ESTIMATION FOR A REGRESSION MODEL WITH AN UNKNOWN COVARIANCE MATRIX

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1. Summary and introduction

A linear regression model is considered under which the residual error vector is assumed to have a multivariate normal distribution with unknown covariance matrix Σ. To estimate Σ, it is assumed that the regression design can be given independent replications. This problem has been considered by Rao, who obtains a point estimator and suggests two classes of confidence regions for the vector β of regression parameters. In the present paper, we find the maximum likelihood estimators of β and of Σ, and derive their distributions. One of Rao's two classes of confidence regions for β had previously been inapplicable due to the lack of tables for upper tail values of the distribution of the pivotal quantity. These tables are now provided, and the performances of the two classes of confidence regions are compared in terms of their expected volumes.

In the classical linear regression model, the vector of observations $y = (y_1, y_2, \cdots, y_p)$ has the form

$$y = \beta X + \varepsilon, \tag{1.1}$$

where β: $1 \times q$ is an unknown vector of regression parameters, X is a known $q \times p$ matrix of rank $q \leqq p$, and ε has a p variate normal distribution with mean vector zero and covariance matrix $\Sigma = \sigma^2 I$. Since the simple structure of the covariance matrix may not be valid for some problems, extensions of the results of the classical model to models where Σ has a more general structure have been considered. Such attempts can be classified in the following hierarchy of complexity:

(i) Σ an arbitrary known matrix,

(ii) Σ known up to a scale factor σ^2,

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(iii) Σ unknown but with some special structure,
(iv) Σ completely unknown and arbitrary.

The maximum likelihood estimators (MLE) for cases (i) and (ii) are well known (see Anderson [1]). In both of these cases, the MLE $\hat{\beta}(\Sigma)$ of β has the form $\hat{\beta}(\Sigma) = y\Sigma^{-1}X'(X\Sigma^{-1}X')^{-1}$ with covariance matrix $(X\Sigma^{-1}X')^{-1}$ yielding the minimum concentration ellipse among all linear unbiased estimators of β. (Note that in case (ii), $\hat{\beta}(\Sigma)$ is independent of the unknown scale factor σ^2.) Watson [22], [23] and Watson and Hannan [24] have investigated the errors involved when the assumptions made concerning Σ in cases (i) and (ii) are violated.

As an example of a model of the type considered in case (iii), assume that Σ has the intraclass correlation structure. This class of linear regression models has been considered by Halperin [10], by Geisser and Greenhouse [5], [6], and by other authors. Alternative possible special models for Σ include the models of autocorrelation, circular symmetry, and compound symmetry. In each of these special cases, as well as in cases (i) and (ii), inference concerning the parameters of the regression model is possible even when only one replication of the random vector y is available.

If, however, we are in complete ignorance of Σ, it is clear that more than one observation must be taken on y in order to estimate both β and Σ. In some problems, one may actually have independent replications of the y's: for example, (a) where each y vector represents a score vector on an examination and the replications are individuals from a particular homogeneous group, or (b) in the analysis of growth curves (see Rao [19], Pothoff and Roy [15], Gleser and Olkin [8], [9]). The replications on y enable us to simultaneously estimate β and Σ.

Versions of case (iv) have been considered by many authors. Cochran and Bliss [2] discuss a variant of this model in connection with the comparison of discriminant functions from two populations. Rao [16], [17], [18] considers the problem of testing the hypothesis that the vector β of regression parameters obeys certain linear constraints, derives the likelihood ratio test statistic for this problem, and obtains its null and nonnull distributions. Further distributional results for the likelihood ratio statistic are given by Narain [13], Olkin and Shrikhande [14], and Kabe [11].

Rao [16], [18], [20], [21] also considers the problem of estimating β. He obtains a certain "least squares" estimator for β which is, in fact, the MLE of β (Gleser and Olkin [7]). They find the MLE of β and Σ, give representations for their densities, and compare the covariance matrices of the MLE of β and the BLUE of β when Σ is known. The comparison shows that for even moderate sample sizes, there is little difference in the accuracies of the two estimators. (Similar results are also given by Rao [21] and Williams [25].) The above results, together with a new and very useful representation for the density of the MLE of β, appear in Section 2 and Appendix A.

Rao ([16]–[21]) has proposed two classes of confidence regions for (linear combinations of) the elements of the vector β—one class based on a statistic closely related to Mahalanobis's distance, the other on the likelihood ratio test statistic for testing that β obeys certain linear restraints. These two procedures are described in Section 3. Distributional difficulties with the former class of confidence regions have up to now severely limited its applicability, and have prevented comparisons with the class of regions based on the likelihood ratio statistic. In Appendix B of this paper, we provide the necessary tables for the application of this confidence procedure in certain cases, and indicate how these tables may be used (and extended) in more general contexts. The availability of these tables permits comparison of the two classes of confidence regions; these comparisons appear in Section 4. An illustrative example is given in Section 5.

2. The regression model: estimators of β and Σ

Let $y^{(1)}, \cdots, y^{(N)}$ be N independent random p dimensional row vectors, each having a multivariate normal distribution with mean vector $\mathscr{E}(y^{(j)}) = \beta X$ and covariance matrix Σ, where X is a known $q \times p$ matrix of rank $q \leqq p$, where β is a $1 \times q$ vector of unknown regression parameters, and where Σ is an unknown positive definite matrix.

We may immediately reduce the data to the sufficient statistic $(\bar{y}, S)$, where $\bar{y} = N^{-1} \Sigma_{j=1}^{N} y^{(j)}$ is the sample mean vector and $S = \Sigma_{i=1}^{N} (y^{(i)} - \bar{y})'(y^{(i)} - \bar{y})$ is the sample cross product matrix. Thus, $\bar{y}$ and S are independently distributed, $\bar{y}$ has a multivariate normal distribution with mean vector βX and covariance matrix $N^{-1}\Sigma$, (denoted $\bar{y} \sim N(\beta X, N^{-1}\Sigma)$), and S has the Wishart distribution with $n \equiv N - 1$ degrees of freedom and expectation $\mathscr{E}(S) = n\Sigma$, (denoted $S \sim W(\Sigma; p, n)$), S being $p \times p$. The joint density of $\bar{y}$ and S is given by

$$p(\bar{y}, S) = c|\Sigma|^{-N/2}|S|^{(n-p-1)/2} \exp\{-\tfrac{1}{2}\mathrm{tr}\, \Sigma^{-1}[S + N(\bar{y} - \beta X)'(\bar{y} - \beta X)]\}, \tag{2.1}$$

where

$$c^{-1} = [2^N \pi^{(p+1)/2} N^{-1}]^{p/2} \prod_{i=1}^{p} \Gamma[\tfrac{1}{2}(n - i + 1)]. \tag{2.2}$$

To obtain the MLE of β and Σ, first maximize $p(\bar{y}, S)$ with respect to Σ; this yields

$$N\hat{\Sigma}(\beta) = S + N(\bar{y} - \beta X)'(\bar{y} - \beta X) \tag{2.3}$$

(see, for example, Anderson [1], p. 46). Inserting $\hat{\Sigma}(\beta)$ for Σ in the joint density yields a constant multiple of

$$|S + N(\bar{y} - \beta X)'(\bar{y} - \beta X)|^{-N/2} = |S|^{-N/2}[1 + N(\bar{y} - \beta X)S^{-1}(\bar{y} - \beta X)']^{-N/2}, \tag{2.4}$$

from which, maximizing with respect to β, we obtain the MLE of β to be

$$\hat{\beta} = \bar{y}S^{-1}X'(XS^{-1}X')^{-1}. \tag{2.5}$$

The MLE of Σ is then $\hat{\Sigma} \equiv \hat{\Sigma}(\hat{\beta})$.

The distribution of $\hat{\beta}$ is obtained in Appendix A. There, it is shown that the following result holds.

THEOREM 2.1. *The probability density of $\hat{\beta}$ is*

$$(2.6)\qquad p(\hat{\beta}) = \sum_{j=0}^{\infty} c_j \frac{|NX\Sigma^{-1}X'|^{1/2}[Q(\hat{\beta})]^j \exp\{-\frac{1}{2}Q(\hat{\beta})\}}{(2\pi)^{q/2}2^j[\Gamma(\frac{1}{2}q + j)/\Gamma(\frac{1}{2}q)]} \equiv \sum_{j=0}^{\infty} c_j h_j(\hat{\beta}),$$

where $Q(\hat{\beta}) = N(\hat{\beta} - \beta)X\Sigma^{-1}X'(\hat{\beta} - \beta)'$,

$$(2.7)\qquad c_j = \frac{c_0}{j!} \frac{\Gamma(\frac{1}{2}(p - q) + j)}{\Gamma(\frac{1}{2}(p - q))} \frac{\Gamma(\frac{1}{2}q + j)}{\Gamma(\frac{1}{2}q)} \frac{\Gamma(\frac{1}{2}(n + q + 1))}{\Gamma(\frac{1}{2}(n - p + q) + j)},$$

the components of $\hat{\beta}$ range from $-\infty$ to ∞, and

$$(2.8)\qquad c_0 = \frac{\Gamma(\frac{1}{2}(n + 2q - p + 1))\Gamma(\frac{1}{2}(n + 1))}{\Gamma(\frac{1}{2}(n + q + 1))\Gamma(\frac{1}{2}(n - p + q + 1))}.$$

Note that $c_j \geqq 0$ for all j. It can be shown that $\Sigma_{j=0}^{\infty} c_j = 1$ and that each $h_j(\hat{\beta})$, $j = 0, 1, \cdots$, is a q variate density. (Indeed, $h_0(\hat{\beta})$ is the density of a q variate normal distribution having mean vector β and covariance matrix $(NX\Sigma^{-1}X')^{-1}$.) Thus, (2.6) is a mixture of the densities $h_j(\hat{\beta})$. Using standard results concerning mixtures of densities, we can conclude that for any measurable set R in q dimensional space,

$$(2.9)\qquad c_0 P\{u \in R\} \leqq P\{\hat{\beta} \in R\} \leqq c_0 P\{u \in R\} + (1 - c_0),$$

where $u \sim N(\beta, (NX\Sigma^{-1}X')^{-1})$. From the fact that for fixed h.

$$(2.10)\qquad \frac{\Gamma(t + h)}{\Gamma(t)} = t^h[1 + o(1)], \qquad t \to \infty,$$

it follows that

$$(2.11)\qquad c_0 = 1 - \frac{q(p - q)}{2N} + O(N^{-2})$$

as $N \to \infty$. From (2.9) and (2.11), we see that $\sqrt{N}(\hat{\beta} - \beta)$ has an asymptotic q variate normal distribution with mean vector zero and covariance matrix $(X\Sigma^{-1}X')^{-1}$, and we also have a measure of the accuracy of the approximation involved in replacing the finite sample distribution of $\hat{\beta}$ with the asymptotic distribution.

Two alternative forms for the density (2.6) of $\hat{\beta}$ in terms of an integral representation and a hypergeometric series may prove helpful (Gleser and Olkin [7]). These are the following:

$$(2.12)\qquad p(\hat{\beta}) = h_0(\hat{\beta}) \int_0^1 \frac{g^{(p-q)/2-1}(1 - g)^{(n+2q-p+1)/2-1} \exp\{\frac{1}{2}gQ(\hat{\beta})\}\, dg}{B(\frac{1}{2}(p - q), \frac{1}{2}(n - p + q + 1))},$$

and

$$p(\hat{\beta}) = c_0 h_0(\hat{\beta})\,{}_1F_1\left(\tfrac{1}{2}(p-q), \tfrac{1}{2}(n+q+1); \tfrac{1}{2}Q(\hat{\beta})\right), \tag{2.13}$$

where ${}_1F_1(a, b; z)$ is the confluent hypergeometric function

$${}_1F_1(a, b; z) = \sum_{j=0}^{\infty} \frac{\Gamma(a+j)}{\Gamma(a)} \frac{\Gamma(b)}{\Gamma(b+j)} \frac{z^j}{j!}. \tag{2.14}$$

From (2.12), a direct computation (involving an interchange of the order of integration between $\hat{\beta}$ and g) yields $\mathscr{E}(\hat{\beta}) = \beta$ (that is, $\hat{\beta}$ is unbiased) and

$$N \operatorname{Cov}(\hat{\beta}) = \frac{n-1}{n-p+q-1}(X\Sigma^{-1}X')^{-1}. \tag{2.15}$$

We have derived the estimators $\hat{\beta}$ and $\hat{\Sigma}$ assuming that Σ is unknown. If Σ is known, then the estimator $\hat{\beta}(\Sigma) = \bar{y}\Sigma^{-1}X'(X\Sigma^{-1}X')^{-1}$ is the Gauss-Markov (BLUE) estimator of β— that is, among all unbiased linear estimators of β, $\hat{\beta}(\Sigma)$ has the smallest ellipsoid of concentration. The covariance matrix of $\hat{\beta}(\Sigma)$ is $(NX\Sigma^{-1}X')^{-1}$; from this fact and (2.15), it follows that for all Σ,

$$\operatorname{Cov}(\hat{\beta}) = \left(1 + \frac{p-q}{n-p+q-1}\right) \operatorname{Cov}[\hat{\beta}(\Sigma)]. \tag{2.16}$$

For n moderately large with respect to $p - q$, $\operatorname{Cov}(\hat{\beta})$ and $\operatorname{Cov}[\hat{\beta}(\Sigma)]$ are nearly equal (more accurately, they are of the same order of magnitude in N). We thus have an estimator $\hat{\beta}$ for β which, regardless of the value Σ of the unknown covariance matrix, has for large enough N approximately the minimal ellipse of concentration achievable by the BLUE of β given that value of Σ. Comparisons similar to the above have been made in Gleser and Olkin [7], Rao [21], and Williams [25].

It is worth noting that as $N \to \infty$ both $\sqrt{N}(\hat{\beta} - \beta)$ and $\sqrt{N}[\hat{\beta}(\Sigma) - \beta]$ have the limiting distribution $N(0, (X\Sigma^{-1}X')^{-1})$. A measure of the error involved in assuming that $\hat{\beta}$ and $\hat{\beta}(\Sigma)$ have the same distribution in small samples can be obtained from (2.9) and (2.11).

The distribution of $\hat{\Sigma}$ is given in Appendix A.

3. Confidence regions for β

From (2.6), (2.12), or (2.13) it can be seen that the density of $\hat{\beta}$ is constant on ellipsoids that have the form

$$Q(\hat{\beta}) = \text{constant}, \tag{3.1}$$

where

$$Q(\hat{\beta}) = N(\hat{\beta} - \beta)(X\Sigma^{-1}X')(\hat{\beta} - \beta)'. \tag{3.2}$$

The regions of form (3.1) are thus ellipsoids of concentration for the distribution of $\hat{\beta}$. Since $\hat{\beta}$ has approximately a q variate normal distribution with mean vector β and covariance matrix $(NX\Sigma^{-1}X')^{-1}$, this suggests using the ellipsoid $\{\beta: Q(\hat{\beta}) \leqq \chi_q^2(\gamma)\}$, where $\chi_q^2(\gamma)$ is the upper tail of a χ_q^2 distribution, as a 100γ per cent confidence interval for β. Unfortunately, this region cannot be used since Σ is unknown. We can, however, replace Σ by its MLE $\hat{\Sigma}$, and form a confidence region for β based on the pivotal quantity

$$\Delta = N(\hat{\beta} - \beta)(X\hat{\Sigma}^{-1}X')(\hat{\beta} - \beta)'. \tag{3.3}$$

Since $(N\hat{\Sigma})^{-1} = S^{-1} - N(1 + r)^{-1}S^{-1}(\bar{y} - \hat{\beta}X)'(\bar{y} - \hat{\beta}X)S^{-1}$, where

$$r = N(\bar{y} - \hat{\beta}X)S^{-1}(\bar{y} - \hat{\beta}X)', \tag{3.4}$$

and since $XS^{-1}(\bar{y} - \hat{\beta}X) = 0$, it follows that $X\hat{\Sigma}^{-1}X' = NXS^{-1}X'$ and

$$\Delta = N^2(\hat{\beta} - \beta)(XS^{-1}X')(\hat{\beta} - \beta)'. \tag{3.5}$$

Although the region $\{\beta: N^2(\hat{\beta} - \beta)(XS^{-1}X')(\hat{\beta} - \beta) \leqq \chi_q^2(\gamma)\}$ has asymptotic confidence γ as $N \to \infty$, it is not an exact 100γ per cent confidence region for β. Thus for moderate sample sizes it may be of value to determine exact confidence regions for β based on the pivotal quantity Δ defined in (3.5).

The problem of finding the constant $b^{(\gamma)}$ for which the region

$$E_1 = \{\beta: N(\hat{\beta} - \beta)(XS^{-1}X')(\hat{\beta} - \beta)' \leqq b^{(\gamma)}\} \tag{3.6}$$

has *exact* confidence γ is quite difficult since $b^{(\gamma)}$ or equivalently $c^{(\gamma)} = b^{(\gamma)}/(1 + b^{(\gamma)})$ is obtained as the solution of the integral equation

$$\int_0^1 dg \int_0^{c^{(\gamma)}} dh \, \frac{g^{a_1 - 1}(1 - g)^{a_2 - 1}h^{d_1 - 1}(1 - h)^{d_2 - 1}(1 - gh)^{-(d_1 + d_2)}}{B(a_1, a_2 - d_1)B(d_1, d_2)} = \gamma, \tag{3.7}$$

where $a_1 = \frac{1}{2}(p - q)$, $a_2 = \frac{1}{2}(n + 2q - p + 1)$, $d_1 = \frac{1}{2}q$, and $d_2 = \frac{1}{2}(n - p + 1)$.

THEOREM 3.1. *If $c^{(\gamma)}$ is chosen to satisfy* (3.7), *then E_1 (with $b^{(\gamma)} = c^{(\gamma)}/(1 - c^{(\gamma)})$) is a 100γ per cent confidence region for β.*

PROOF. From (3.4) and Lemma 2 of Appendix A, $(n - p + 1)\Delta/q(1 + r)$ has, conditional upon r, Snedecor's F distribution with q and $n - p + 1$ degrees of freedom. Also $(1 + r)^{-1}$ has a Beta distribution with parameters $\frac{1}{2}(n - p + q + 1)$ and $\frac{1}{2}(p - q)$. It follows, therefore, that for $P\{\beta \in E_1\}$ to be equal to γ, we must have

$$\begin{aligned} \gamma = P\{\beta \in E_1\} &= P\left\{\frac{(n - p + 1)\Delta}{q(1 + r)} \leqq \left(\frac{n - p + 1}{q}\right)\frac{b^{(\gamma)}}{1 + r}\right\} \\ &= \int_0^\infty \frac{r^{(p - q)/2 - 1}\,dr}{B\left(\frac{1}{2}(p - q), \frac{1}{2}(n + q - p + 1)\right)(1 + r)^{(n + 1)/2}} \\ &\quad \cdot \int_0^{b^{(\gamma)}/(1 + r)} \frac{x^{q/2 - 1}\,dx}{B\left(\frac{1}{2}q, \frac{1}{2}(n - p + 1)\right)(1 + x)^{(n - p + q + 1)/2}}. \end{aligned} \tag{3.8}$$

By a change of variables to $g = r/(1+r)$, $h = (x + xr)/(1 + x + xr)$, we obtain (3.7). *Q.E.D.*

Another expression for $P\{\beta \in E_1\}$ has been given by Rao [17] in terms of the hypergeometric function. However, in either form it is difficult to solve for the cutoff point $b^{(\gamma)}$. A computer program has been written utilizing a certain mixture representation for the integral (3.7). This program is described in Appendix B.

Notice that the statistic $r = N(\bar{y} - \hat{\beta}X)S^{-1}(\bar{y} - \hat{\beta}X)'$ is a function of the sufficient statistic $(\bar{y}, S)$ and has a distribution which is functionally independent of the parameters β and Σ under the model (1.1). Thus, r is an ancillary statistic. Indeed, the statistic r can be used to test the goodness of fit of the model (1.1) (see Rao [20]). Following a somewhat standard practice, we might agree to find a confidence region for β which has probability of coverage γ, conditional upon r for each possible value of r. Returning to the distributional fact used in the proof of Theorem 3.1, we see that one such region is

$$E_2 = \left\{\beta : \frac{(n - p + 1)N(\hat{\beta} - \beta)(XS^{-1}X')(\hat{\beta} - \beta)'}{q(1 + r)} \leqq F^{(\gamma)}_{q,n-p+1}\right\}, \tag{3.9}$$

where $F^{(\gamma)}_{q,n-p+1}$ is the upper tail of Snedecor's F distribution with q and $n - p + 1$ degrees of freedom. Since E_2 has, conditional upon r, coverage γ for β, it is also a 100γ per cent unconditional confidence region for β. Because tables of the F distribution are easily available, the region E_2 has been preferred by statisticians. However, in certain circumstances the performance of region E_1 may be superior to that of region E_2. Without values of $b^{(\gamma)}$, comparisons of these two confidence regions are difficult, if not impossible, to do. Using the tables of $b^{(\gamma)}$, such comparisons can now be made.

Before leaving the present section, however, it is worth noting that the region E_2 is the set of all vectors β_0 in q dimensional space for which the null hypothesis $H: \beta = \beta_0$ is not rejected by the appropriate likelihood ratio test at level $\alpha = 1 - \gamma$. The likelihood ratio test of $H: \beta = \beta_0$ versus general alternatives has rejection region

$$\frac{(n - p + 1)N(\hat{\beta} - \beta_0)(XS^{-1}X')(\hat{\beta} - \beta_0)'}{q(1 + r)} \geqq F^{(\gamma)}_{q,n-p+1} \tag{3.10}$$

(Rao [21]), so that for given values of $\bar{y}$ and S (and thus of $\hat{\beta}$, S, and r), we accept $H: \beta = \beta_0$ if and only if β_0 is in E_2.

4. Comparison of the two procedures

Historically, there have been two main sets of criteria for the comparison of confidence regions—those based on concepts of power and those based on volume considerations. Since every confidence region can generate a test for such hypotheses as $H: \beta = \beta_0$, it seems reasonable to apply power considerations in the comparison of confidence regions. However, the difficulty involved in obtaining and analyzing the nonnull distributions of Δ and $\Delta(1 + r)^{-1}$ discourage comparisons based on power concepts (see Rao [18], [21]).

Comparisons of confidence regions through consideration of their volumes also have intuitive appeal, since the volume of a region can be viewed as a measure of the "quantity" of models (parameters) which are accepted by (included in) the confidence procedure. For example, in the case of two confidence intervals A and B of confidence γ we would prefer interval A to interval B if the length of A were always less than the length of B, because intuitively we we would feel that A would give us a more precise picture of which models are reasonable, given the data.

In the present situation our regions are ellipsoids in q dimensional Euclidean space. Since the volume of an ellipsoid

$$(u_1, u_2, \cdots, u_m) A^{-1} (u_1, u_2, \cdots, u_m)' \leqq 1 \tag{4.1}$$

is $c(m)|A|^{1/2}$, where $c(m) = (2\pi)^{m/2}\Gamma(m/2)$, we conclude that

$$\begin{aligned} \text{volume } E_1 &= c(q)|NXS^{-1}X'|^{-1/2} b_0^{q/2}, \\ \text{volume } E_2 &= c(q)|NXS^{-1}X'|^{-1/2}(1 + r)^{q/2}[qF_0/(n - p + 1)]^{q/2}, \end{aligned} \tag{4.2}$$

where $F_0 = F^{(\gamma)}_{q,n-p+1}$ and $b_0 = b^{(\gamma)}$.

Since these volumes are random variables, we may compare their expected values. Thus, we say that region E_1 is preferable to region E_2 if and only if $\mathscr{E}[\text{volume } E_1] \leqq \mathscr{E}[\text{volume } E_2]$, or equivalently if and only if the ratio

$$I_{1,2} = \left[\frac{(n - p + 1)b_0}{qF_0}\right]^{q/2} \frac{\mathscr{E}[|NXS^{-1}X'|^{1/2}]}{\mathscr{E}[|NXS^{-1}X'|^{1/2}(1 + r)^{q/2}]} \tag{4.3}$$

is less than or equal to 1. By Lemma 2 of Appendix A, $XS^{-1}X'$ and r are independently distributed and $(1 + r)^{-1}$ has a Beta distribution with $\frac{1}{2}(n - p + q + 1)$ and $\frac{1}{2}(p - q)$ degrees of freedom. Thus (4.3) becomes

$$I_{1,2} = \frac{\Gamma[\frac{1}{2}(n - q + 1)]\,\Gamma[\frac{1}{2}(n - p + q + 1)]}{\Gamma[\frac{1}{2}(n + 1)]\,\Gamma[\frac{1}{2}(n - p + 1)]} \left[\frac{(n - p + 1)b_0}{qF_0}\right]^{q/2}. \tag{4.4}$$

From equation 4.4 (and Table III of Appendix B), values of $I_{1,2}$ are computed for $n = 10(2)30(5)35$, $p = 2(1)\frac{1}{2}n$, $q = 1(1)p - 2$, and $\gamma = 0.90, 0.95, 0.975, 0.99$. In the resulting table we have observed certain patterns. (A selection from this table appears in Table I below.) First, if we fix n, p, and q, and allow γ to increase, then the ratio $I_{1,2}$ increases, becoming greater than 1 for large enough γ. The larger q is, the smaller the value of γ at which $I_{1,2}$ changes from less than 1 to greater than 1. Saying this another way, for fixed r, p, γ, the ratio $I_{1,2}$ is nearly monotonically decreasing in q (the decrease of $I_{1,2}$ in q is reversed in the third decimal place for $q \geqq p - 4$).

TABLE I

RATIO OF THE EXPECTED VOLUME OF E_1 TO E_2

$n = 14$

p	q	0.90	0.95	0.99
2	1	1.00+	1.00+	1.01
3	1	1.00+	1.00+	1.01
4	1	1.00+	1.01	1.02
	2	1.00+	1.01	1.02
5	1	1.00+	1.01	1.03
	2	1.00+	1.01	1.04
	3	0.99	1.00+	1.03
6	1	1.01	1.02	1.04
	2	1.00+	1.02	1.05
	3	0.99	1.01	1.04
	4	0.98	0.99	1.02
7	1	1.01	1.02	1.05
	2	1.00+	1.02	1.07
	3	0.98	1.01	1.06
	4	0.96	0.98	1.03
	5	0.95	0.97	1.00+

$n = 24$

p	q	0.90	0.95	0.99
2	1	1.00	1.00+	1.00+
3	1	1.00+	1.00+	1.00+
4	1	1.00+	1.00+	1.01
	2	1.00+	1.00+	1.01
5	1	1.00+	1.00+	1.01
	2	1.00+	1.00+	1.01
	3	1.00−	1.00+	1.01
6	1	1.00+	1.00+	1.01
	2	1.00+	1.01	1.02
	3	1.00−	1.01	1.02
	4	1.00−	1.00+	1.01
7	1	1.00+	1.01	1.02
	2	1.00+	1.01	1.03
	3	1.00−	1.01	1.03
	4	0.99	1.00+	1.02
	5	0.99	1.00−	1.01

$n = 24$ (*continued*)

p	q	0.90	0.95	0.99
8	1	1.00+	1.01	1.02
	2	1.00+	1.01	1.03
	3	1.00−	1.01	1.03
	4	0.99	1.00+	1.03
	5	0.98	1.00−	1.02
	6	0.98	0.99	1.01
9	1	1.01+	1.01	1.02
	2	1.00	1.01	1.04
	3	1.00−	1.01	1.04
	4	0.99	1.00+	1.04
	5	0.98	0.99	1.03
	6	0.97	0.98	1.00+
	7	0.97	0.98	1.00+
10	1	1.00+	1.01	1.03
	2	1.00+	1.02	1.05
	3	1.00−	1.01	1.05
	4	0.98	1.00+	1.05
	5	0.97	0.99	1.03
	6	0.96	0.97	1.01
	7	0.95	0.97	1.00−
	8	0.95	0.96	0.99
11	1	1.00	1.01	1.03
	2	1.00+	1.02	1.05
	3	0.99	1.01	1.06
	4	0.98	1.00+	1.05
	5	0.96	0.98	1.04
	6	0.94	0.97	1.01
	7	0.93	0.95	1.00−
	8	0.93	0.94	0.98
	9	0.93	0.94	0.97
12	1	1.01	1.01	1.04
	2	1.00+	1.02	1.06
	3	0.99	1.02	1.07
	4	0.97	1.00−	1.06
	5	0.95	0.98	1.04
	6	0.92	0.95	1.01
	7	0.91	0.93	0.99
	8	0.90	0.92	0.96
	9	0.90	0.91	0.95
	10	0.91	0.92	0.94

TABLE I (Continued)

RATIO OF THE EXPECTED VOLUME OF E_1 TO E_2

$n = 35$

p	q	0.90	0.95	0.99
2	1	1.00	1.00+	1.00+
3	1	1.00−	1.00+	1.00+
4	1	1.00−	1.00+	1.00+
	2	1.00−	1.00+	1.00+
5	1	1.00+	1.00+	1.00+
	2	1.00−	1.00+	1.00+
	3	1.00−	1.00+	1.01
6	1	1.00+	1.00+	1.00+
	2	1.00+	1.00+	1.01
	3	1.00−	1.00+	1.01
	4	1.00−	1.00+	1.01
7	1	1.00+	1.00+	1.01
	2	1.00−	1.00+	1.01
	3	1.00−	1.00+	1.01
	4	1.00−	1.00+	1.01
	5	1.00−	1.00−	1.01
8	1	1.00+	1.00+	1.01
	2	1.00+	1.00+	1.01
	3	1.00−	1.00+	1.01
	4	1.00−	1.00+	1.01
	5	0.99	1.00−	1.01
	6	0.99	1.00−	1.01
9	1	1.00+	1.00+	1.01
	2	1.00+	1.01	1.02
	3	1.00−	1.01	1.02
	4	0.99	1.01+	1.02
	5	0.99	1.00+	1.02
	6	0.99	1.00−	1.01
	7	0.99	0.99	1.01+
10	1	1.00+	1.00+	1.01
	2	1.00+	1.01	1.02
	3	1.00−	1.01	1.02
	4	0.99	1.00+	1.02
	5	0.99	1.00−	1.02
	6	0.98	0.99	1.02
	7	0.98	0.99	1.01
	8	0.98	0.99	1.00+
11	1	1.00+	1.00+	1.01
	2	1.00+	1.01	1.02
	3	1.00−	1.01	1.03
	4	0.99	1.00+	1.03
	5	0.99	0.99	1.02
	6	0.98	0.99	1.02
	7	0.98	0.99	1.01
	8	0.97	0.98	1.00+
	9	0.97	0.98	1.00−
12	1	1.00+	1.01	1.01
	2	1.00+	1.01	1.03
	3	1.00−	1.01	1.03
	4	0.99	1.00+	1.03
	5	0.98	1.00−	1.03
	6	0.97	0.98	1.02
	7	0.97	0.98	1.02
	8	0.96	0.98	1.01
	9	0.96	0.97	1.00−
	10	0.97	0.97	0.99

Second, if we fix p, q, and γ, and allow n to increase, then the ratio $I_{1,2}$ converges to 1. This result is not at all surprising since the pivotal quantities Δ and $\Delta(1 + r)^{-1}$ converge to one another in probability at an exponential rate as $n \to \infty$, regardless of the values of p, q, and γ.

Finally, if we fix n, q, and γ, and allow p to increase, then the ratio $I_{1,2}$ may increase or decrease depending on whether the initial value of $I_{1,2}$ in the series is greater or less than one. The actual pattern of movement of $I_{1,2}$ in p is probably a slowly undulating one, offering little practical guidance in the choice of procedure.

Recalling that values of $I_{1,2}$ greater than one favor procedure E_2, that values of $I_{1,2}$ less than one favor procedure E_1, and that a value of $I_{1,2}$ equal to one favors neither procedure, the patterns which we have noted in our table of $I_{1,2}$ suggest that the confidence region E_1 should be used if the requirements for probability of coverage are modest ($\gamma = 0.90$, or even 0.95), the number q of regression parameters is not much less than the dimension p of a single replication y of the model (1.1), and/or if N is of moderate size. However, it should be kept in mind that $I_{1,2}$ is a dimensionless quantity (a ratio of volumes), so that if a large saving in expected volume is of interest, the $I_{1,2}$ tells us little unless we also know the expected volume of one of the two confidence regions.

It should also be remarked that in our table of $I_{1,2}$, values very rarely are less than 0.88 or greater than 1.07. Thus, unless one is greatly concerned about keeping the expected volume of the region as low as possible, the choice between the regions E_1 and E_2 can be governed by computational convenience, by other aspects of the context of the given research problem, or by personal conviction.

REMARK. One advantage in using the conditional region E_2 is that its conditional probability of coverage given r is independent of r. Since r is a monotone function of the likelihood ratio test statistic for the goodness of fit of model (1.1), one can perform a preliminary test for the fit of the model without affecting the coverage probability of the confidence region for the parameters of the model (assuming the model is accepted by the likelihood ratio test). The expected volume of E_2 would, of course, be affected by such a two stage procedure. A similar two stage procedure based on r and E_1 could be constructed, but this would require new tables of $b^{(\gamma)}$. To our knowledge, no satisfactory criterion for comparing such two stage procedures has yet been proposed, so that balancing this advantage of region E_2 against a possibly smaller expected volume for E_1 must be left entirely to the individual.

5. An illustrative example

To illustrate the computation of the point estimators of β and Σ and the construction of the two confidence regions E_1 and E_2 for β, we make use of the growth curve data reported earlier by Potthoff and Roy [15]. In a study performed at the University of North Carolina Dental School, measurements were made of the distance (in mm.) from the center of the pituitary to the pteryomaxillary fissure for eleven girls and sixteen boys at ages 8, 10, 12, and 14 years. The resulting data for the boys is given in Table II below.

TABLE II

DISTANCE IN MM FROM CENTER OF PITUITARY TO PTERYOMAXILLARY FISSURE

	Age in Years			
Subject Age	8	10	12	14
1	26	25	29	31
2	21.5	22.5	23	26.5
3	23	22.5	24	27.5
4	25.5	27.5	26.5	27
5	20	23.5	22.5	26
6	24.5	25.5	27	28.5
7	22	22	24.5	26.5
8	24	21.5	24.5	25.5
9	23	20.5	31	26
10	27.5	28	31	31.5
11	23	23	23.5	25
12	21.5	23.5	24	28
13	17	24.5	26	29.5
14	22.5	25.5	25.5	26
15	23	24.5	26	30
16	22	21.5	23.5	25

In the present analysis we adopt a linear model for the growth curve; namely,

$$y_i = \beta_1 + \tfrac{1}{3}\beta_2(t_i - 11), \tag{5.1}$$

where y_i is the distance (in mm.) measured at time t_i with $i = 1, 2, 3, 4$. We have chosen to represent the model in terms of the orthogonal polynomials for the sake of computational convenience. In terms of the model (1.1),

$$X = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix}, \tag{5.2}$$

with $p = 4$ and $q = 3$. The sample size $N = 16$, so that $n = 15$. Computation yields the following:

$$\begin{gathered} \bar{y} = (22.88, 23.81, 25.72, 27.47), \\ S = \begin{pmatrix} 90.25 & 34.37 & 42.16 & 24.19 \\ 34.37 & 68.44 & 32.91 & 42.16 \\ 54.44 & 32.91 & 105.48 & 48.61 \\ 24.19 & 42.16 & 48.61 & 65.23 \end{pmatrix}, \end{gathered} \tag{5.3}$$

from which

$$\hat{\beta} = (25.00, 0.83),$$

(5.4)
$$\hat{\Sigma} = \begin{pmatrix} 5.78 & 2.02 & 2.59 & 1.50 \\ 2.02 & 4.40 & 2.10 & 2.65 \\ 2.59 & 2.10 & 6.61 & 3.04 \\ 1.50 & 2.65 & 3.04 & 4.08 \end{pmatrix}.$$

The 95 per cent confidence region of form E_1 for (β_1, β_2) is given by

$$\text{(5.5)} \quad \begin{aligned} \mathrm{E}_1 = \{(\beta_1, \beta_2)\colon & 0.34(\beta_1 - 25.00)^2 \\ & + 0.10(\beta_1 - 25.00)(\beta_2 - 0.83) + 7.27(\beta_2 - 0.83)^2 \leqq 0.767\}, \end{aligned}$$

where $b^{(.95)} = 0.767$ is obtained by linearly interpolating the values of $c^{(.95)}$ for $n = 14$ and $n = 16$, and then from the resulting c forming $b = c(1 - c)^{-1}$. The 95 per cent confidence region of form E_2 for (β_1, β_2) is given by

$$\text{(5.6)} \quad \begin{aligned} E_2 = \{(\beta_1, \beta_2)\colon & 0.34(\beta_1 - 25.00)^2 \\ & + 0.10(\beta_1 - 25.00)(\beta_2 - 0.83) + 7.27(\beta_2 - 0.83)^2 \leqq 0.611\}, \end{aligned}$$

since $r = 0.144$, $F_{2,14}^{(.95)} = 3.74$. Notice that the volume of E_2 is less than the volume of E_1 for this example. Although this result will not always occur if this particular example is replicated (since r is a random variable), the tables of $I_{1,2}$ described in Section 4 would lead us to expect the result we have obtained (since for both $n = 14$ and $n = 16$, with $p = 4$, $q = 2$, and $\gamma = 0.95$, the value of $I_{1,2}$ is 1.01).

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APPENDIX A

Distributional Results

A.1. Introduction

In this appendix we derive the distributions of $\hat{\beta}$ and $\hat{\Sigma}$ by means of a certain canonical distributional representation of these statistics. As a first step in obtaining this representation, note that $\hat{\beta}$ is invariant under the transformation $\tilde{y} = \bar{y}A$, $\tilde{S} = A'SA$, $\tilde{X} = XA$ for A nonsingular. Consequently, if we choose A so that $A'\Sigma A = I$ (that is, $A = \Sigma^{-1/2}$), then $\tilde{y} \sim N(\beta\tilde{X}, N^{-1}I)$, $\tilde{S} \sim W(I; p, n)$, $\tilde{y}$ and $\tilde{S}$ are independently distributed. In terms of $\tilde{y}$, $\tilde{S}$, and $\tilde{X}$,

$$\text{(A.1)} \quad \begin{aligned} \hat{\beta} &= \tilde{y}\tilde{S}^{-1}\tilde{X}'(\tilde{X}\tilde{S}^{-1}\tilde{X}')^{-1}, \\ N\hat{\Sigma} &= \Sigma^{1/2}[\tilde{S} + N(\tilde{y} - \hat{\beta}\tilde{X})'(\tilde{y} - \hat{\beta}\tilde{X})]\Sigma^{1/2}. \end{aligned}$$

Further simplification is possible. There exists a nonsingular $q \times q$ matrix T and a $p \times p$ orthogonal matrix Γ such that

$$\text{(A.2)} \quad \tilde{X} = T(I_q, 0)\Gamma',$$

(MacDuffee p. 77 [12]), where I_q is the $q \times q$ identity matrix. This has the effect of reducing the dimensionality of the space as follows. Transform from $\tilde{y}, \tilde{S}$ to

$$z = \sqrt{N}\tilde{y}\Gamma, \qquad V = \Gamma'\tilde{S}\Gamma. \tag{A.3}$$

Let $z = (\dot{z}, \ddot{z})$, where $\dot{z}$ consists of the first q components of z, and then partition V as

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \qquad V_{11}: q \times q, \qquad V_{22}: (p-q) \times (p-q). \tag{A.4}$$

It is easily verified that $\dot{z}$, $\ddot{z}$, and V are stochastically independent, that $\dot{z} \sim N(\sqrt{N}\beta T, I_q)$, that $\ddot{z} \sim N(0, I_{p-q})$, and that $V \sim W(I; p, n)$. Furthermore,

$$\begin{aligned} b &\equiv \sqrt{N}\,\hat{\beta}T = \dot{z} - \ddot{z}V_{22}^{-1}V_{21}, \\ \tilde{\Sigma} &\equiv \Gamma'\Sigma^{-1/2}\hat{\Sigma}\Sigma^{-1/2}\Gamma = V + \begin{pmatrix} V_{12}V_{22}^{-1}\ddot{z}' \\ \ddot{z}' \end{pmatrix} (\ddot{z}V_{22}^{-1}V_{21}, \ddot{z}), \end{aligned} \tag{A.5}$$

where

$$\tilde{\Sigma} = \begin{pmatrix} \tilde{\Sigma}_{11} & \tilde{\Sigma}_{12} \\ \tilde{\Sigma}_{21} & \tilde{\Sigma}_{22} \end{pmatrix} \tag{A.6}$$

is partitioned in the manner of V. Let $\mu = \sqrt{N}\,\beta T$. The following lemma is known (and easily verified).

LEMMA A.1. *If V has a $W(I; p, n)$ distribution, then $M = V_{11} - V_{12}V_{22}^{-1}V_{21} \sim W(I_q; q, n - p + q)$, $V_{22} \sim W(I_{p-q}; p - q, n)$, and the $q(p - q)$ elements of $L = V_{22}^{-1/2}V_{21}$ are independently distributed as $N(0, 1)$. Furthermore, M, V_{22}, and L are mutually stochastically independent.*

A.2. The distribution of $\hat{\beta}$

Since $\sqrt{N}\hat{\beta} = bT^{-1}$, to obtain the distribution of $\hat{\beta}$ it is sufficient to find the distribution of b. From (A.5) and Lemma A.1, we see that

$$b = \dot{z} - wL \equiv \dot{z} - \ddot{z}V_{22}^{-1/2}L, \tag{A.7}$$

where w, $\dot{z}$, and L are independent. Again from Lemma A.1, it follows that the conditional distribution of b given w is $N(\mu, (1 + ww')I_q)$. Let $r = ww'$ and note that $r = \ddot{z}V_{22}^{-1}\ddot{z}' = N(\bar{y} - \beta X)S^{-1}(\bar{y} - \beta X)'$. Since $\ddot{z} \sim N(0, I_{p-q})$ and $V_{22} \sim W(I; p - q, n)$ and $\ddot{z}$ and V_{22} are independent, it can be shown in a straightforward manner using Hsu's theorem (Anderson [1], p. 319) that r has the density

$$p(r) = \frac{r^{(p-q)/2-1}}{B\big(\frac{1}{2}(p-q), \frac{1}{2}(n-p+q+1)\big)(1+r)^{(n+1)/2}}. \tag{A.8}$$

The distribution of b given w is the same as that of b given r (since the former conditional distribution depends upon w only through $r = ww'$), namely $N\big(\mu, (1+r)I_q\big)$, so that

$$p(b, r) = p(b|r)p(r) = \frac{r^{(p-q)/2-1} \exp\left\{-\frac{1}{2}\frac{(b-\mu)(b-\mu)'}{1+r}\right\}}{(2\pi)^{q/2} B\big(\frac{1}{2}(p-q), \frac{1}{2}(n-p+q+1)\big)(1+r)^{(n+q+1)/2}}. \tag{A.9}$$

Transforming from b to $\hat{\beta} = N^{-1/2} bT^{-1}$ and from r to $g = r/(1+r)$, noting that $TT' = X\Sigma^{-1}X'$, that $\mu = \sqrt{N}\beta T$, and integrating over g, where $0 \leqq g \leqq 1$, yields (2.12). The expansion of the integral form (2.12) of $p(\hat{\beta})$ in terms of the confluent hypergeometric function ${}_1F_1\big(\frac{1}{2}(p-q), \frac{1}{2}(n+q+1); Q(\hat{\beta})\big)$ (equation (2.13)) is well known (for example, see Erdélyi [4], p. 255). Finally by grouping terms appropriately in the infinite sum representation of ${}_1F_1$ in the representation (2.13) for $p(\hat{\beta})$, we obtain (2.11) and the result of Theorem 2.5.

REMARK. The representations (2.12) and (2.13) for $p(\hat{\beta})$ were obtained by a slightly more complicated proof in Gleser and Olkin [7]. The representation (2.11) is new. As demonstrated in Section 2, the new representation is useful in finding approximations to $p(\hat{\beta})$ for moderate values of the sample size N.

As a byproduct of the above derivations and from Lemma A.1, we have the following result which is useful in Sections 3 and 4.

LEMMA A.2. *The distribution of* $(n-p+1)N(\hat{\beta}-\beta)XS^{-1}X'(\hat{\beta}-\beta)'/q(1+r)$ *given* $r = N(\bar{y} - \hat{\beta}X)S^{-1}(\bar{y} - \hat{\beta}X)'$ *is* $F_{q,n-p+1}$. *Further,* r *and* $XS^{-1}X'$ *are stochastically independent,* $(1+r)^{-1}$ *has a beta distribution with parameters* $\frac{1}{2}(n-p+q+1)$ *and* $\frac{1}{2}(p-q)$, *and* $(XS^{-1}X')^{-1} \sim W\big((X\Sigma^{-1}X')^{-1}; q, n-p+q\big)$.

PROOF. From (A.5), $\sqrt{N}(\hat{\beta} - \beta) = (b-\mu)T^{-1}$. Since, as shown above, the conditional distribution of b given r is $N\big(\mu, (1+r)I_q\big)$, since from Lemma A.1, M is independent of $\dot{z}$, $\ddot{z}$, L, and V_{22} (thus of $\hat{\beta}$ and r), and since

$$\Delta = N(\hat{\beta}-\beta)(XS^{-1}X')(\hat{\beta}-\beta)' = (b-\mu)M^{-1}(b-\mu)', \tag{A.10}$$

it follows that $(n-p+1)\Delta/q(1+r)$ given r has Snedecor's F distribution with q and $n-p+1$ degrees of freedom (see Anderson [1], Theorem 5.2.2). That $(1+r)^{-1}$ has the beta distribution with parameters $\frac{1}{2}(n-p+q+1)$ and $\frac{1}{2}(p-q)$ follows from (A.8). Finally

$$(XS^{-1}X')^{-1} = (T')^{-1}MT^{-1}, \tag{A.11}$$

and thus $(XS^{-1}X')^{-1} \sim W\big((TT')^{-1}; q, n-p+q\big)$. Since $(TT')^{-1} = (X\Sigma^{-1}X')^{-1}$, the proof of the lemma is completed.

A.3. The distribution of $\hat{\Sigma}$

From Equation (A.5),

$$\tilde{\Sigma} = \begin{pmatrix} \tilde{\Sigma}_{11} & \tilde{\Sigma}_{12} \\ \tilde{\Sigma}_{21} & \tilde{\Sigma}_{22} \end{pmatrix} = V + \begin{pmatrix} V_{12} V_{22}^{-1} \\ I \end{pmatrix} \ddot{z}' \ddot{z} (V_{22}^{-1} V_{21}, I), \tag{A.12}$$

where $V \sim W(I; p, n)$ is independently distributed of $\ddot{z} \sim N(0, I_{p-q})$. Let

$$\begin{aligned} M &= V_{11} - V_{12} V_{22}^{-1} V_{21}, \\ \tilde{\Sigma}_{12} &= V_{12} V_{22}^{-1} (V_{22} + \ddot{z}'\ddot{z}), \qquad \tilde{\Sigma}_{22} = V_{22} + \ddot{z}'\ddot{z}, \end{aligned} \tag{A.13}$$

be a transformation from (V_{11}, V_{12}, V_{22}) to $(M, \tilde{\Sigma}_{12}, \tilde{\Sigma}_{22})$. Noting that

$$V_{12} V_{22}^{-1} V_{21} = \tilde{\Sigma}_{12} \tilde{\Sigma}_{22}^{-1} (\tilde{\Sigma}_{22} - \ddot{z}'\ddot{z}) \tilde{\Sigma}_{22}^{-1} \tilde{\Sigma}'_{12} \tag{A.14}$$

and that $\tilde{\Sigma}_{12} = \tilde{\Sigma}'_{21}$, it follows by a direct computation that

$$\begin{aligned} p(\ddot{z}, M, \tilde{\Sigma}_{12}, \tilde{\Sigma}_{22}) &= \frac{C(p, n)}{(2\pi)^{(p-q)/2}} |\tilde{\Sigma}_{22}|^{-q} |\tilde{\Sigma}_{22} - \ddot{z}'\ddot{z}|^{(n+2q-p-1)/2} |M|^{(n-p-1)/2} \\ &\quad \cdot \exp\{-\tfrac{1}{2}[\operatorname{tr} \tilde{\Sigma}_{22} + \operatorname{tr} M + \operatorname{tr} \tilde{\Sigma}_{12} \tilde{\Sigma}_{22}^{-1} (\tilde{\Sigma}_{22} - \ddot{z}'\ddot{z}) \tilde{\Sigma}_{22}^{-1} \tilde{\Sigma}_{21}]\}, \end{aligned} \tag{A.15}$$

where $M > 0$, $\tilde{\Sigma}_{22} - \ddot{z}'\ddot{z} > 0$,

$$C^{-1}(p, n) = 2^{np/2} \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\big(\tfrac{1}{2}(n - i + 1)\big), \tag{A.16}$$

and the elements of $\tilde{\Sigma}_{12}$ and $\ddot{z}$ are unrestricted.

Now let

$$v = \ddot{z} \tilde{\Sigma}_{22}^{-1/2}, \qquad \tilde{\Sigma}_{11} = M + \tilde{\Sigma}_{12} \tilde{\Sigma}_{22}^{-1} \tilde{\Sigma}_{21} \tag{A.17}$$

be a transformation from $(\ddot{z}, M)$ to $(v, \tilde{\Sigma}_{11})$. Then

$$\begin{aligned} p(\tilde{\Sigma}, v) &= \frac{C(p, n)}{(2\pi)^{(p-q)/2}} |\tilde{\Sigma}_{22}|^{(n-p-1)/2} |\tilde{\Sigma}_{11} - \tilde{\Sigma}_{12} \tilde{\Sigma}_{22}^{-1} \tilde{\Sigma}_{21}|^{(n-p-1)/2} \\ &\quad \cdot |\tilde{\Sigma}_{22}|^{1/2} (1 - vv')^{(n+2q-p-1)/2} \\ &\quad \cdot \exp\{-\tfrac{1}{2}[\operatorname{tr} \tilde{\Sigma}_{11} + \operatorname{tr} \tilde{\Sigma}_{22} - \operatorname{tr} v\, \tilde{\Sigma}_{22}^{-1/2} \tilde{\Sigma}_{21} \tilde{\Sigma}_{12} \tilde{\Sigma}_{22}^{-1/2}\, v']\} \\ &= [C(p, n) |\tilde{\Sigma}|^{(n-p-1)/2} \exp\{-\tfrac{1}{2} \operatorname{tr} \tilde{\Sigma}\}] \\ &\quad \cdot \left[\frac{|\tilde{\Sigma}_{22}|^{1/2} (1 - vv')^{(n+2q-p-1)/2}}{(2\pi)^{(p-q)/2}} \exp\{+\tfrac{1}{2} v \Xi(\tilde{\Sigma}) v'\} \right], \end{aligned} \tag{A.18}$$

where $\Xi(H)$ is, for any positive definite

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \tag{A.19}$$

defined by $\Xi(H) = H_{22}^{-1/2}H_{21}H_{12}H_{22}^{-1/2}$, and where the range of definition is $\tilde{\Sigma} > 0$, $vv' \leqq 1$. We make use of the invariance of vv' under the transformation $v \to v\Gamma$, Γ orthogonal, to reduce the expression still further. Let U be the orthogonal matrix such that

$$U\Xi(\tilde{\Sigma})U' = \operatorname{diag}(\nu_1, \cdots, \nu_{p-q}) \equiv D_\nu, \tag{A.20}$$

where the values of ν_i are the characteristic roots of $\Xi(\tilde{\Sigma})$. Hence, letting $s = vU'$ where $s\colon 1 \times (p - q)$, we obtain

$$\begin{aligned} p(\tilde{\Sigma}) = {} & [C(p, n)|\tilde{\Sigma}|^{(n-p-1)/2} \exp\{-\tfrac{1}{2}\operatorname{tr}\tilde{\Sigma}\}] \\ & \cdot\left[\frac{|\tilde{\Sigma}_{22}|^{1/2}}{(2\pi)^{(p-q)/2}}\int_{ss'\leqq 1}(1 - ss')^{(n+2q-p-1)/2}\exp\{\tfrac{1}{2}sD_\nu s'\}\,ds\right]. \end{aligned} \tag{A.21}$$

An alternative expression for $p(\tilde{\Sigma})$ may be obtained by noting that

(A.22)

$$\begin{aligned} & \int_{ss'\leqq 1}(1 - ss')^{(n+2q-p-1)/2}\exp\{\tfrac{1}{2}sD_\nu s'\} \\ & = 2^{q-p}\Gamma[\tfrac{1}{2}(n + 2q - 2 + 1)] \\ & \quad \cdot \sum_{j_1,\cdots,j_{p-q}=0}^{\infty}\left[\Gamma\left(\sum_{i=1}^{p-q} j_i + \tfrac{1}{2}(n + q + 1)\right)\right]^{-1}\prod_{i=1}^{p-q}\left(\frac{\nu_i}{2}\right)^{j_i}\frac{\Gamma(j_i + \frac{1}{2})}{j_i!}. \end{aligned}$$

When $p - q \geqq q$, some of the ν_i are 0 with probability one, so that the expressions for $p(\tilde{\Sigma})$ can be somewhat simplified.

The distribution of $\hat{\Sigma}$ may be determined by making the transformation from $\tilde{\Sigma}$ to $\hat{\Sigma} = N^{-1}\Sigma^{1/2}\Gamma\tilde{\Sigma}\Gamma'\Sigma^{1/2}$.

APPENDIX B

Tables for Applying Confidence Region E_1

Table III gives values of $c^{(\gamma)}$ (see equation (3.7)) needed in order to construct 100γ per cent confidence regions of the form E_1 (see (3.6)). The present tables are calculated for $n = 10(2)30(5)35$, $p = 2(1)\frac{1}{2}n$, $q = 1(1)\,p - 2$, and $\gamma = 0.90$, 0.95, 0.975, 0.99. These values of n, p, q, and γ have been chosen as illustrative, but not exhaustive, examples of situations met in practice. For example, it is usually desirable for n to be somewhat larger than p so that sufficient degrees of freedom are available to accurately estimate Σ. For the distribution of $\hat{\Sigma}$ to be nonsingular, we must have $n \geqq p + 1$; the assumption $n \geqq 2p$ provides a comfortable number of degrees of freedom for $\hat{\Sigma}$. When n is large (say, over 40), this assumption is unnecessarily strict and can be replaced by the condition that $n - p - 1$ be of a reasonable magnitude.

The values of n given are not uncommon in practice. Values of n less than 10 are rarely practical (unless $p = 2$) for reasons already indicated. If n is larger than 35 or 40, large sample approximations may be appropriate (unless p is too large). Simple linear or quadratic interpolation in the tables should give enough accuracy in most situations for the application of the confidence region E_1 when n is odd, $11 \leqq n \leqq 34$.

The coverage probabilities γ chosen for Table III are those customarily given in standard tables for upper tail probabilities. Finally, the values of q which have been chosen reflect the fact that (at least in the context of growth curves) the most desirable models are those which require estimation of the fewest parameters.

Starting with equation (3.7), the table was constructed as follows. First the expansion

$$(B.1) \qquad (1 - gh)^{-(d_1+d_2)} = \sum_{j=0}^{\infty} \frac{\Gamma(d_1 + d_2 + j)}{\Gamma(d_1 + d_2)} \frac{(gh)^j}{j!}$$

enables us to expand the double integral in (3.7) in the following infinite series:

$$(B.2) \qquad \gamma = \sum_{j=0}^{\infty} \frac{\Gamma(d_1 + d_2 + j)}{\Gamma(d_1 + d_2)j!} \cdot \frac{\int_0^1 g^{a_1+j-1}(1 - g)^{a_2-1}\, dg \int_0^{c(\gamma)} h^{d_1+j-1}}{B(a_1, a_2 - d_1)B(d_1, d_2)} (1 - h)^{d_2-1}\, dh$$

$$= \sum_{j=0}^{\infty} c_j I_{c(\gamma)}(d_1 + j, d_2),$$

where the values of c_j with $j = 0, 1, \cdots$ have already been defined in Theorem 2.1; where for constants $f_1, f_2 > 0$, $0 \leqq z \leqq 1$,

$$(B.3) \qquad I_z(f_1, f_2) = \int_0^z \frac{w^{f_1-1}(1 - w)^{f_2-1}\, dw}{B(f_1, f_2)},$$

and where $a_1 = \frac{1}{2}(p - q)$, $a_2 = \frac{1}{2}(n + 2q - p + 1)$, $d_1 = \frac{1}{2}q$, $d_2 = \frac{1}{2}(n - p + 1)$. The interchange of summation and integration used to obtain equation (B.2) is readily justified from Fubini's theorem by noting that $c_j \geqq 0$, all j, $\Sigma_{j=0}^{\infty} c_j = 1$, and $0 \leqq I_z(f_1, f_2) \leqq 1$. These facts also support the following inequality:

$$(B.4) \qquad \sum_{j=0}^{M} c_j I_z(d_1 + j, d_2) \leqq \sum_{j=0}^{\infty} c_j I_z(d_1 + j, d_2) \leqq \sum_{j=0}^{M} c_j I_z(d_1 + j, d_2) + \left(1 - \sum_{j=M+1}^{\infty} c_j\right),$$

which holds for all nonnegative integers M. This inequality permits evaluation of the error involved in truncating the infinite sum $\gamma(z) \equiv \Sigma_{j=0}^{\infty} c_j I_z(d_1 + j, d_2)$

after M terms have been computed. Using this inequality, a grid of values for the infinite sum $\gamma(z)$ was computed (to five place accuracy) for each n, p, q chosen, and for values of z ranging by jumps of 0.02 from 0.50 to 0.98. After such a grid was formed, a value of γ was chosen ($\gamma = 0.90, 0.95, 0.975, 0.99$) and the grid was searched for that value z^* of z which yielded a calculated value of $\gamma(z)$ closest to γ. Since $\gamma(z)$ is monotonic increasing in z, z was allowed to move in increments of 0.001 down or up from z^* depending on whether $\gamma(z^*)$ was greater or less than γ. This movement was terminated once the size of $\gamma(z) - \gamma$ reversed from that of $\gamma(z^*) - \gamma$. A similar incremental movement in steps of 0.0001 from this new value of z was terminated when once again $\gamma(z) - \gamma$ reversed sign. The value of z computed in this entire series for which $|\gamma(z) - \gamma|$ was a minimum was then chosen to be $c^{(\gamma)}$. The resulting values of $c^{(\gamma)}$ are accurate to within $\pm 5 \times 10^{-5}$—assuming that we want $c^{(\gamma)}$ to give us coverage γ up to an error of $\pm 5 \times 10^{-6}$ and ignoring errors in the calculation of the individual terms $c_j I_z(d_1 + j, d_2)$. The value of $c^{(\gamma)}$ was checked by evaluating (B.2) within a six place accuracy.

The computations are simplified by noting the recursion

$$c_{j+1} = c_j \frac{(q + 2j)(p - q + 2j)}{(n + q + 1 + 2j)(2 + 2j)}. \tag{B.5}$$

Users of Table III should note that $n = N - 1$, where N is the sample size, and that for the value of γ selected, E_1 is to be used with $b^{(\gamma)} = c^{(\gamma)}/(1 - c^{(\gamma)})$.

TABLE III

TABLES OF CRITICAL VALUES $c^{(\gamma)}$ FOR CONFIDENCE REGION E_1

$n = 10$						$n = 12$					
p	q	0.90	0.95	0.975	0.99	p	q	0.90	0.95	0.975	0.99
2	1	29496	39059	47606	57308	2	1	24342	32672	40351	49397
3	1	35507	46173	55314	65209	3	1	28535	37844	46197	55735
4	1	43063	54628	63985	73502	4	1	33711	44003	52913	62682
	2	54450	63812	71169	78562		2	44962	53881	61302	69251
5	1	52419	64336	73269	81642	5	1	40093	51253	60466	70060
	2	63314	72388	79077	85361		2	51696	60903	68255	75786
	3	67957	75669	81371	86783		3	57359	65398	71765	78283
						6	1	47901	59595	68679	77545
							2	59364	68491	75411	82124
							3	64550	72346	78246	84002
							4	67525	74504	79803	85010

TABLE III (Continued)

$n = 14$							$n = 16$					
p	q	0.90	0.95	0.975	0.99		p	q	0.90	0.95	0.975	0.99
2	1	20693	28031	34938	43276		2	1	17984	24524	30771	38451
3	1	23765	31921	39457	48357		3	1	20322	27541	34346	42574
4	1	27477	36511	44656	54013		4	1	23097	31059	38435	47178
	2	38079	46318	53429	61362			2	32930	40479	47159	54827
5	1	31983	41909	50586	60213		5	1	26406	35162	43098	52279
	2	43220	51937	59250	67157			2	36931	44993	51982	59824
	3	49251	57076	63548	70497			3	42981	50394	56708	63708
6	1	37456	48211	57247	66846		6	1	30367	39938	48383	57856
	2	49136	58174	65494	73116			2	41527	50044	57246	65109
	3	55061	63005	69373	75981			3	47673	55363	61765	68688
	4	58707	65907	71661	77642			4	51664	58735	64581	70888
7	1	44077	55460	64555	73717		7	1	35118	45478	54313	63865
	2	55857	64953	72011	79036			2	46779	55638	63904	70583
	3	61453	69278	75315	81334			3	52891	60733	67084	73756
	4	64764	71780	77196	82622			4	56735	63864	69611	75644
	5	66952	73405	78398	83428			5	59396	66001	71320	76919
							8	1	40806	51832	60851	70173
								2	52726	61736	68862	76107
								3	58629	66442	72567	78787
								4	62215	69238	74736	80331
								5	64635	71100	76165	81343
								6	66372	72419	77164	82032

TABLE III (Continued)

n = 18					
p	q	0.90	0.95	0.975	0.99
2	1	15893	21779	27464	34542
3	1	17729	24180	30347	37928
4	1	19873	26949	33626	41713
	2	28957	35872	42099	42092
5	1	22389	30141	37341	45904
	2	32137	39541	46107	53665
	3	38050	45004	51055	57928
6	1	25357	33834	41557	50542
	2	35768	43650	50511	58250
	3	41871	49158	55388	62329
	4	46002	52797	58552	64931
7	1	28870	38103	46317	55622
	2	39906	48222	55304	63096
	3	46134	53693	60025	66925
	4	50242	57215	63011	69303
	5	53177	59690	65081	70926
8	1	33035	43014	51636	61092
	2	44608	53279	60469	68160
	3	50860	58595	64919	71642
	4	54864	61918	67653	73730
	5	57672	64218	69526	75158
	6	59751	65899	70879	76170
9	1	37974	48629	57509	66882
	2	49917	58805	65947	73341
	3	56047	63818	69996	76366
	4	59856	66866	72420	78150
	5	62468	68928	74044	79335
	6	64371	70413	75200	80166
	7	65819	71533	76069	80787

n = 20					
p	q	0.90	0.95	0.975	0.99
2	1	14238	19590	24806	31370
3	1	15718	21546	27182	34208
4	1	17420	23771	29850	37339
	2	25820	32179	37986	44899
5	1	19392	26317	32866	40826
	2	28397	35203	41343	48555
	3	34083	40582	46326	52965
6	1	21683	29231	36268	44676
	2	31317	38574	45030	52489
	3	37242	44086	50055	56855
	4	41388	47850	53423	59728
7	1	24360	32576	40103	48919
	2	34629	42334	49074	56719
	3	40755	47915	54061	60944
	4	44952	51647	57337	63668
	5	48042	54354	59688	65609
8	1	27497	36413	44407	53554
	2	38379	46504	53469	61201
	3	44651	52078	58335	65202
	4	48852	55729	61475	67753
	5	51888	58325	63678	69516
	6	54195	60276	65321	70819
9	1	31181	40800	49208	58558
	2	42619	51103	58209	65902
	3	48956	56577	62861	69603
	4	53093	60073	65791	71909
	5	56027	62514	67810	73472
	6	58226	64327	69301	74625
	7	59333	65718	70435	75488
10	1	35513	45799	54517	63894
	2	47388	56131	63255	70746
	3	53677	61383	67582	74062
	4	57669	64648	70237	76071
	5	60447	66893	72046	77434
	6	62493	68526	73347	78398
	7	64066	69771	74337	79131
	8	65306	70743	75098	79680

TABLE III (Continued)

$n = 22$

p	q	0.90	0.95	0.975	0.99
2	1	12890	17791	22597	28693
3	1	14103	19405	24570	31066
4	1	15488	21234	26790	33713
	2	23275	29142	34552	41068
5	1	17072	23306	29280	36648
	2	25400	31666	37390	44208
	3	30840	36915	42350	48720
6	1	18888	25652	32060	39859
	2	27795	34480	40522	47633
	3	33480	39886	45555	52119
	4	37573	43691	49042	55188
7	1	20983	28325	35187	43415
	2	30489	37598	43941	51299
	3	36409	43139	49022	55742
	4	40595	46967	52475	58720
	5	43749	49806	55007	60878
8	1	23410	31369	38693	47319
	2	33522	41052	47666	55207
	3	39653	46686	52747	59565
	4	43901	50500	56127	62416
	5	47055	53287	58568	64449
	6	49497	55418	60416	65965
9	1	26227	34834	42605	51561
	2	36941	44875	51720	59373
	3	43235	50533	56717	63551
	4	47504	54283	59975	66229
	5	50626	56987	62302	68131
	6	53011	59025	64031	69512
	7	54899	60624	65382	70587
10	1	29510	38781	46964	56164
	2	40787	49081	56091	63754
	3	47177	54680	60915	67665
	4	51417	58317	64009	70148
	5	54459	60888	66170	71859
	6	56747	62808	67767	73108
	7	58553	64293	68993	74055
	8	59999	65481	69969	74807
11	1	33342	43262	51787	61098
	2	45095	53674	60748	68287
	3	51497	59122	65320	71876
	4	55635	62572	68180	74099
	5	58550	64967	70141	75602
	6	60722	66736	71579	76697
	7	62402	68090	72672	77521
	8	63738	69158	73525	78154
	9	64830	70027	74219	78669

$n = 24$

p	q	0.90	0.95	0.975	0.99
2	1	11775	16293	20748	26436
3	1	12791	17656	22428	28483
4	1	13937	19181	24295	30734
	2	21182	26623	31685	37842
5	1	15231	20889	26363	33193
	2	22965	28764	34119	40579
	3	28146	33836	38978	45074
6	1	16706	22818	28682	35927
	2	24953	31125	36772	43512
	3	30385	36387	41764	48083
	4	34377	40162	45278	51227
7	1	18387	24996	31272	38938
	2	27177	33738	39680	46686
	3	32852	39165	44764	51266
	4	36965	43008	48301	54397
	5	40122	45909	50937	56691
8	1	20308	27450	34150	42215
	2	29673	36637	42870	50130
	3	35575	42193	47996	54645
	4	39787	46070	51513	57705
	5	42983	48967	54115	59942
	6	45502	51222	56117	61636
9	1	22518	30234	37367	45815
	2	32466	39830	46330	53783
	3	38580	45489	51466	58224
	4	42864	49364	54928	61167
	5	46071	52223	57453	63297
	6	48572	54426	59380	64900
	7	50583	56181	60906	66163
10	1	25066	33387	40952	49744
	2	35594	43345	50073	57649
	3	41889	49060	55172	61974
	4	46209	52894	58534	64766
	5	49396	55680	60951	66757
	6	51853	57803	62776	68240
	7	53813	59484	64215	69413
	8	55408	60840	65364	70333

TABLE III (Continued)

$n = 24$ (*continued*)

p	q	0.90	0.95	0.975	0.99
11	1	28009	36953	44922	53975
	2	29100	47207	54116	61738
	3	45519	52906	59092	65850
	4	49831	56647	62305	68452
	5	52965	59331	64592	70297
	6	55348	61347	66288	71641
	7	57227	62923	67609	72682
	8	58750	64194	68670	73521
	9	60004	65230	69526	74181
12	1	31420	40990	49313	58528
	2	43017	51427	58439	66006
	3	49486	57020	63204	69816
	4	53737	60619	66229	72206
	5	56769	63151	68336	73856
	6	59048	65036	69891	75062
	7	60824	66491	71083	75978
	8	62250	67651	72029	76700
	9	63420	68599	72799	77286
	10	64393	69381	73427	77758

$n = 26$

p	q	0.90	0.95	0.975	0.99
2	1	10836	15026	19178	24506
3	1	11698	16188	20617	26272
4	1	12661	17478	22205	28205
	2	19425	24489	29232	35041
5	1	13743	18918	23967	30331
	2	20939	26321	31330	37422
	3	25870	31208	36066	41874
6	1	14959	20525	25917	32661
	2	22617	28336	33621	39998
	3	27786	33410	38494	44522
	4	31664	37135	42015	47745
7	1	16331	22320	28072	35192
	2	24483	30557	36123	42778
	3	29892	35811	41117	47356
	4	33896	39613	44672	50560
	5	37028	42546	47387	52988
8	1	17888	24340	30479	37997
	2	26557	32996	38839	45745
	3	32206	38422	43944	50372
	4	36328	42289	47516	53542
	5	39517	45236	50213	55914
	6	42071	47571	52330	57759
9	1	19661	26613	33156	41070
	2	28868	35684	41799	48936
	3	34747	41253	46969	53544
	4	38975	45168	50546	56679
	5	42202	48110	53200	58970
	6	44764	50419	55265	60741
	7	46851	52282	56922	62151
10	1	21679	29165	36114	44389
	2	31452	38651	45029	52378
	3	37538	44323	50211	56893
	4	41847	48251	53749	59937
	5	45098	51172	56335	62155
	6	47648	53436	58344	63829
	7	49709	55247	59931	65155
	8	51412	56736	61229	66235

TABLE III (Continued)

n = 26 (*continued*)

p	q	0.90	0.95	0.975	0.99
11	1	23994	32043	39405	48021
	2	34331	41902	48512	56006
	3	40604	47646	53674	60416
	4	44963	51550	57131	63326
	5	48207	54415	59641	65423
	6	50726	56613	61550	66998
	7	52742	58356	63051	68226
	8	54396	59777	64271	69221
	9	55779	60960	65283	70044
12	1	26652	35289	43049	51951
	2	37538	45460	52257	59818
	3	43961	51225	57347	64082
	4	48333	55062	60681	66825
	5	51538	57837	63069	68773
	6	53998	59944	64864	70220
	7	55951	61602	66271	71352
	8	57538	62941	67399	72250
	9	58856	64046	68328	72986
	10	59967	64972	69101	73599
13	1	29708	38941	47059	56159
	2	41112	49345	56274	63824
	3	47623	55053	61206	67845
	4	51958	58774	64371	70384
	5	55090	61429	66614	72175
	6	57462	63416	68274	73481
	7	59323	64962	69556	74481
	8	60830	66209	70591	75295
	9	62069	67226	71427	75939
	10	63109	68077	72126	76481
	11	63990	68793	72711	76928

n = 28

p	q	0.90	0.95	0.975	0.99
2	1	10036	13943	17829	22841
3	1	10775	14942	19071	24371
4	1	11596	16047	20439	26045
	2	17936	22671	27130	32630
5	1	12510	17272	21947	27882
	2	19234	24251	28950	34708
	3	23932	28955	33558	39102
6	1	13530	18630	23606	29884
	2	20667	25985	30937	36962
	3	25588	30873	35688	41446
	4	29339	34521	39178	44692
7	1	14672	20140	25442	32078
	2	22250	27886	33098	39394
	3	27399	32955	37981	43947
	4	31282	36697	41531	47210
	5	34362	39620	44271	49702
8	1	15956	21824	27469	34474
	2	24002	29972	35451	42010
	3	29383	35218	40455	46618
	4	33393	39042	44047	49880
	5	36542	42001	46795	52346
	6	39100	44377	48985	54293
9	1	17400	23697	29700	37067
	2	25944	32263	38010	44825
	3	31556	37673	43117	49463
	4	35682	41562	46726	52688
	5	38892	44543	49466	55115
	6	41476	46917	51629	57014
	7	43609	48860	53391	58556
10	1	19037	25802	32184	39926
	2	28095	34769	40775	47089
	3	33937	40333	45968	52474
	4	38168	44270	49581	55649
	5	41422	47254	52288	58008
	6	44019	49611	54411	59844
	7	46146	51522	56122	61317
	8	47922	53108	57534	62526

TABLE III (Continued)

$n = 28$ (*continued*)

p	q	0.90	0.95	0.975	0.99
11	1	20894	28160	34934	43045
	2	30484	37520	43775	51005
	3	36540	43204	49010	55627
	4	40862	47173	52607	58750
	5	44141	50135	55256	61012
	6	46735	52456	57319	62767
	7	48843	54325	58971	64165
	8	50591	55865	60325	65306
	9	52066	57155	61454	66249
12	1	23004	30795	37960	46398
	2	33145	40542	47033	54433
	3	39388	46303	52251	58935
	4	43773	50265	55788	61950
	5	47061	53191	58372	64126
	6	49631	55455	60353	65777
	7	51702	57263	61928	67087
	8	53407	58741	63207	68142
	9	54839	59977	64274	69021
	10	56057	61022	65172	69757
13	1	25414	33758	41309	50040
	2	36096	43837	50527	58026
	3	42501	49641	55694	62396
	4	46915	53556	59131	65262
	5	50180	56411	61610	67307
	6	52706	58598	63495	68851
	7	54722	60328	64978	70058
	8	56370	61732	66174	71025
	9	57746	62900	67167	71828
	10	58909	63881	67997	72495
	11	59905	64718	68701	73058
14	1	28170	37084	44997	53963
	2	39370	47423	54260	61780
	3	45892	53214	59325	65975
	4	50296	57041	62620	68658
	5	53504	59791	64963	70547
	6	55958	61874	66727	71960
	7	57902	63513	68110	73068
	8	59475	64828	69209	73936
	9	60779	65914	70115	74651
	10	61878	66825	70873	75249
	11	62814	67598	71515	75752
	12	63621	68259	72059	76171

$n = 30$

p	q	0.90	0.95	0.975	0.99
2	1	09346	13004	16655	21382
3	1	09987	13875	17741	22729
4	1	10696	14834	18935	24204
	2	16656	21100	25305	30521
5	1	11476	15885	20234	25791
	2	17784	22481	26906	32367
	3	22257	26995	31361	36649
6	1	12344	17045	21663	27527
	2	19020	23986	28641	34350
	3	23703	28683	33247	38746
	4	27324	32238	36681	41977
7	1	13308	18329	23234	29426
	2	20376	25627	30518	36477
	3	25277	30505	35270	40971
	4	29030	34163	38779	44246
	5	32046	37060	41528	46784
8	1	14382	19750	24960	31490
	2	21871	27424	32562	38776
	3	26992	32478	37444	43345
	4	30876	36232	41017	46645
	5	33969	39178	43790	49178
	6	36508	41568	46020	51193
9	1	15584	21328	36864	33745
	2	23519	29387	34780	41247
	3	28864	34613	39779	45866
	4	32873	38451	43399	49170
	5	36038	41437	46185	51689
	6	38617	43844	48412	53683
	7	40765	45829	50234	55299
10	1	16931	23082	28960	36198
	2	25338	31536	37184	43895
	3	30908	36925	42286	48549
	4	35035	40834	45937	51838
	5	38262	43846	48719	54320
10	6	40869	46253	50922	56265
	7	43031	48232	52724	57852
	8	44852	49885	54220	59157

TABLE III (Continued)

$n = 30$ (*continued*)

p	q	0.90	0.95	0.975	0.99
11	1	18444	25029	31259	38840
	2	27349	33888	39790	46733
	3	33141	39426	44976	51394
	4	37373	43385	48628	54631
	5	40650	46409	51391	57065
	6	43275	48804	53559	58952
	7	45437	50760	55322	60485
	8	47247	52386	56777	61740
	9	48786	53759	57999	62782
12	1	20155	27209	33810	41744
	2	29570	36452	42594	49728
	3	35579	42125	47846	54392
	4	39903	46118	51484	57565
	5	43213	49131	54202	59919
	6	45842	51498	56319	61735
	7	47988	53414	58024	63189
	8	49778	55003	59432	64389
	9	51290	56336	60606	65381
	10	52587	57472	61598	66211
13	1	22092	29642	36623	44898
	2	32029	39255	45623	42918
	3	38235	45026	50892	57515
	4	42635	49033	54498	60621
	5	45957	52012	57145	62868
	6	48572	54332	59193	64594
	7	50690	56197	60830	65970
	8	52444	57733	62174	67099
	9	53918	59014	63287	68019
	10	55178	60104	64232	68807
	11	56264	61039	65037	69468
14	1	24285	32356	39705	48270
	2	34757	42321	48898	56324
	3	41128	48144	54123	60777
	4	45573	52124	57651	63760
	5	48890	55054	60221	65913
	6	51469	57306	62177	67529
	7	53542	59102	63731	68810
	8	55246	60570	64995	69848
	9	56671	61791	66042	70704
	10	57882	62823	66924	71423
	11	58923	63706	67676	72032
	12	59825	64468	68320	72547

$n = 30$ (*continued*)

p	q	0.90	0.95	0.975	0.99
15	1	26782	35392	43100	51915
	2	47771	45649	52390	59875
	3	44278	51484	57538	64172
	4	48732	55400	60948	66992
	5	52010	58243	63401	69004
	6	54533	60409	65254	70510
	7	56542	62120	66710	71685
	8	58181	63508	67886	72635
	9	59545	64657	68856	73411
	10	60696	65622	69669	74060
	11	61682	66445	70359	74611
	12	62535	67156	70954	75085
	13	63277	67768	71459	75476

$n = 35$

p	q	0.90	0.95	0.975	0.99
2	1	79720	11125	14293	18425
3	1	08440	11766	15099	19435
4	1	08950	12461	15972	20527
	2	14127	17972	21644	26246
5	1	09504	13214	16913	21693
	2	14949	18990	22838	27644
	3	18935	23078	26937	31669
6	1	10109	14034	17933	22950
	2	15839	20088	24119	29135
	3	19999	24335	28359	33275
	4	23305	27643	31613	36407
7	1	10773	14930	19046	24321
	2	16803	21271	25492	30717
	3	21147	25685	29881	34985
	4	24574	29097	33219	38178
	5	27396	31866	35901	40716
8	1	11500	15907	20249	25790
	2	17853	22554	26975	32422
	3	22386	27134	31503	36792
	4	25936	30648	34923	40042
	5	28842	33485	37658	42616
	6	31283	35841	39909	44709
9	1	12301	16978	21567	27390
	2	18997	23943	28574	34245
	3	23725	28691	33238	38709
	4	27400	32305	36735	42010
	5	30388	35204	39513	44607
	6	32884	37597	41785	46704
	7	35009	39618	43689	48453

TABLE III (Continued)

$n = 35$ (*continued*)

p	q	0.90	0.95	0.975	0.99
10	1	13182	18148	22994	29105
	2	20245	25450	30297	36197
	3	25174	30366	35092	40746
	4	28974	34077	38662	44089
	5	32040	37029	41469	46687
	6	34588	39458	43761	48792
	7	36746	41494	45669	50530
	8	38600	43231	47285	51988
11	1	14157	19437	24558	30970
	2	21607	27083	32152	38279
	3	26745	32170	37080	42916
	4	30665	35968	40701	46267
	5	33808	38972	43542	48879
	6	36401	41423	45836	50964
	7	38586	43469	47740	52682
	8	40456	45207	49347	54122
	9	42078	46707	50727	55353
12	1	15238	20855	26268	32995
	2	23095	28855	34147	40491
	3	28444	34106	39195	45197
	4	32486	37989	42872	48568
	5	35698	41033	45724	51164
	6	38331	43500	48015	53226
	7	40537	45549	49907	54919
	8	42415	47281	51496	56329
	9	44038	48769	52856	57531
	10	45452	50058	54026	58559
13	1	16439	22418	28137	35182
	2	24727	30783	36306	42872
	3	30288	36190	41454	47613
	4	34444	40146	45166	50980
	5	37718	43219	48022	53549
	6	40383	45694	50302	55579
	7	42603	47737	52172	57238
	8	44481	49452	53729	58600
	9	46099	50923	55064	59773
	10	47502	52188	56203	60760
	11	48735	53298	57201	61628
14	1	17777	24144	30184	37554
	2	26517	32878	38630	45405
	3	32286	38428	43860	50156
	4	36548	42442	47590	53498
	5	39876	45533	50435	56027
	6	42565	48007	52694	58022
	7	44787	50030	54527	59621
	8	46660	51725	56054	60949
	9	48263	53167	57349	62070
	10	49649	54406	58454	63019
	11	50864	55489	59422	63853
	12	51933	56438	60264	64572

p	q	0.90	0.95	0.975	0.99
15	1	19271	26052	32422	40113
	2	28481	35154	41130	48098
	3	34455	40834	46425	52840
	4	38811	44890	50152	56134
	5	42179	47982	52967	58602
	6	44877	50434	55181	60528
	7	47096	52435	56980	62089
	8	48952	54095	58459	63353
	9	53534	55503	59710	64423
	10	51895	56706	60773	65324
	11	53083	57755	61700	66113
	12	54126	58671	62504	66790
	13	55050	59480	63213	67387
16	1	20940	28158	34861	42856
	2	30639	37626	43817	50955
	3	36806	43415	49159	55660
	4	41240	47488	52843	58867
	5	44633	50565	55612	61262
	6	47328	52983	57710	63112
	7	49529	54945	59515	64605
	8	51359	56562	60942	65812
	9	52911	57928	62143	66826
	10	54240	59089	63158	67674
	11	55395	60095	64036	68410
	12	56407	60976	64804	69055
	13	57300	61748	65474	69611
	14	58093	62430	66063	70096
17	1	22818	30498	37544	45824
	2	33005	40298	46681	53940
	3	39354	46176	52030	58599
	4	43847	50246	55673	61711
	5	47245	53284	58371	64003
	6	49921	55653	60459	65766
	7	52087	57556	62128	67168
	8	53882	59123	63495	68311
	9	55394	60438	64641	69269
	10	56681	61547	65595	70052
	11	57797	62509	66428	70744
	12	58769	63341	67143	71327
	13	59626	64075	67772	71846
	14	60386	64721	68324	72298
	15	61064	65297	68817	72697

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