i \in N\}$.

3.27 Theorem. *$\{\mathscr{L}_i: i \in N\}$ has a uniform sequential lattice table.*

Proof. We must construct a function $h: N \to N$ and an array $\{\Theta_{i,j}: i \in N \,\&\, j \geqslant h(i)\}$ satisfying 3.26(i) and (ii). We will also need, for each $i \in N$, a function f_i as in 3.25(i).

Let $\{\Psi_i^*: i \in N\}$ be a recursive list of all finite usl tables for finite lattices such that for each $i \in N$, $\{j: \Psi_i^* = \Psi_j^*\}$ is infinite. Such a list must exist since we can recursively identify whether a given finite set of k-tuples is a usl table for some lattice. Furthermore, given Ψ_i^*, we can recursively specify the lattice $\mathscr{L}_i^*$ such that Ψ_i^* is a usl table for $\mathscr{L}_i^*$.

Let $\{\langle \Gamma_i, \Gamma_i^*, \Lambda_i\rangle: i \in N\}$ be a recursive list of all triples of finite usl tables for lattices such that if Γ_i is a usl table for the lattice $\mathscr{M}_i$, then Γ_i^* is a usl table for $\mathscr{M}_i$ which is an admissible extension of Γ_i, and Λ_i is a usl table for a lattice $\mathscr{N}_i \subseteq \mathscr{M}_i$ and is an admissible extension of $\Gamma_i \upharpoonright \mathscr{N}_i$. As in the preceding paragraph, such a list must exist, and we may assume without loss of generality that for each $i \in N$, $\{j: \langle \Gamma_j, \Gamma_j^*, \Lambda_j\rangle = \langle \Gamma_i, \Gamma_i^*, \Lambda_i\rangle\}$ is infinite.

The construction will proceed by induction on $\{i: i \in N\}$. At the end of stage i of the induction, we will have constructed $\{\Theta_{k,j}: k \leqslant i \,\&\, j \geqslant h(i)\}$ satisfying the following induction hypotheses:

(1) $\forall j \geqslant h(i)(\Theta_{i,j+1}$ is an admissible extension of $\Theta_{i,j})$.

(2) $i > 0 \to \forall j \geqslant h(i)(\Theta_{i-1,j}$ is an admissible extension of $\Theta_{i,j} \upharpoonright \mathscr{L}_{i-1})$.

(3) If Ψ is any finite usl table for a finite lattice $\mathscr{M} \supseteq \mathscr{L}_i$, then there are $j > h(i)$ and $\Psi^* \simeq \Psi$ such that $\Theta_{i,j}$ is an admissible extension of $\Psi^* \upharpoonright \mathscr{L}_i$.

(4) For all $j \geqslant h(i)$ and all finite usl tables Γ and Γ^* for $\mathscr{M} \supseteq \mathscr{L}_i$ such that Γ^* is an admissible extension of Γ and $\Theta_{i,j}$ is an admissible extension of $\Gamma \upharpoonright \mathscr{L}_i$, there is a $k > j$ and an isomorphic copy Γ^+ of Γ^* such that the following diagram commutes:

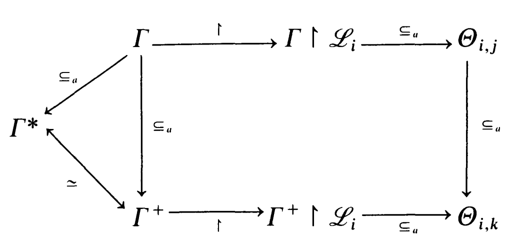

Fig. 3.11

Stage i. We proceed by substages, $\{r: r \in N\}$.

Substage 0. We wish to define $h(i)$, $f_i(0)$ and $\Theta_{i,h(i)}$. If $i = 0$, let $h(i) = f_i(0) = 0$ and let $\Theta_{0,0}$ be any finite usl table for $\mathscr{L}_0$. The existence of $\Theta_{0,0}$ follows from

Lemma 2.2. Suppose that $i > 0$. Let Ψ be any finite usl table for $\mathscr{L}_i$. By (3), there is a $k > h(i-1)$ and a usl table $\Psi^* \simeq \Psi$ such that $\Theta_{i-1,k}$ is an admissible extension of $\Psi^* \upharpoonright \mathscr{L}_{i-1}$. Fix the least such k, set $h(i) = f_i(0) = k$, and let $\Theta_{i,h(i)} = \Psi^*$. (Note that (2) will hold in this case.)

The ability to complete Substage 0 depended on the satisfaction of (3) for $i-1$ in place of i. Thus we want the construction of a sequential table for $\mathscr{L}_{i-1}$ to guarantee the ability to admissibly extend restrictions of tables for $\mathscr{L}_i$. In order to make the construction more uniform and thus allow for more applications, we want this to be done before we determine $\mathscr{L}_i$. Hence the sequential table for $\mathscr{L}_{i-1}$ considers tables for all possible finite $\mathscr{M} \supseteq \mathscr{L}_{i-1}$.

Substage $r+1$. We assume that $\Theta_{i,f_i(r)}$ has been defined. We wish to define $f_i(r+1)$ and tables $\Theta_{i,j}$ for all j such that $f_i(r) < j \leqslant f_i(r+1)$. This definition is broken down into a sequence of three steps. In the first step, we try to insure that $\Theta_{i,f_i(r+1)}$ contains all the interpolants needed by $\Theta_{i,f_i(r)}$ so that it can be a successor of $\Theta_{i,f_i(r)}$ in a weakly homogeneous sequential lattice table for $\mathscr{L}_i$. In the second and third steps, we insure that all instances of (3) and (4) are satisfied.

Step 1. Let $\Theta^*_{i,r}$ be a type 1 extension of $\Theta_{i,f_i(r)}$. By Corollary 3.12, such an extension exists and is, by definition, admissible.

Step 2. Let Ψ^*_r be a usl table for the lattice $\mathscr{M}^*_r$. If $\mathscr{L}_i \nsubseteq \mathscr{M}^*_r$, set $\Theta^\#_{i,r} = \Theta^*_{i,r}$. Assume that $\mathscr{L}_i \subseteq \mathscr{M}^*_r$. (The reader can follow the remainder of this step in Fig. 3.12.) By Lemma 3.21, there are finite usl tables $\Psi^+_r \simeq \Psi^*_r$ for $\mathscr{M}^*_r$ and $\Theta^+_{i,r}$ for $\mathscr{L}_i$ such that $\Theta^+_{i,r}$ is an admissible extension of both $\Psi^+_r \upharpoonright \mathscr{L}_i$ and $\Theta^*_{i,r}$. By the Transitivity Lemma (3.9), $\Theta^+_{i,r}$ is an admissible extension of $\Theta_{i,f_i(r)}$. If $i = 0$, set $\Theta^\#_{i,r} = \Theta^+_{i,r}$ and $k^\# = r+1$. If $i > 0$, then by (2), $\Theta_{i-1,f_i(r)}$ is an admissible extension of $\Theta_{i,f_i(r)} \upharpoonright \mathscr{L}_{i-1}$. Hence by (4), there is a usl table $\Theta^\#_{i,r}$ for $\mathscr{L}_i$ such that $\Theta^\#_{i,r} \simeq \Theta^+_{i,r}$, $\Theta^\#_{i,r}$ is an admissible extension of $\Theta_{i,f_i(r)}$, and there is a $k^\# > f_i(r)$ such that $\Theta_{i-1,k^\#}$ is an admissible extension of both $\Theta_{i-1,f_i(r)}$ and $\Theta^\#_{i,r} \upharpoonright \mathscr{L}_{i-1}$. Fix the least such $k^\#$ and fix $\Theta^\#_{i,r}$.

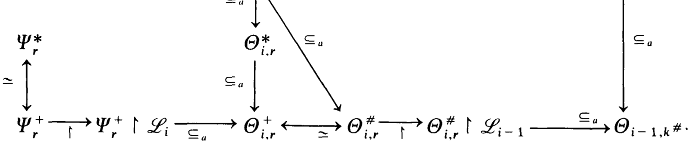

Fig. 3.12

Step 3. Let $f_i(r+1) = k^\#$, $\Theta_{i,j} = \Theta_{i,f_i(r)}$ for all j such that $f_i(r) < j < k^\#$, and $\Theta_{i,f_i(r+1)} = \Theta^\#_{i,r}$ unless $\Lambda_r = \Theta_{i,j}$ for some $j \leqslant f_i(r)$. Thus it remains to consider the case where $\Lambda_r = \Theta_{i,j}$ for some $j \leqslant f_i(r)$. (The reader should follow the progress of this case in Fig. 3.13.) By the Transitivity Lemma (3.9), $\Theta^\#_{i,r}$ is an admissible extension of $\Gamma_r \upharpoonright \mathscr{L}_i$. By Lemma 3.22, there is a usl table $\Gamma^\#_r \simeq \Gamma^*_r$ such that $\Gamma^\#_r$ is an admissible extension of Γ_r, and a usl table $\Theta'_{i,r}$ for $\mathscr{L}_i$ such that $\Theta'_{i,r}$ is an admissible extension of both $\Gamma^\#_r \upharpoonright \mathscr{L}_i$ and $\Theta^\#_{i,r}$. By the Transitivity Lemma (3.9), $\Theta'_{i,r}$ is an admissible extension of $\Theta_{i,j}$. If $i = 0$, set $f_i(r+1) = r+1$ and $\Theta_{i,r+1} = \Theta'_{i,r}$.

Suppose that $i > 0$. By (2), $\Theta_{i-1,j}$ is an admissible extension of $\Theta_{i,j} \upharpoonright \mathscr{L}_{i-1}$. We now apply (4) inductively to obtain an isomorphic copy $\Theta''_{i,r}$ of $\Theta'_{i,r}$ such that $\Theta''_{i,r}$ is an admissible extension of $\Theta_{i,j}$, and a $k > f_i(r)$ such that $\Theta_{i-1,k}$ is an admissible extension of both $\Theta_{i-1,j}$ and $\Theta''_{i,r} \upharpoonright \mathscr{L}_{i-1}$. Fix the least such k and let $f_i(r+1) = k$. Set $\Theta_{i,s} = \Theta_{i,f_i(r)}$ for all s such that $f(r) < s < k$ and let $\Theta_{i,k} = \Theta''_{i,r}$.

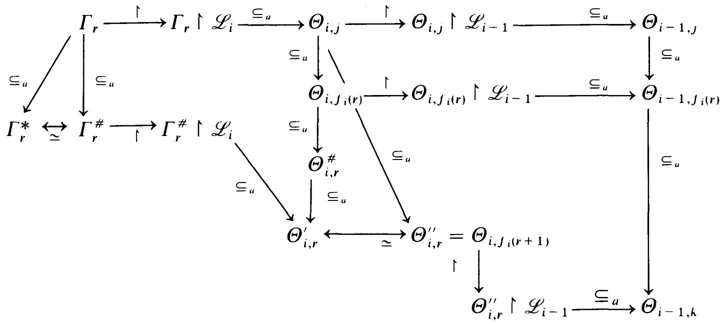

Fig. 3.13

The construction is now complete. Note that Fig. 3.12 describes the situation at the end of Step 2 regardless of the procedure followed to define $\Theta^{\#}_{i,r}$; and Fig. 3.13 describes the situation at the end of Step 3 regardless of the procedure followed to define $\Theta_{i,f_i(r+1)}$. Hence (1) and (2) follow from Figs. 3.12 and 3.13 and the Transitivity Lemma (3.9).

Let Ψ be any finite usl table for a finite lattice $\mathscr{M} \supseteq \mathscr{L}_i$. Then $\Psi = \Psi^*_r$ for some $r > h(i)$. Hence by Fig. 3.12, $\Psi^*_r \simeq \Psi^+_r$ and $\Psi^+_r \upharpoonright \mathscr{L}_i \subseteq_a \Theta^+_{i,r} \simeq \Theta^{\#}_{i,r}$. By the First Pullback Lemma (3.23), there is a usl table $\Psi^{\#}$ for $\mathscr{M}$ such that $\Psi^{\#} \simeq \Psi^+_r$ and $\Psi^{\#} \upharpoonright \mathscr{L}_i \subseteq_a \Theta^{\#}_{i,r}$. By Fig. 3.13, $\Theta^{\#}_{i,r} \subseteq_a \Theta'_{i,r}$, so by the Transitivity Lemma (3.9), $\Psi^{\#} \upharpoonright \mathscr{L}_i \subseteq_a \Theta'_{i,r}$. By Fig. 3.13, $\Theta'_{i,r} \simeq \Theta''_{i,r}$. We again apply the First Pullback Lemma (3.23) to conclude that there is a finite usl table Ψ'' for $\mathscr{M}$ such that $\Psi'' \simeq \Psi^{\#}$ and $\Psi'' \upharpoonright \mathscr{L}_i \subseteq_a \Theta''_{i,r} = \Theta_{i,f_i(r+1)}$. Since $\Psi'' \simeq \Psi^*_r$, (3) is now seen to hold.

Let $j \in N$ be given, and let Γ and Γ^* be finite usl tables for $\mathscr{M} \supseteq \mathscr{L}_i$ such that Γ^* is an admissible extension of Γ and $\Theta_{i,j}$ is an admissible extension of $\Gamma \upharpoonright \mathscr{L}_i$. Then there is an $r \in N$ such that $f_i(r) \geqslant j$ and $\langle \Gamma_r, \Gamma^*_r, \Lambda_r \rangle = \langle \Gamma, \Gamma^*, \Theta_{i,j} \rangle$. Fix such an r. By Fig. 3.13 and the Transitivity Lemma (3.9), the following diagram commutes:

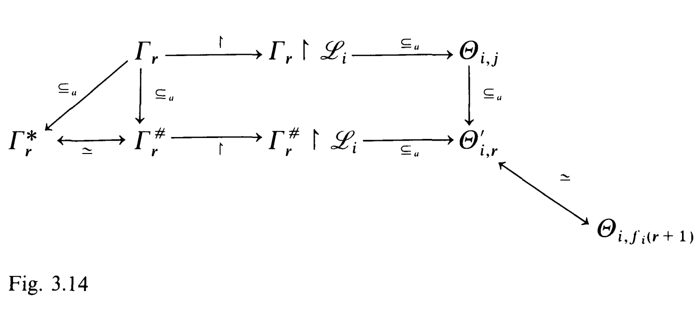

Fig. 3.14

(4) now follows from the Second Pullback Lemma (3.24).

We note that the construction of $\{\Theta_{i,j}: i \leqslant k \,\&\, j \geqslant h(i)\}$ can be carried out recursively. Furthermore, there is an isomorphic copy of $\Theta^*_{i,r}$ embedded into $\Theta_{i,f_i(r+1)}$. Hence $\{\Theta_{i,j}: i \in N \,\&\, j \geqslant h(i)\}$ is a uniform sequential lattice table. ▯

3.28 Remark. Let $\mathscr{L} = \{\mathscr{L}_i: i \in N\}$ and $\mathscr{M} = \{\mathscr{M}_i: i \in N\}$ be countable usls such that $\mathscr{L}_i = \mathscr{M}_i$ for all $i \leqslant n$, and for all $j \in N$, both $\mathscr{L}_j$ and $\mathscr{M}_j$ are finite. Let $\{\Theta_{i,j}: i,j \in N\}$ and $\{\Psi_{i,j}: i,j \in N\}$ be the uniform sequential tables constructed in Theorem 3.27 for $\mathscr{L}$ and $\mathscr{M}$ respectively. Then $\Theta_{i,j} = \Psi_{i,j}$ for all $i \leqslant n$ and $j \in N$.

3.29 Remark. Let $\{\mathscr{L}_i: i \in N\}$ be an approximation to the usl $\mathscr{L}$ with least element such that each $\mathscr{L}_i$ is finite and $\{\mathscr{L}_i: i \in N\}$ is recursive in a set of degree **a**. Then there is a uniform sequential usl table $\{\Theta_{i,j}: i,j \in N\}$ and a function $h: N^2 \to N$ which is recursive in a set of degree **a** such that for all $i,j \in N$, $h(i,j)$ is an index for $\Theta_{i,j}$ as a canonically finite set.

Theorem 3.27 was proved by Lachlan and Lebeuf [1976].