1. ELEMENTARY PROPERTIES OF DIRECT PRODUCTS

We shall concern ourselves with systems

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

constituted by an arbitrary set A, a binary operation $+$ (the operation of addition), and arbitrarily many other operations arranged in a sequence $O_0, O_1, \ldots, O_\xi, \ldots$ of a type τ (where τ is a finite or transfinite ordinal). Each of these operations O_ξ is assumed to be defined for finite or transfinite sequences of elements $x_0, x_1, \ldots, x_\kappa, \ldots$ of a well-determined type ρ_ξ called the <u>rank</u> of the operation. Thus, O_ξ may be a unary operation ($\rho_\xi = 1$), a binary operation ($\rho_\xi = 2$), a ternary operation ($\rho_\xi = 3$), an operation on simple infinite sequences ($\rho_\xi = \omega$), etc. An operation O_ξ with the rank $\rho_\xi = \mu$ will be referred to for brevity as a μ-ary operation. Two systems

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle \text{ with } \xi < \tau$$

and

$$\underline{A}' = \langle A', +', O_0', O_1', \ldots, O_\xi', \ldots \rangle \text{ with } \xi < \tau'$$

are called <u>similar</u> if the sequences of ranks ρ_ξ and ρ_ξ' are identical, i.e., if

$$\tau = \tau', \text{ and } \rho_\xi = \rho_\xi' \text{ for every } \xi < \tau$$

The sequence of ranks ρ_ξ will sometimes be referred to as the <u>similarity type</u> of the system $\underline{A}$.

The symbolic expression

$$x, y, \ldots \in A$$

will express, as usual, the fact that $x, y, \ldots$ are elements of A. We shall speak occasionally of elements of the system $\underline{A}$ having actually in mind elements of the set A; and we shall call the system $\underline{A}$ finite (or infinite) in case the set A is finite (or infinite).

Definition 1.1. By an algebra we understand a system

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle \text{ with } \xi < \tau$$

subjected to the following conditions:
(i) the set A is closed under all the operations $+, O_0, O_1, \ldots, O_\xi, \ldots$, i.e.,
(i') if x and y are in A, then $x + y$ exists and is in A;
(i") if O_ξ with $\xi < \tau$ is a μ-ary operation and if $x_0, x_1, \ldots, x_\kappa, \ldots$, with $\kappa < \mu$ are in A, then $O_\xi(x_0, x_1, \ldots, x_\kappa, \ldots)$ exists and is in A;
(ii) there is an element $z \varepsilon A$ such that
(ii') z is a zero element with respect to the operation +, i.e., $x + z = z + x = x$ for every $x \varepsilon A$;
(ii") z is an idempotent element with respect to each of the operations O_ξ with $\xi < \tau$; i.e., if O_ξ is a μ-ary operation and $x_0 = x_1 = \ldots = x_\kappa = \ldots = z$ for $\kappa < \mu$, then $O_\xi(x_0, x_1, \ldots x_\kappa, \ldots) = z$.

Under the same conditions we shall say that the set A is an algebra under the operations $+, O_0, O_1, \ldots, O_\xi, \ldots$

Sometimes we shall call the set A itself an algebra, without explicitly mentioning the operations involved; we shall do so when it will be clearly seen from the context which operations are referred to.

Groups without operators are thus algebras in the sense of 1.1 with $\tau = 0$. Groups with operators are algebras in which τ is different from 0, and $O_0, O_1, \ldots, O_\xi, \ldots$ with $\xi < \tau$ are unary operations subjected to the condition

$$O_\xi(x + y) = O_\xi(x) + O_\xi(y) \text{ for all } x, y \varepsilon A.$$

Rings are algebras with $\tau = 1$ and $\rho_0 = 2$. Lattices and Boolean algebras are often characterized as algebras with $\tau = 1$ and $\rho_0 = 2$, but they can also be treated as algebras with $\tau = 0$.

Definition 1.2. Let

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

be an algebra.
(i) The uniquely determined element $z \varepsilon A$ satisfying condition 1.1 (ii') is denoted by 0.

(ii) The sum of a finite sequence of elements $x_0, x_1, \dots, x_\kappa, \dots \varepsilon A$, $\kappa < \nu$, *is defined recursively by means of the formulas*:

$$\sum_{\kappa<0} x_\kappa = 0, \ \textit{and} \sum_{\kappa<\nu+1} x_\kappa = \sum_{\kappa<\nu} x_\kappa + x_\nu \ \textit{for every finite } \nu.$$

We shall apply to algebras in the sense of 1.1 various familiar notions of a general algebraic nature. In the first place we here have in mind the notion of a *subalgebra*:

Definition 1.3. *By a subalgebra of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \dots, O_\xi, \dots \rangle$$

we understand an arbitrary algebra

$$\underline{B} = \langle B, +, O_0, O_1, \dots, O_\xi, \dots \rangle$$

formed by a subset B *of* A *containing the zero element of* A *and by the fundamental operation* $+, O_0, O_1, \dots, O_\xi, \dots$ *of* A. *Under the same conditions the set* B *itself is called a subalgebra of* $\underline{A}$, *or a subalgebra of* A (*under the operations* $+, O_0, O_1, \dots, O_\xi, \dots$).

Thus, given an algebra

$$\underline{A} = \langle A, +, O_0, O_1, \dots, O_\xi, \dots \rangle,$$

the set A as well as the set $\{0\}$ (containing 0 as the sole element) are obviously subalgebras of $\underline{A}$. In discussing subalgebras of a given algebra, we shall use a familiar set-theoretical symbolism. For instance, the formula

$$B \subseteq C$$

will express the fact that the subalgebra B is included in the subalgebra C; the symbolic expression

$$B \quad C, \text{ or } \bigcap_{i \varepsilon I} B_i,$$

will denote the intersection (common part) of the subalgebras B and C, or of all subalgebras B_i correlated with elements i of an arbitrary set I. The intersection of arbitrarily many subalgebras is clearly a subalgebra.

The notions of *homomorphism* and *isomorphism* (or *one-to-one homomorphism*)--as applied to algebras in the sense of 1.1--are assumed to be known. We shall speak of functions f which map a given algebra

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

homomorphically, or isomorphically, onto another (similar) algebra

$$\underline{A}' = \langle A', +', O_0', O_1', \ldots, O_\xi', \ldots \rangle.$$

Such a function f maps the set A onto the set A', i.e., the domain (the set of argument values) of f includes A, and A' consists of all function values f(x) correlated with elements $x \varepsilon A$. Moreover, f satisfies certain familiar conditions involving the fundamental operations of $\underline{A}$ and $\underline{A}'$; and in the case of isomorphism f is biunique when restricted to elements of A, i.e., for arbitrary elements x and y in A, $f(x) = f(y)$ implies $x = y$. In general, given a function f and a set A included in the domain of f, we denote by $f^*(A)$ the set onto which f maps A, i.e., the sets of all elements f(x) with $x \varepsilon A$. Thus, in case f maps $\underline{A}$ homomorphically or isomorphically onto $\underline{A}'$, we can write

$$f^*(A) = A'.$$

A function f is called an $\underline{A}$, $\underline{A}'$ - homomorphism if it maps $\underline{A}$ homomorphically onto a subalgebra of $\underline{A}'$. It is called an $\underline{A}$, $\underline{A}'$ - isomorphism if it maps $\underline{A}$ isomorphically onto the whole algebra $\underline{A}'$; and if such a function f exists, the algebras $\underline{A}$ and $\underline{A}'$ are said to be isomorphic, in symbols,

$$\underline{A} \simeq \underline{A}'.$$

The notions just discussed are often applied to subalgebras

$$\underline{B} = \langle B, +, O_0, O_1, \ldots O_\xi, \ldots \rangle, \quad \underline{C} = \langle C, +, O_0, O_1, \ldots O_\xi, \ldots, \ldots \rangle, \ldots$$

of a given algebra $\underline{A}$. In this connection the terms "$\underline{B}$, $\underline{C}$ - homomorphism (isomorphism)" and "$\underline{B}$, $\underline{C}$-homomorphism (isomorphism)", as well as the formulas

$$\underline{B} \simeq \underline{C} \quad \text{and} \quad B \simeq C,$$

can be used interchangeably.

With any two similar algebras

$$\underline{A}' = \langle A', +', O_0', O_1', \ldots, O_\xi', \ldots \rangle \text{ and } \underline{A}'' = \langle A'', +'', O_0'', O_1'', \ldots, O_\xi'' \ldots \rangle$$

we correlate in a familiar way a new similar algebra

$$\underline{A} = \langle A, +, O_0, O_1, \dots, O_\xi, \dots \rangle$$

called the *cardinal product* (or *outer direct product*) of $\underline{A}'$ and $\underline{A}''$, in symbols,

$$\underline{A} = \underline{A}' \times \underline{A}''.$$

The set A consists of all ordered couples $\langle a', a'' \rangle$ with $a' \varepsilon A'$ and $a'' \varepsilon A''$, and the fundamental operations of $\underline{A}$ are defined by means of formulas:

$$\langle a', a'' \rangle + \langle b', b'' \rangle = \langle a' +' b', a'' +'' b'' \rangle,$$

$$O_\xi(\langle a_0', a_0'' \rangle, \langle a_1', a_1'' \rangle, \dots) = \langle O_\xi'(a_0', a_1', \dots), O_\xi''(a_0'', a_1'', \dots) \rangle.$$

This construction can clearly be extended to an arbitrary, finite or infinite, system of algebras $\underline{A}^{(i)}$.

Almost all the fundamental results of this work can be expressed in terms of cardinal products. However, the discussion of cardinal multiplication can be greatly simplified by reducing this operation on algebras to a related operation on subalgebras of one algebra -- an operation which will be referred to as *(inner) direct multiplication*. The possibility and usefulness of such a reduction is well known from group-theoretical discussion.

Definition 1.4. *Let*

$$\underline{A} = \langle A, +, O_0, O_1, \dots, O_\xi, \dots \rangle$$

be an algebra. By the (inner) direct product of subalgebras B *and* C, *in symbols* $B \times C$, *we understand the (uniquely determined) subalgebra* D *of* $\underline{A}$ *which satisfied the following conditions:*
(i) D *consists of those and only those elements* $d \varepsilon A$ *which can be represented in the form:*

$$d = b + c \text{ with } b \varepsilon B \text{ and } c \varepsilon C;$$

(ii) *if* $b_1, b_2 \varepsilon B$, $c_1, c_2 \varepsilon C$, *and* $b_1 + c_1 = b_2 + c_2$, *then*

$$b_1 = b_2 \text{ and } c_1 = c_2;$$

(iii) *if* $b_1, b_2 \varepsilon B$ *and* $c_1, c_2 \varepsilon C$, *then*

$$(b_1 + c_1) + (b_2 + c_2) = (b_1 + b_2) + (c_1 + c_2);$$

(iv) <u>if</u> O_ξ <u>is a</u> μ-<u>ary operation</u>, <u>and</u> $b_\kappa \varepsilon B$ <u>and</u> $c_\kappa \varepsilon C$ <u>for every</u> $\kappa < \mu$, <u>then</u>

$$O_\xi(b_0 + c_0, b_1 + c_1, \ldots, b_\kappa + c_\kappa, \ldots) = O_\xi(b_0, b_1, \ldots, b_\kappa, \ldots) + O_\xi(c_0, c_1, \ldots, c_\kappa, \ldots).$$

Thus the operation of (inner) direct multiplication applies to subalgebras treated simply as sets; while that of cardinal multiplication applies to algebras, that is, to systems formed by a set and a sequence of operations. Hence no confusion will arise from the fact that the same symbol $\times$ is used to denote both of these notions.[5]

It is easily seen that, in case the operation + in A satisfies the associative law, Definition 1.4 can be simplified by replacing condition (iii) by the following one:

(iii') <u>if</u> $b \varepsilon B$ <u>and</u> $c \varepsilon C$, <u>then</u> $b + c = c + b$.

Hence, when applied to groups with or without operators, Definition 1.4 proves to be equivalent to the usual definition of a direct product (or direct sum) of subgroups.[6] The importance of our definition in the general case and its adequacy for the purposes of this work is evident from the following

Theorem 1.5. Let

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

<u>be an algebra, and let</u> $\underline{B}$ <u>and</u> $\underline{C}$ <u>be two algebras similar to</u> $\underline{A}$. <u>In order that</u>

(i) $$\underline{A} \cong \underline{B} \times \underline{C}$$

<u>it is necessary and sufficient that there exist subalgebras</u> B <u>and</u> C <u>of</u> $\underline{A}$ <u>such that</u>

(ii) $$A = B \times C,$$

(iii) $\langle B, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle \cong \underline{B}$ and $\langle C, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle \cong \underline{C}$.

Proof: Let

$$\underline{B} = \langle \bar{B}, +', O'_0, O'_1, \ldots, O'_\xi, \ldots \rangle \text{ and } \underline{C} = \langle \bar{C}, +'', O''_0, O''_1, \ldots, O''_\xi, \ldots \rangle.$$

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5. Unfortunately, the two notions are themselves very often confused in the literature.

6. Compare, e.g., van der Waerden [1], vol. 1, pp. 141 ff.

If (i) holds, we consider a $\underline{B} \times \underline{C}$, $\underline{A}$ - isomorphism f. Thus, f maps the set of all couples $< b, c >$ with $b \varepsilon \bar{B}$ and $c \varepsilon \bar{C}$ onto the set A. Let $\bar{B}$ be the set of all element $f(< b, 0'' >)$ with $b \varepsilon B$, and $\bar{C}$ the set of all elements $f(< 0', c >)$ with $c \varepsilon C$; $0'$ and $0''$ are the zero elements of $\underline{B}$ and $\underline{C}$, respectively. The proof that B and C are subalgebras of $\underline{A}$ which satisfy conditions (ii) and (iii) is based upon 1.1 - 1.4, and presents no difficulty. If, conversely, B and C are any subalgebras of $\underline{A}$ satisfying (ii) and (iii), we first show by means of (ii) and 1.4 that

$$\underline{A} \simeq < B, +, 0_0, 0_1, \ldots, 0_\xi, \ldots > \times < C, +, 0_0, 0_1, \ldots, 0_\xi, \ldots >,$$

and hence, with the help of (iii), we obtain (i). Thus, our theorem holds in both directions.

It should be emphasized that the notion of a cardinal product -- like those of homomorphism and isomorphism -- applies to arbitrary systems $\underline{A}$ formed by a set and a sequence of operations, and does not depend on restrictive conditions imposed on these systems in 1.1. On the other hand, the definition of a direct product implicitly involves the notion of a zero element, and it cannot be applied to systems without a zero unless we agree to use the term "subalgebra" in a more general sense. What is even more important, a detailed examination of the proof of 1.5 reveals that condition 1.1 (ii) plays an essential role in this proof. It can easily be shown by means of examples that, in general, Theorem 1.5 does not apply to systems $\underline{A}$ in which either part (ii') or part (ii") of this condition fails; for it may then happen that $\underline{A}$ is isomorphic to the cardinal product of two systems $\underline{B}$ and $\underline{C}$, without containing any subsystem isomorphic to $\underline{B}$ or $\underline{C}$. We can thus say that only by restricting ourselves to systems which satisfy 1.1 (ii) are we in a position to introduce an adequate notion of an inner direct product.

As regards condition 1.1 (i), the situation is more involved. Algebraic systems in which 1.1 (i) fails, i.e., which lack the closure property, occur quite frequently in application and therefore deserve more attention than is usually paid to them. Theorem 1.5 can easily be extended to such systems. This requires but a slight modification of Definition 1.4; conditions 1.4 (ii) - (iv) must be replaced by somewhat different conditions, which are equivalent to the original ones when applied to algebras in the sense of 1.1. Of course, the systems under discussion are assumed to satisfy 1.1 (ii), and hence also those particular cases of

1.1 (i) which are implied by 1.1 (ii). So far, however, we have not succeeded in extending to such systems all the fundamental results of this work. We know only that our results apply to systems which are not supposed to satisfy 1.1 (i'), and in which 1.1 (i") is restricted to cases where at least one of the elements $x_0, x_1, \ldots, x_\kappa, \ldots$ involved equals 0.

We take this opportunity to discuss briefly the possibility of extending the notions and results of our work to still more comprehensive types of algebraic systems. In algebras considered in 1.1 the "main" operation, +, is assumed to be a binary one. This circumstance, however, --which, of course has influenced the formulation of 1.4 --does not seem to be essential; the operation + can presumably be replaced by an operation with an arbirtary rank $\mu > 2$, under appropriate modifications of 1.1 and 1.4. On the other hand, it is well known that operations can be regarded as relations of a special kind -- in fact, unary operations as special binary relations, binary operations as special ternary relations, and, in general, operations with a rank μ as special relations with the rank $\mu + 1$ (i.e., as sets cf sequences of type $\mu + 1$). Hence the problem arises whether some or all of the operations constituting algebraic systems under discussion cannot be replaced by arbitrary relations subjected to conditions analogous to 1.1 (i), (ii). We have not yet investigated this problem.

In consequence of 1.5 the notion of a cardinal product will be entirely eliminated from the main body of our further discussion. (We shall use it only in Section 4 in introducing the notion of the cardinal product of isomorphism types.) We shall undertake instead a detailed study of (inner) direct products. It may be noticed that the order of operation $O_0, O_1, \ldots, O_\xi, \ldots$ in an algebra $\underline{A}$, which plays an essential role in the definition of a cardinal product, is not involved at all in the definition of a direct product. Hence we could now modify our original conception of an algebra, and regard algebras as systems constituted by a set A, a binary operation +, and a set Ω of other operations. This would bring us into complete agreement with the point of view generally accepted in the discussion of groups with operators.

In the next few theorems we formulate the most elementary properties of direct products.

<u>Theorem</u> 1.6. <u>Let</u> B <u>and</u> C <u>be subalgebras of an algebra</u>

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

such that $B \times C$ exist.

(i) If $b \varepsilon B$ and $c \varepsilon C$, then $b + c = c + b$.

(ii) If $b_1, b_2 \varepsilon B$ and $c \varepsilon C$, then $(b_1 + b_2) + c = b_1 + (b_2 + c) = (b_1 + c) + b_2$. Similarly, if $b \varepsilon B$ and $c_1, c_2 \varepsilon C$, then $b + (c_1 + c_2) = (b + c_1) + c_2 = c_1 + (b + c_2)$.

(iii) If $a \varepsilon B \times C$, $b \varepsilon B$, and if either $a + b \varepsilon B$ or $b + a \varepsilon B$, then $a \varepsilon B$.

Proof: If $b_1, b_2 \varepsilon B$ and $c \varepsilon C$, then, by 1.3 and 1.4 (iii),

$$(b_1 + 0) + (b_2 + c) = (b_1 + b_2) + (0 + c) \text{ and } (b_1 + c) + (b_2 + 0) = (b_1 + b_2) + (c + 0).$$

Hence the first part of (ii) follows by 1.1 (ii') and 1.2 (i). The proof of the second part is similar. To prove (i), put $b_1 = 0$ and $b_2 = b$, and apply (ii), 1.1 (ii'), 1.2 (i), and 1.3. If $a \varepsilon B \times C$ and $b \varepsilon B$, then, by 1.4 (i),

$$a = b' + c \text{ where } b' \varepsilon B \text{ and } c \varepsilon C$$

Therefore, by (ii),

$$a + b = (b' + b) + c \text{ and } b + a = (b + b') + c.$$

By 1.1 (ii') and 1.2 (i),

$$a + b = (a + b) + 0 \text{ and } b + a = (b + a) + 0.$$

Hence, if either $a + b \varepsilon B$ or $b + a \varepsilon B$, we have by 1.3 and 1.4 (ii)

$$c = 0.$$

Thus (iii) holds, and the proof is complete.

Theorem 1.7. Let B and C be subalgebras of an algebra

$$\underline{A} = \langle A, +, 0_0, 0_1, \ldots, 0_\xi, \ldots \rangle$$

such that $B \times C$ exists. We then have:

(i) $B \times C$ is a subalgebra of A, and in fact the smallest subalgebra which includes both B and C.

(ii) $$B \times C = C \times B.$$

(iii) If B' and C' are subalgebras of B and C, respectively, $B' \times C'$ exists and $B' \times C' \subseteq B \times C$.

Proof: (i) is an immediate consequence of 1.4; (ii) follows from 1.3, 1.4, and 1.6 (i). To prove (iii), let D' be the set consisting of all elements of the form $b + c$ where $b \in B'$ and $c \in C'$. By 1.1 (ii') and 1.2 (i) we see that $0 \in D'$, and hence we easily conclude by 1.3 and 1.4 (iii), (iv) that D' is a subalgebra of $\underline{A}$. Finally, by 1.4,

$$D' = B' \times C' \text{ and } D' \subseteq B \times C;$$

and the proof is complete.

Theorem 1.8. Let B and C be subalgebras of an algebra

$$\underline{A} = \langle A, +, 0_0, 0_1, \ldots, 0_\xi, \ldots \rangle.$$

(i) If $B \times C$ exists, then $B \cap C = \{0\}$.
(ii) $B \times C = C$ if, and only if, $B = \{0\}$.
(iii) $B \times C = \{0\}$ if, and only if, $B = C = \{0\}$.

Proof: Suppose $B \times C$ exists and $a \in B \times C$. By 1.1 (ii') and 1.2 (i),

$$a + 0 = 0 + a.$$

Hence, by 1.3 and 1.4 (ii),

$$a = 0.$$

Therefore

$$B \cap C = \{0\},$$

and (i) holds. Suppose now that

(1) $$B \times C = C.$$

Then, by 1.7 (i), C includes B, and hence, by (i),

(2) $$B = \{0\}.$$

Conversely, (2) implies (1) by 1.1, 1.2 (i), and 1.4, and the proof of (ii) is thus complete. (iii) follows from (ii) and 1.7 (i).

Theorem 1.9. Let B, C, and D be subalgebras of an algebra

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle.$$

If either $B \times C$ and $(B \times C) \times D$ exist, or else $C \times D$ and $B \times (C \times D)$ exist, then all the direct products involved exist, and

$$(B \times C) \times D = B \times (C \times D).$$

Proof: If $B \times C$ and $(B \times C) \times D$ exist, then $C \times D$ exists by 1.7 (i), (iii), and the proof is easily completed with the aid of 1.1 (i), 1.4, 1.6 (ii), and 1.7 (i). The proof under the alternative assumption is analogous

In view of this theorem, we shall usually omit parentheses in expressions like

$$(B \times C) \times D \text{ and } B \times (C \times D).$$

Definition 1.10. Let

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

be an algebra. The (inner) direct product of a finite sequence of subalgebras $B_0, B_1, \ldots, B_\kappa, \ldots, \kappa < \nu$, is defined recusively as follows:

$$\prod_{\kappa<0} B_\kappa = \{0\};$$

$\prod_{\kappa<\nu+1} B_\kappa = \prod_{\kappa<\nu} B_\kappa \times B_\nu$ for every $\nu < \omega$ (assuming that $\prod_{\kappa<\nu} B_\kappa$ and $\prod_{\kappa<\nu} B_\kappa \times B_\nu$ exist).

The problem of extending the notion of a direct product to infinite systems of subalgebras is somewhat involved, and will not be discussed here.[7]

Theorem 1.11. Let B_0 and B_1 be subalgebras of an algebra

7. For algebras with one operation the problem is discussed in Jónsson-Tarski [2].

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle.$$

We then have:

(i)
$$\prod_{\kappa<1} B_\kappa = B_0;$$

(ii)
$$\prod_{\kappa<2} B_\kappa = B_0 \times B_1 \text{ in case } B_0 \times B_1 \text{ exists.}$$

Proof: by 1.8 (ii) and 1.10.

The following theorems 1.12 - 1.14 are inductive generalizations of various parts of 1.4, 1.7, 1.8, and 1.9.

Theorem 1.12. Let $B, B_0, B_1, \ldots, B_\kappa, \ldots$ with $\kappa < \nu < \omega$ be subalgebras of an algebra

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle.$$

We then have

$$B = \prod_{\kappa<\nu} B_\kappa$$

if, and only if, the following conditions are satisfied:

(i) B consists of those and only those elements $b \varepsilon A$ which can be represented in the form

$$b = \sum_{\kappa<\nu} b_\kappa \text{ with } b_\kappa \varepsilon B_\kappa \text{ for } \kappa < \nu;$$

(ii) if $b_\kappa, b'_\kappa \varepsilon B_\kappa$ for $\kappa < \nu$, and if $\sum_{\kappa<\nu} b_\kappa = \sum_{\kappa<\nu} b'_\kappa$, then

$$b_\kappa = b'_\kappa \text{ for } \kappa < \nu;$$

(iii) if $b_\kappa, b'_\kappa \varepsilon B_\kappa$ for $\kappa < \nu$, then

$$\sum_{\kappa<\nu} (b_\kappa + b'_\kappa) = \sum_{\kappa<\nu} b_\kappa + \sum_{\kappa<\nu} b'_\kappa;$$

(iv) if O_ξ is a μ-ary operation, and $b_{\kappa,\iota} \varepsilon B_\kappa$ for $\kappa < \nu$ and $\iota < \mu$, then

$$O_\xi\Big(\sum_{\kappa<\nu} b_{\kappa,0}, \sum_{\kappa<\nu} b_{\kappa,1}, \ldots, \sum_{\kappa<\nu} b_{\kappa,\iota}, \ldots\Big) = \sum_{\kappa<\nu} O_\xi(b_{\kappa,0}, b_{\kappa,1}, \ldots b_{\kappa,\iota}, \ldots).$$

Proof: by induction, using 1.2, 1.4 and 1.10.

Theorem 1.13. Let $B_0, B_1, \ldots, B_\kappa, \ldots$ with $\kappa < \nu < \omega$ be subalgebras of an algebra

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

such that $\prod_{\kappa<\nu} B_\kappa$ exists.

We then have:

(i) $\prod_{\kappa<\nu} B_\kappa$ is a subalgebra of $\underline{A}$, and in fact the smallest subalgebra of $\underline{A}$ which includes all the subalgebras B_κ with $\kappa < \nu$.

(ii) If B'_κ is a subalgebra of B_κ for $\kappa < \nu$, then $\prod_{\kappa<\nu} B'_\kappa$ exists and $\prod_{\kappa<\nu} B'_\kappa \subseteq \prod_{\kappa<\nu} B_\kappa$.

(iii) If $\kappa_0 < \kappa_1 < \ldots < \kappa_\lambda < \ldots < \nu$ for $\lambda < \pi$, then $\prod_{\lambda<\pi} B_{\kappa_\lambda}$ exists and $\prod_{\lambda<\pi} B_{\kappa_\lambda} \subseteq \prod_{\kappa<\nu} B_\kappa$.

(iv) If $\pi < \nu$, then $\prod_{\kappa<\pi} B_\kappa = \prod_{\kappa<\pi} B_\kappa \times \prod_{\kappa<\nu-\pi} B_{\pi+\kappa}$.

Proof: by induction, using 1.1, 1.2, 1.3, 1.7, 1.9, 1.10.

Theorem 1.14. Let $B_0, B_1, \ldots, B_\kappa, \ldots$ with $\kappa < \nu < \omega$ be subalgebras of an algebra

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle.$$

(i) For every $\lambda < \nu$ we have $\prod_{\kappa<\nu} B_\kappa = B_\lambda$ if, and only if,

$$B_\kappa = \{0\} \text{ for } \kappa < \nu \text{ and } \kappa \neq \lambda.$$

(ii) We have $\prod_{\kappa<\nu} B_\kappa = \{0\}$ if, and only if,

$$B_\kappa = \{0\} \text{ for } \kappa < \nu.$$

Proof: by 1.8 (ii), (iii) and 1.10.

We shall now give a theorem of a somewhat less obvious nature, which is of fundamental significance for our further discussion and leads directly to several important consequences. To state this theorem (1.17) more conveniently, we introduce the notion of a subtractive subalgebra.

Definition 1.15. A subalgebra B of an algebra

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi \ldots \rangle$$

is called a subtractive subalgebra if it satisfies the following condition: if $a \in A$ and $b \in B$, and if either $a + b \in B$ or $b + a \in B$, then $a \in B$.

Corollary 1.16. *If* B *and* C *are subalgebras of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle,$$

and if A = B × C, *then* B *and* C *are subtractive subalgebras of* $\underline{A}$.

Proof: by 1.5 (iii), 1.7 (ii), and 1.15.

Theorem 1.17 (Modular law)[8] *If* B *and* C *are subalgebras of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

such that B × C *exists, and if* D *is a subtractive subalgebra of* A *which includes* B, *then*

$$(B \times C) \cap D = B \times (C \cap D).$$

Proof: By 1.1, 1.3, and 1.7 (i), $C \cap D$ and $(B \times C) \cap D$ are subalgebras of $\underline{A}$. Hence, by 1.7 (i), (iii), $B \times (C \cap D)$ exists and

(1) $$B \times (C \cap D) \subseteq (B \times C) \cap D.$$

Suppose

(2) $$a \in (B \times C) \cap D.$$

Then, by 1.4 (i),

(3) $$a = b + c \text{ where } b \in B \text{ and } c \in C.$$

Therefore, by our hypothesis and (2),

$$b \in D \text{ and } b + c \in D.$$

Consequently, by (3) and 1.15,

$$c \in C \cap D.$$

From this and (3) we conclude by 1.4 (i) that

(4) $$a \in B \times (C \cap D).$$

Thus, for any given element $a \in A$, (2) implies (4). Hence, the inclusion symbol in (1) can be replaced by the equality symbol, and the proof is complete.

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8. The modular law in its application to various special algebras can be found in the literature. Compare Birkhoff [1] pp. 34 ff., where bibliographical references to earlier publications (by R. Dedekind and others) can be found; see also Baer [1], p. 455.

Corollary 1.18. *If* B *and* C *are subalgebras of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

such that $B \times C$ *exists, and if* D *is a subtractive subalgebra of* A *with*

$$B \subseteq D \subseteq B \times C,$$

then

$$D = B \times (C \cap D).$$

Proof: by 1.17.

A subalgebra B is called a *factor* of a subalgebra D if, for some subalgebra C, $B \times C = D$. Using this terminology we conclude from 1.18 (with the help of 1.16) that the factor relation between factors of a given algebra coincides with the relation of set-theoretical inclusion.

Theorem 1.19. *If* B, C, *and* D *are subalgebras of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

such that $B \times C$ *and* $(B \times C) \times D$ *exist, then*

$$(B \times C) \cap (B \times D) = B.$$

Proof: By 1.7 (i), (iii), $B \times D$ exists and

(1) $$D \subseteq B \times D \subseteq (B \times C) \times D.$$

By 1.7 (ii), 1.9, and 1.16, $B \times D$ is a subtractive subalgebra of $(B \times C) \times D$, and therefore, by (1), 1.7 (ii), and 1.18,

(2) $$B \times D = [(B \times C) \cap (B \times D)] \times D.$$

By 1.7 (i),

(3) $$B \subseteq (B \times C) \cap (B \times D) \subseteq B \times D.$$

By (2) and 1.16, $(B \times C) \cap (B \times D)$ is a subtractive subalgebra of $B \times D$. Consequently, by (3) and 1.18,

(4) $$(B \times C) \cap (B \times D) = B \times [(B \times C) \cap (B \times D) \cap D].$$

By (2) and 1.8 (i),

(5) $$(B \times C) \cap (B \times D) \cap D = \{0\}.$$

The conclusion follows from (4) and (5) by 1.7 (ii) and 1.8 (ii).

We conclude this section with two elementary theorems which establish certain connections between the notion of a direct product and those of homomorphism and isomorphism.

Theorem 1.20. *Let* B, B_0, $B_1, \ldots, B_\kappa, \ldots$ *with* $\kappa < \nu < \omega$ *be subalgebras of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, C_\xi, \ldots \rangle,$$

and let

$$B = \prod_{\kappa < \nu} B_\kappa.$$

Then there exist functions $f_0, f_1, \ldots, f_\kappa, \ldots$ *with* $\kappa < \nu$ *such that*

(i) $$b = \sum_{\kappa < \nu} f_\kappa(b) \text{ for } b \in B,$$

(ii) f_κ *is a* B, B_κ – *homomorphism and* $f^*_\kappa(B) = B_\kappa$ *for* $\kappa < \nu$.

Proof: Given an element b in B, there exists by 1.12 (i), (ii) a uniquely determined finite sequence $b_0, b_1, \ldots, b_\kappa, \ldots$ with $b_\kappa \in B_\kappa$ for $\kappa < \nu$ such that

$$b = \sum_{\kappa < \nu} b_\kappa.$$

Putting

$$f_\kappa(b) = b_\kappa \text{ for } \kappa < \nu,$$

we see that condition (i) of the conclusion is satisfied; and, by 1.1 (ii'), 1.2, and 1.12, the same applies to (ii).

Theorem 1.21. *If* B, C, *and* D *are subalgebras of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle,$$

and if $C \times D$ *exists, then we have* $B \simeq C \times D$ *if, and only if, there exist subalgebras* C' *and* D' *of* A *such that* $B = C' \times D'$, $C' \simeq C$, *and* $D' \simeq D$.

Proof: by 1.4.

Theorem 1.21 can obviously be extended to direct products of arbitrary finite sequences of subalgebras.

In our further discussion we shall frequently apply various definitions and theorems of this section without referring to them explicitly.