

A NEW APPROACH TO THE SCHAUDER ESTIMATES FOR LINEAR ELLIPTIC EQUATIONS

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In this talk we describe a relatively simple approach to the Schauder estimates for general elliptic systems of the type considered by Douglis and Nirenberg [1]. Our method, as presented in the lectures [6], requires neither preliminary singular integral estimates as in [1] nor auxiliary existence results as in those proposed by Campanato, (see [2]), and Safonov [5]. Instead our procedure involves the direct deduction of the Hölder estimates from the corresponding L^2 estimates for the constant coefficient case, by means of mollification. It is also readily extended to more general classes of operators. In the special case of a single second order equation, the classical mean value inequality for subharmonic functions can be used in place of the L^2 estimates.

To illustrate the technique, we confine attention here to elliptic systems of the form,

$$(1) \quad L^i u = \sum_{j=1}^N \sum_{\substack{|\alpha|=s_i, \\ |\beta|=t_j}} D^\alpha \left[a_{\alpha\beta}^{ij} D^\beta u_j \right] = \sum_{|\alpha|=s_i} D^\alpha f_\alpha^i$$

with complex valued coefficients $a_{\alpha\beta}^{ij}$ and inhomogeneous terms f_α^i , $i, j = 1, \dots, N$, $|\alpha| = s_i$, $|\beta| = t_j$, in $C^\gamma(\mathbb{R}^n)$, $0 < \gamma < 1$, with s_i, t_i , $i = 1, \dots, N$ non-negative integers. The full generality of [1] may be recovered by some modification together with the standard interpolation inequalities, ([1], Section 2, [3], Section 6.8). Indeed we

shall only describe the proof here for the case of constant s_i, t_i and solutions $u = (u_1, \dots, u_N)$ with compact support.

The system (1) is elliptic if the determinant of its symbol doesn't vanish, that is

$$(2) \quad \mathcal{D}(x, \xi) = \det \begin{bmatrix} a_{ij} & \\ & \xi^{\alpha+\beta} \end{bmatrix} (x) \\ \neq 0 ,$$

for any $\xi \in \mathbb{R}^n \setminus \{0\}$, $x \in \mathbb{R}^n$. Since \mathcal{D} is a homogeneous polynomial in ξ of degree $m = \sum (s_i + t_i)$, it is convenient to assume

$$(3) \quad |\mathcal{D}(x, \xi)| \geq \lambda |\xi|^m ,$$

for all $\xi, x \in \mathbb{R}^n$ and some positive constant λ .

In order to formulate the Schauder estimates we recall from [3], the following notation for Hölder norms and seminorms on an open subset Ω of \mathbb{R}^n :

$$(4) \quad |u|_{0; \Omega} = \sup_{\Omega} |u| ; \\ [u]_{\gamma; \Omega} = \sup_{x, y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\gamma} ; \\ |u|_{k, \gamma; \Omega} = \sum_{|\beta| \leq k} |D^\beta u|_{0; \Omega} + \sum_{|\beta| = k} [D^\beta u]_{\gamma; \Omega} .$$

When $\Omega = \mathbb{R}^n$, we shall write simply $|u|_0$, $[u]_\gamma$, $|u|_{k, \gamma}$. We can now assert the following Schauder estimates.

THEOREM 1 Let $u = (u_1, \dots, u_N)$, $u_i \in C_0^{t_i, \gamma}(\mathbb{R}^n)$, satisfy (1). Then we have the estimates,

$$(5) \quad [D^{t_i} u_i]_{\gamma} \leq C \left\{ \sum_{i=1}^N [a]_{\gamma}^{1+t_i/\gamma} |u_i|_0 + [f]_{\gamma} \right\},$$

where C is a constant depending only on $n, N, \gamma, \lambda, m, |a|_0$

and where $a = \{a_{\alpha, \beta}^{ij}, i, j = 1, \dots, N, |\alpha| = s_i, |\beta| = t_j\}$,

$f = \{f_{\alpha}^i, i = 1, \dots, N, |\alpha| = s_i\}$.

We will deduce Theorem 1 (at least for constant s_i, t_i) from the corresponding L^2 estimate for the constant coefficient case.

Lemma 2 Suppose the coefficients $a_{\alpha\beta}^{ij}$ are constant and $f_{\alpha}^i \in L^2(\mathbb{R}^n)$, $i, j = 1, \dots, N, |\alpha| = s_i, |\beta| = t_j$. Then if $u_i \in L^2(\mathbb{R}^n)$, $i = 1, \dots, N$, satisfy the system (1) we have the estimates,

$$(6) \quad \|D^{t_i} u_i\|_2 \leq C \|f\|_2,$$

where C is a constant depending only on n, N, m, λ and $|a|_0$.

The estimate (6) follows immediately by Fourier transformation of (1). In order to use Lemma 2, let us specialize now to the case $s_i = s, t_i = t$, fix a point $x_0 \in \mathbb{R}^n$ and write the system (1) in the form,

$$(7) \quad \begin{aligned} & \sum D^{\alpha} \left\{ a_{\alpha\beta}^{ij}(x_0) D^{\beta} u_j - f_{\alpha}^i(x_0) \right\} \\ &= \sum D^{\alpha} \left\{ [a_{\alpha\beta}^{ij}(x_0) - a_{\alpha\beta}^{ij}(x)] D^{\beta} u_j + f_{\alpha}^i(x) - f_{\alpha}^i(x_0) \right\} \\ &= \sum D^{\alpha} g_{\alpha}^i. \end{aligned}$$

For positive τ , we now invoke the mollification of u , given by

$$(8) \quad u(x, \tau) = u_\tau(x) = \tau^{-n} \int \rho\left(\frac{x-y}{\tau}\right) u(y) dy$$

where the mollifier ρ is a fixed, non-negative $C^\infty(\mathbb{R}^n)$ function, vanishing outside the unit ball and such that $\int \rho = 1$. By mollifying equation (7) we get the corresponding equation for u_τ , namely

$$(9) \quad \sum D^\alpha \left\{ a_{\alpha\beta}^{ij}(x_0) D^\beta u_j(x, \tau) - f_\alpha^i(x_0) \right\} = \sum D^\alpha g_\alpha^i(x, \tau).$$

Since the mollified functions u_τ and g_τ are smooth on $\mathbb{R}^n \times \mathbb{R}_+^n$ we can differentiate equation (9), $k \geq 1$ times to obtain for $v = D^k u_\tau$ the equations,

$$(10) \quad \sum a_{\alpha\beta}^{ij}(x_0) D^{\alpha+\beta} v_j = \sum D^\alpha D^k g_\alpha^i(x, \tau) \equiv h^i$$

We now apply a localized version of Lemma 2 which follows by replacement of u by ηu for a suitable cut-off function η .

Lemma 3. Let $B_R = B_R(x_0)$ and $v = (v_1, \dots, v_N)$ satisfy (10) with the ellipticity condition (3) holding at x_0 . Then for any $0 < \sigma < \sigma' < 1$ we have

$$(11) \quad \|D^{s+t} v\|_{L^2(B_{\sigma R})} \leq C \left\{ \frac{1}{R^{s+t}(\sigma'-\sigma)^{s+t}} \|v\|_{L^2(B_{\sigma'R})} + \|h\|_{L^2(B_R)} \right\}$$

where C is a constant depending on n, N, s, t, λ and $\|\mathcal{A}\|_0$.

Taking $k \geq t$ and using Lemma 3, in conjunction with interpolation, we obtain an estimate

$$\begin{aligned}
 (12) \quad \|D^{k+s+t} u_\tau\|_{L^2(B_{R/2})} &\leq C \left\{ R^{-(k+s)} \|D^t u_\tau\|_{L^2(B_R)} + \|h\|_{L^2(B_R)} \right\} \\
 &\leq CR^{n/2} \left\{ R^{-(k+s)} \sup_{B_R} |D^t u_\tau| + \tau^{-(k+s)} \sup_{B_R} |g| \right\} \\
 &\leq CR^{n/2} \left\{ R^{-(k+s)} \operatorname{osc}_{B_{2R}} D^t u + \tau^{-(k+s)} A R^\gamma \right\},
 \end{aligned}$$

provided u is normalized so that $D^t u(x_0) = 0$, $\tau \leq R$ and A is given by

$$(13) \quad A = [a]_\gamma |D^t u|_0 + [f]_\gamma.$$

But now, fixing $k + s > 1 + \frac{n}{2}$ and using the Sobolev imbedding theorem, we obtain

$$\sup_{B_{R/2}} |D^{1+t} u_\tau| \leq \frac{C}{R} \left\{ \operatorname{osc}_{B_{2R}} D^t u + \tau^{-(k+s)} R^{k+s+\gamma} A \right\}$$

so that, in particular,

$$(14) \quad \tau^{1-\gamma} |D^{1+t} u(x_0, \tau)| \leq C \left\{ \left(\frac{\tau}{R}\right)^{1-\gamma} [D^t u]_\gamma + \left(\frac{R}{\tau}\right)^{k+s+\gamma} A \right\}$$

where as above C depends only on n, N, s, t, λ and $|A|_0$. The final key to our approach is the equivalence of the semi-norms $[v]_\gamma$ and

$$(15) \quad [v]_\gamma' = \sup_{\tau > 0} \tau^{1-\gamma} |Dv_\tau|.$$

(In fact, in our notes [6], we employed the operator, $\mathcal{D} = (D, \frac{\partial}{\partial \tau})$ instead of D but Professor G.C. Dong kindly pointed out that the spatial gradient is sufficient for this equivalence). We therefore deduce from (14),

$$[D^t u]_\gamma \leq C \left\{ \left(\frac{\tau}{R} \right)^{1-\gamma} [D^t u]_\gamma + \left(\frac{R}{\tau} \right)^{k+s+\gamma} A \right\},$$

where now C also depends on γ , so that by fixing τ to make the ratio τ/R sufficiently small, we obtain the Hölder estimate,

$$(16) \quad [D^t u]_\gamma \leq C \left\{ [A]_\gamma |D^t u|_0 + [f]_\gamma \right\},$$

from which (5), (in the case $t_i \equiv t$), follows by interpolation.

Remarks(i) Second order equations. In the case of a single second order equation, $s + t = 2, N = 1$, the classical mean value inequality for subharmonic functions, ([3], Section 2.1) can be applied directly to equation (10) with $k = 3$, to yield the estimate (14). Details are carried out in [6].

(ii) **Estimates at the boundary.** For the single second order Dirichlet problem, these can be deduced from the corresponding interior estimates by means of reflection. The oblique boundary value problem is readily reduced to the Dirichlet problem (with s replaced by $s+1$) and interior estimation (with n replaced by $n-1$). Again, details are carried out in the lecture notes [6]. In more general cases, we have so far not been able to avoid the use of auxiliary existence results.

(iii) **L^p estimates.** As is known (see for example [2]), these can be deduced by interpolation of L^2 and Schauder estimates, utilizing the space BMO of functions of bounded mean oscillation. As an immediate corollary of Theorem 1, (with $\mathcal{A} = \text{constant}$) and Lemma 2, we can for example extend (6) to

$$(17) \quad \|D^i u_i\|_p \leq C \|f\|_p,$$

with the constant C also depending on p , $1 < p < \infty$.

(iv) **Nonlinear equations.** The mollification characterization of Hölder continuity can also be used to recover Safonov's general results on second derivative Hölder estimates for fully nonlinear elliptic equations [5], although in this situation the procedure is closer to that of Safonov. We also employed this technique in [6] to extend his results to oblique boundary value problems; (see also [4]).

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