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Middle convolution for completely integrable systems with logarithmic singularities along hyperplane arrangements

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Abstract.

The middle convolution for completely integrable systems with logarithmic singularities along hyperplane arrangements is defined as a natural generalization of the middle convolution for Fuchsian ordinary differential equations. Additivity of the generalized middle convolution is proved. It is observed that the singular locus may increase by the generalized middle convolution. Examples concerning with hypergeometric series in several variables are given.

§1. Introduction

The middle convolution, which is introduced by Katz [7], is an operation for local systems on a punctured complex line, and plays a fundamental role in the theory of rigid local systems. By Riemann–Hilbert correspondence, it induces an operation for Fuchsian ordinary differential equations, which we also call the middle convolution. In this paper we extend the latter one to the operation for completely integrable systems in several variables with logarithmic singularities along hyperplane arrangements. The extended middle convolution possesses similar properties as the original one, such as additivity, and, on the other hand, gives several different features from the original one. In particular, by combining with the prolongation-restriction process, the extended middle convolution may change the index of rigidity, which is invariant under the original one.

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According to Dettweiler–Reiter [1], we give a definition of the middle convolution for a Fuchsian system

$$\frac{dU}{dx} = \left(\sum_{i=1}^{r} \frac{A_i}{x - a_i} \right) U, \tag{1}$$

where $a_1, a_2, \ldots, a_r \in \mathbb{C}$ are distinct points, and $A_1, A_2, \ldots, A_r$ are constant $n \times n$-matrices. Take $\lambda \in \mathbb{C}$. Define $rn \times rn$-matrices $G_1, G_2, \ldots, G_r$ by

$$G_i = \sum_{j=1}^{r} E_{ij} \otimes (A_j + \delta_{ij}\lambda) \quad (1 \leq i \leq r),$$

where E_{ij} denotes the $r \times r$-matrix with the only non-zero entry 1 at (i,j)-th position $(1 \leq i, j \leq r)$. The operation which sends the system (1) to the system

$$\frac{d\hat{U}}{dx} = \left(\sum_{i=1}^{r} \frac{G_i}{x - a_i} \right) \hat{U} \tag{2}$$

is called the *convolution* with parameter λ, and is denoted by c_λ.

Let $\mathcal{K}$ and $\mathcal{L}$ be the subspaces of $\mathbb{C}^{rn}$ defined by

$$\mathcal{K} = \left\{ \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix} ; v_i \in \mathrm{Ker} A_i \ (1 \leq i \leq r) \right\},$$
$$\mathcal{L} = \mathrm{Ker}(G_1 + G_2 + \cdots + G_r).$$

It is easy to see that $\mathcal{K}$ and $\mathcal{L}$ are invariant subspaces of $\mathbb{C}^{rn}$ for $(G_1, G_2, \ldots, G_r)$. Then $(G_1, G_2, \ldots, G_r)$ induces the action $(\bar{G}_1, \bar{G}_2, \ldots, \bar{G}_r)$ on the quotient space $\mathbb{C}^{rn}/(\mathcal{K} + \mathcal{L})$. The operation which sends (1) to the system

$$\frac{d\bar{U}}{dx} = \left(\sum_{i=1}^{r} \frac{\bar{G}_i}{x - a_i} \right) \bar{U} \tag{3}$$

is called the *middle convolution* with parameter λ, and is denoted by mc_λ.

It is shown that the middle convolution keeps the index of rigidity invariant, and if (1) is irreducible then the middle convolution system (3) is also irreducible. Moreover, if the residue matrices of (1) satisfies the conditions

$$\bigcap_{j \neq i} \mathrm{Ker} A_j \cap \mathrm{Ker}(A_i - c) = 0 \quad (\forall i, \ \forall c \in \mathbb{C})$$

and

$$\sum_{j\neq i} \mathrm{Im}A_j + \mathrm{Im}(A_i - c) = \mathbb{C}^n \quad (\forall i, \forall c \in \mathbb{C}),$$

we have the additivity

$$mc_0 = \mathrm{id}, \quad mc_{\lambda_2} \circ mc_{\lambda_1} = mc_{\lambda_1+\lambda_2}.$$

It is also shown that the middle convolution keeps the deformation equation invariant ([5]).

One of the main result of the theory of rigid local systems is the following.

Theorem 1.1. ([1, Theorem A.14]) *Any irreducible rigid Fuchsian system can be obtained from a Fuchsian equation of rank 1 by a finite iteration of middle convolutions and additions.*

It is shown in [2] that the middle convolution is analytically realized in the following way. For a solution $U(x)$ of (1), define rn-column vector $V(x)$ by

$$V(x) = \begin{pmatrix} \frac{U(x)}{x-a_1} \\ \frac{U(x)}{x-a_2} \\ \vdots \\ \frac{U(x)}{x-a_r} \end{pmatrix}.$$

Let $\hat{U}(x)$ be a Riemann–Liouville transform with exponent λ of $V(x)$:

$$\hat{U}(x) = \int_\Delta V(t)(t-x)^\lambda \, dt,$$

where Δ is an appropriate 1-cycle. Then $\hat{U}(x)$ makes a solution of the convolution system (2). Solutions of the middle convolution system (3) can be obtained from $\hat{U}(x)$ by the linear transformation induced by the projection $\mathbb{C}^{rn} \to \mathbb{C}^{rn}/(\mathcal{K}+\mathcal{L})$.

We shall use this analytic realization to extend the middle convolution for completely integrable systems in several variables. This will be done in Section 2. In Section 3 we show that the additivity of the middle convolution also holds for the extended case. In Section 4 we give two examples concerning with hypergeometric series in several variables.

In our previous work [4] we defined the middle convolution for completely integrable systems of KZ type. The results in the present paper is a generalization to completely integrable systems with any arrangements of hyperplanes as logarithmic singularities.

§2. Middle convolution

We consider an arrangement $\mathcal{A}$ of hyperplanes in $\mathbb{C}^l$. We fix a coordinate $(x_1, x_2, \ldots, x_l)$ of $\mathbb{C}^l$. For each hyperplane $H \in \mathcal{A}$, let f_H be a defining linear polynomial for H. For each $H \in \mathcal{A}$, we take a constant $n \times n$-matrix A_H.

We consider the Pfaffian system

$$dU = \Omega U, \tag{4}$$

where Ω is a 1-form given by

$$\Omega = \sum_{H \in \mathcal{A}} A_H d \log f_H. \tag{5}$$

We assume that the system (4) is completely integrable:

$$\Omega \wedge \Omega = O.$$

This condition will be stated in terms of A_H in Theorem 2.1 below.

For each i $(1 \leq i \leq l)$, we set

$$\mathcal{A}_{x_i} = \{H \in \mathcal{A};\ (f_H)_{x_i} \neq 0\}.$$

Then the system (4) can be written as

$$\frac{\partial U}{\partial x_i} = \left(\sum_{H \in \mathcal{A}_{x_i}} A_H \frac{(f_H)_{x_i}}{f_H} \right) U \tag{6}$$

for $1 \leq i \leq l$. In this paper we call the system (6) the x_i-equation.

Take any i $(1 \leq i \leq l)$, and set $x_i = x$. For each $H \in \mathcal{A}_x$ we define a_H by

$$f_H = (f_H)_x (x - a_H). \tag{7}$$

Then the x-equation can be written as

$$\frac{\partial U}{\partial x} = \left(\sum_{H \in \mathcal{A}_x} \frac{A_H}{x - a_H} \right) U. \tag{8}$$

Let j $(1 \leq j \leq l)$ be another index, and set $x_j = y$. For each $H \in \mathcal{A}_y$, we define b_H by

$$f_H = (f_H)_y (y - b_H).$$

For $H, H' \in \mathcal{A}_x$ such that $(a_H - a_{H'})_y \neq 0$, we define $c_{HH'}$ by

$$a_H - a_{H'} = (a_H - a_{H'})_y (y - c_{HH'}).$$

Then obviously

$$c_{HH'} = c_{H'H}$$

holds. For $H \in \mathcal{A}_x$ we set

$$\mathcal{C}_{H,y} = \{H' \in \mathcal{A}_x \setminus \{H\}\,;\, (a_H - a_{H'})_y \neq 0\}.$$

Theorem 2.1. *The integrability condition $\Omega \wedge \Omega = O$ for the system (4) holds if and only if, for any pair (i, j) $(1 \leq i, j \leq l)$ of distinct indices, by setting $x_i = x$ and $x_j = y$, the following conditions hold:*

$$\left[A_H, \sum_{\substack{H'' \in \mathcal{C}_{H,y} \\ c_{HH''} = b_{H'}}} A_{H''} + A_{H'} \right] = O \tag{9}$$

for $H \in \mathcal{A}_x \cap \mathcal{A}_y$ and $H' \in \mathcal{A}_x^{\mathrm{c}} \cap \mathcal{A}_y$,

$$\left[A_H, \sum_{\substack{H'' \in \mathcal{C}_{H,y} \\ c_{HH''} = c_{HH'}}} A_{H''} \right] = O \tag{10}$$

for $H \in \mathcal{A}_x \cap \mathcal{A}_y$ and $H' \in \mathcal{C}_{H,y}$ such that $c_{HH'} \neq b_K$ for any $K \in \mathcal{A}_x^{\mathrm{c}} \cap \mathcal{A}_y$,

$$\left[A_H, \sum_{\substack{H'' \in \mathcal{A}_x \cap \mathcal{A}_y \\ c_{HH''} = b_{H'}}} A_{H''} + A_{H'} \right] = O \tag{11}$$

for $H \in \mathcal{A}_x \cap \mathcal{A}_y^{\mathrm{c}}$ and $H' \in \mathcal{A}_x^{\mathrm{c}} \cap \mathcal{A}_y$, and

$$\left[A_H, \sum_{\substack{H'' \in \mathcal{A}_x \cap \mathcal{A}_y \\ c_{HH''} = c_{HH'}}} A_{H''} \right] = O \tag{12}$$

for $H \in \mathcal{A}_x \cap \mathcal{A}_y^{\mathrm{c}}$ and $H' \in \mathcal{A}_x \cap \mathcal{A}_y$ such that $c_{HH'} \neq b_K$ for any $K \in \mathcal{A}_x^{\mathrm{c}} \cap \mathcal{A}_y$.

Proof. We regard the coefficients of the 2-form $\Omega \wedge \Omega$ as rational functions in x. The principal part of the Laurent series expansion at each pole can be regarded as a rational function in y, and then we get the left hand sides of (9) to (12) as the residue matrices of the poles in y, from which the assertion follows. Q.E.D.

Let $\mathcal{F}$ be the local system on $\mathbb{C} \setminus \{a_H\,;\, H \in \mathcal{A}_x\}$ of local solutions of (8). Let λ be a complex number. For each $H \in \mathcal{A}_x$, we define $V_H(x_1, x_2, \ldots, x_l)$ by

$$V_H(x_1, x_2, \ldots, x_l) = \int_\Delta \frac{U(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_l)}{t - a_H}(t - x)^\lambda\, dt, \tag{13}$$

where $\Delta \in H_1(\mathbb{C} \setminus \{a_H\,;\, H \in \mathcal{A}_x\}, \mathcal{F})$ and $x = x_i$. We set

$$V(x_1, x_2, \ldots, x_l) = (V_H(x_1, x_2, \ldots, x_l))_{H \in \mathcal{A}_x}. \tag{14}$$

Proposition 2.1. *The function $V = (V_H)_{H \in \mathcal{A}_x}$ defined by (13) satisfies the following systems of differential equations:*
For $x = x_i$, the x-equation for V is given by

$$\frac{\partial V_H}{\partial x} = \frac{1}{x - a_H} \sum_{H' \in \mathcal{A}_x} (A_{H'} + \delta_{HH'}\lambda) V_{H'}. \tag{15}$$

For other x_j, we set $x_j = y$. Then the y-equation for V is given by

$$\begin{aligned} \frac{\partial V_H}{\partial y} &= \frac{1}{y - b_H} \sum_{H' \in \mathcal{A}_x} (A_{H'} + \delta_{HH'}\lambda) V_{H'} \\ &\quad + \sum_{H' \in \mathcal{C}_{H,y}} \frac{A_{H'}}{y - c_{HH'}}(V_H - V_{H'}) + \sum_{H' \in \mathcal{A}_x^c \cap \mathcal{A}_y} \frac{A_{H'}}{y - b_{H'}} V_H \end{aligned} \tag{16}$$

for $H \in \mathcal{A}_x \cap \mathcal{A}_y$, and

$$\frac{\partial V_H}{\partial y} = \sum_{H' \in \mathcal{A}_x \cap \mathcal{A}_y} \frac{A_{H'}}{y - c_{HH'}}(V_H - V_{H'}) + \sum_{H' \in \mathcal{A}_x^c \cap \mathcal{A}_y} \frac{A_{H'}}{y - b_{H'}} V_H \tag{17}$$

for $H \in \mathcal{A}_x \cap \mathcal{A}_y^c$.

Proof. As explained in Introduction, the x-equation for V is nothing but the convolution of the x-equation for U, and then is given by (15).

To obtain the y-equation with $y = x_j$, we regard x_k $(k \neq i, j)$ as a parameter, and use the notation $U(x_1, x_2, \ldots, x_l) = U(x_i, x_j) = U(x, y)$. The partial derivative of V_H with respect to y is

$$\frac{\partial V_H}{\partial y} = \int_\Delta \frac{\partial U}{\partial y}(t, y) \frac{(t-x)^\lambda}{t - a_H}\, dt + \int_\Delta U(t, y)(t-x)^\lambda \frac{\partial}{\partial y} \left(\frac{1}{t - a_H} \right) dt.$$

Note that

$$\frac{\partial U}{\partial y}(t, y) = \left(\sum_{H' \in \mathcal{A}_y} \frac{A_{H'}}{y - b_{H'}} \right) U(t, y).$$

If $H \notin \mathcal{A}_y$, a_H is independent of y, and hence we have

$$\begin{aligned}
\frac{\partial V_H}{\partial y} &= \sum_{H' \in \mathcal{A}_y} A_{H'} \int_\Delta \frac{U(t,y)}{y - b_{H'}} \cdot \frac{(t-x)^\lambda}{t - a_H}\, dt \\
&= \sum_{H' \in \mathcal{A}_x \cap \mathcal{A}_y} A_{H'} \int_\Delta U(t,y)(t-x)^\lambda \frac{(f_{H'})_y}{(f_{H'})_x (t - a_{H'})} \cdot \frac{1}{t - a_H}\, dt \\
&\quad + \sum_{H' \in \mathcal{A}_x^c \cap \mathcal{A}_y} \frac{A_{H'}}{y - b_{H'}} \int_\Delta \frac{U(t,y)}{t - a_H} (t-x)^\lambda\, dt \\
&= \sum_{H' \in \mathcal{A}_x \cap \mathcal{A}_y} \frac{(f_{H'})_y}{(f_{H'})_x} \cdot \frac{A_{H'}}{a_H - a_{H'}} (V_H - V_{H'}) + \sum_{H' \in \mathcal{A}_x^c \cap \mathcal{A}_y} \frac{A_{H'}}{y - b_{H'}} V_H.
\end{aligned}$$

Since $(a_H)_y = 0$, we have

$$\frac{(f_{H'})_y}{(f_{H'})_x} \cdot \frac{1}{a_H - a_{H'}} = \frac{1}{y - c_{HH'}}$$

for $H' \in \mathcal{A}_x \cap \mathcal{A}_y$, and hence we get (17).

If $H \in \mathcal{A}_y$, there appear the two terms

$$\int_\Delta A_H \frac{U(t,y)}{y - b_H} \cdot \frac{(t-x)^\lambda}{t - a_H}\, dt = \int_\Delta A_H \frac{(f_H)_y}{(f_H)_x} \cdot \frac{U(t,y)}{(t - a_H)^2} (t-x)^\lambda\, dt$$

and

$$-\int_\Delta \frac{(f_H)_x}{(f_H)_y} \cdot \frac{U(t,y)}{(y - b_H)^2} (t-x)^\lambda\, dt = -\int_\Delta \frac{(f_H)_y}{(f_H)_x} \cdot \frac{U(t,y)}{(x - a_H)^2} (t-x)^\lambda\, dt$$

in $\partial V_H/\partial y$. Since Δ is a twisted 1-cycle, we have

$$\begin{aligned}\int_\Delta \frac{U(t,y)}{(t-a_H)^2}(t-x)^\lambda\,dt &= \int_\Delta \frac{\partial}{\partial t}\left(-\frac{1}{t-a_H}\right)U(t,y)(t-x)^\lambda\,dt\\ &= \int_\Delta \frac{1}{t-a_H}\frac{\partial}{\partial t}\left(U(t,y)(t-x)^\lambda\right)\,dt\\ &= \int_\Delta \frac{1}{t-a_H}\left(\sum_{H'\in\mathcal{A}_x}\frac{A_{H'}}{t-a_{H'}}U(t,y)(t-x)^\lambda\right.\\ &\qquad\left.+\lambda U(t,y)(t-x)^{\lambda-1}\right)\,dt\\ &= \int_\Delta \frac{A_H}{(t-a_H)^2}U(t,y)(t-x)^\lambda\,dt\\ &\qquad+\sum_{H'\in\mathcal{A}_x\setminus\{H\}}\frac{A_{H'}}{a_H-a_{H'}}(V_H-V_{H'})-\frac{\partial V_H}{\partial x},\end{aligned}$$

and then we get

$$\begin{aligned}&\int_\Delta A_H\frac{(f_H)_y}{(f_H)_x}\cdot\frac{U(t,y)}{(t-a_H)^2}(t-x)^\lambda\,dt-\int_\Delta\frac{(f_H)_y}{(f_H)_x}\cdot\frac{U(t,y)}{(x-a_H)^2}(t-x)^\lambda\,dt\\ &\quad=\frac{(f_H)_y}{(f_H)_x}\left\{\frac{\partial V_H}{\partial x}-\sum_{H'\in\mathcal{A}_x\setminus\{H\}}\frac{A_{H'}}{a_H-a_{H'}}(V_H-V_{H'})\right\}.\end{aligned}$$

Thus we have, for $H\in\mathcal{A}_x\cap\mathcal{A}_y$,

$$\begin{aligned}\frac{\partial V_H}{\partial y} &= \frac{(f_H)_y}{(f_H)_x}\frac{\partial V_H}{\partial x}-\frac{(f_H)_y}{(f_H)_x}\sum_{H'\in\mathcal{A}_x\setminus\{H\}}\frac{A_{H'}}{a_H-a_{H'}}(V_H-V_{H'})\\ &\qquad+\sum_{H'\in\mathcal{A}_x\cap\mathcal{A}_y\setminus\{H\}}\frac{(f_{H'})_y}{(f_{H'})_x}\frac{A_{H'}}{a_H-a_{H'}}(V_H-V_{H'})\\ &\qquad+\sum_{H'\in\mathcal{A}_x^c\cap\mathcal{A}_y}\frac{A_{H'}}{y-b_{H'}}V_H\\ &\quad=\frac{(f_H)_y}{(f_H)_x}\frac{1}{x-a_H}\sum_{H'\in\mathcal{A}_x}(A_{H'}+\delta_{HH'}\lambda)V_{H'}\\ &\qquad+\sum_{H'\in\mathcal{A}_x\cap\mathcal{A}_y\setminus\{H\}}\left(\frac{(f_{H'})_y}{(f_{H'})_x}-\frac{(f_H)_y}{(f_H)_x}\right)\frac{A_{H'}}{a_H-a_{H'}}(V_H-V_{H'})\\ &\qquad-\frac{(f_H)_y}{(f_H)_x}\sum_{H'\in\mathcal{A}_x\cap\mathcal{A}_y^c}\frac{A_{H'}}{a_H-a_{H'}}(V_H-V_{H'})+\sum_{H'\in\mathcal{A}_x^c\cap\mathcal{A}_y}\frac{A_{H'}}{y-b_{H'}}V_H.\end{aligned}$$

By the definitions of a_H, b_H and $c_{HH'}$, we have

$$\frac{(f_H)_y}{(f_H)_x}\frac{1}{x-a_H}=\frac{1}{y-b_H}$$

for $H \in \mathcal{A}_x \cap \mathcal{A}_y$,

$$\left(\frac{(f_{H'})_y}{(f_{H'})_x}-\frac{(f_H)_y}{(f_H)_x}\right)\frac{1}{a_H-a_{H'}}=\frac{1}{y-c_{HH'}}$$

for $H \in \mathcal{A}_x \cap \mathcal{A}_y$ and $H' \in \mathcal{C}_{H,y}$,

$$\frac{(f_{H'})_y}{(f_{H'})_x}-\frac{(f_H)_y}{(f_H)_x}=0$$

for $H \in \mathcal{A}_x \cap \mathcal{A}_y$ and $H' \notin \mathcal{C}_{H,y}$, and

$$-\frac{(f_H)_y}{(f_H)_x}\cdot\frac{1}{a_H-a_{H'}}=\frac{1}{y-c_{HH'}}$$

for $H \in \mathcal{A}_x \cap \mathcal{A}_y$ and $H' \in \mathcal{A}_x \cap \mathcal{A}_y^c$. Thus we obtain (16). Q.E.D.

We see that the systems for $V=(V_H)_{H\in\mathcal{A}_x}$ given in Proposition 2.1 make a Pfaffian system

$$dV=\hat{\Omega}V \tag{18}$$

with 1-form $\hat{\Omega}$ of the form

$$\hat{\Omega}=\sum_{\hat{H}\in\hat{\mathcal{A}}} G_{\hat{H}}\, d\log f_{\hat{H}},$$

where $\hat{\mathcal{A}}$ is the arrangement of hyperplanes obtained from $\mathcal{A}$ by adding the hyperplanes of the form $\{y-c_{HH'}=0\}$.

Definition 2.1. We call the operation which sends the Pfaffian system (4) to the system (18) the *convolution* in x_i-direction with parameter λ, and denote it by $c_\lambda^{x_i}$.

Remark 2.1. If there is a pair $H, H' \in \mathcal{A}_x$ such that $H' \in \mathcal{C}_{H,y}$ and $c_{HH'} \neq b_K$ for any $K \in \mathcal{A}_x^c \cap \mathcal{A}_y$, we have $\{y-c_{HH'}=0\} \in \hat{\mathcal{A}} \setminus \mathcal{A}$, and hence $\hat{\mathcal{A}} \supsetneq \mathcal{A}$. Thus the singular locus may increase by the convolution, which is a different feature from the case of one variable.

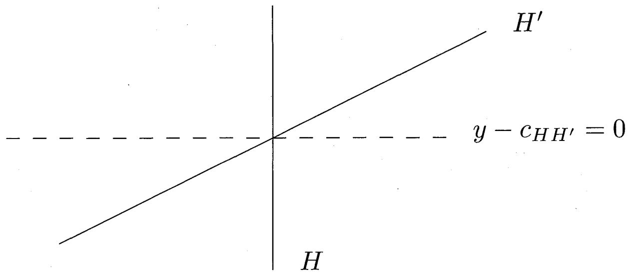

Fig. 1.

Let $\mathcal{K}$ and $\mathcal{L}$ be the subspaces of $(\mathbb{C}^n)^{\mathcal{A}_x} \simeq \mathbb{C}^{n\#\mathcal{A}_x}$ given by

$$\mathcal{K} = \{v = (v_H)_{H\in\mathcal{A}_x}\,;\, v_H \in \mathrm{Ker} A_H\},$$
$$\mathcal{L} = \{v = (v_H)_{H\in\mathcal{A}_x}\,;\, \sum_{H'\in\mathcal{A}_x} A_{H'}v_{H'} + \lambda v_H = 0 \ (\forall H \in \mathcal{A}_x)\}.$$

It is easy to see that

$$\mathcal{L} = \{v = (v_0)\,;\, \left(\sum_{H\in\mathcal{A}_x} A_H\right) v_0 + \lambda v_0 = 0\}$$

if $\lambda \neq 0$, and

$$\mathcal{L} = \{v = (v_H)_{H\in\mathcal{A}_x}\,;\, \sum_{H\in\mathcal{A}_x} A_H v_H = 0\}$$

if $\lambda = 0$. As mentioned in Introduction, $\mathcal{K}$ and $\mathcal{L}$ are invariant subspaces for the x-equation of the convolution system (18). For $y = x_j$ $(j \neq i)$, we write the y-equation (16) and (17) of the system (18) as

$$\frac{\partial V}{\partial y} = G(y)V.$$

Proposition 2.2. *For any x_j $(j \neq i)$, the subspaces $\mathcal{K}$ and $\mathcal{L}$ are invariant for the x_j-equation of the convolution system (18). Namely, by setting $x_j = y$,*

$$G(y)\mathcal{K} \subset \mathcal{K}, \quad G(y)\mathcal{L} \subset \mathcal{L}$$

hold for any value of y.

Proof. Take any $v = (v_H)_{H\in\mathcal{A}_x} \in \mathcal{K}$, and set

$$G(y)v = w = (w_H)_{H\in\mathcal{A}_x}.$$

For $H \in \mathcal{A}_x \cap \mathcal{A}_y$, noting that $v_{H'} \in \mathrm{Ker} A_{H'}$, we have

$$\begin{aligned} w_H &= \frac{1}{y-b_H} \sum_{H' \in \mathcal{A}_x} (A_{H'} + \delta_{HH'}\lambda) v_{H'} \\ &\quad + \sum_{H' \in \mathcal{C}_{H,y}} \frac{A_{H'}}{y - c_{HH'}} (v_H - v_{H'}) + \sum_{H' \in \mathcal{A}_x^c \cap \mathcal{A}_y} \frac{A_{H'}}{y - b_{H'}} v_H \\ &= \left(\frac{\lambda}{y-b_H} + \sum_{H' \in \mathcal{C}_{H,y}} \frac{A_{H'}}{y - c_{HH'}} + \sum_{H' \in \mathcal{A}_x^c \cap \mathcal{A}_y} \frac{A_{H'}}{y - b_{H'}} \right) v_H. \end{aligned}$$

By using the integrability conditions (9) and (10), we get

$$\begin{aligned} A_H w_H &= \left(\frac{\lambda}{y-b_H} + \sum_{H' \in \mathcal{C}_{H,y}} \frac{A_{H'}}{y - c_{HH'}} + \sum_{H' \in \mathcal{A}_x^c \cap \mathcal{A}_y} \frac{A_{H'}}{y - b_{H'}} \right) A_H v_H \\ &= 0, \end{aligned}$$

and hence $w_H \in \mathrm{Ker} A_H$. Similarly, by the help of the integrability conditions (11) and (12), we get $w_H \in \mathrm{Ker} A_H$ for $H \in \mathcal{A}_x \cap \mathcal{A}_y^c$. Thus we obtain $w \in \mathcal{K}$.

Next we assume $\lambda \neq 0$, and take any $v = (v_H)_{H \in \mathcal{A}_x} \in \mathcal{L}$. Then we have $v_H = v_0$ $(\forall H)$ and $\left(\sum_{H \in \mathcal{A}_x} A_H\right) v_0 + \lambda v_0 = 0$. Set

$$G(y) v = w = (w_H)_{H \in \mathcal{A}_x}.$$

Then for $H \in \mathcal{A}_x \cap \mathcal{A}_y$, we have

$$\begin{aligned} w_H &= \frac{1}{y-b_H} \sum_{H' \in \mathcal{A}_x} (A_{H'} + \delta_{HH'}\lambda) v_0 \\ &\quad + \sum_{H' \in \mathcal{C}_{H,y}} \frac{A_{H'}}{y - c_{HH'}} (v_0 - v_0) + \sum_{H' \in \mathcal{A}_x^c \cap \mathcal{A}_y} \frac{A_{H'}}{y - b_{H'}} v_0 \\ &= \left(\sum_{H' \in \mathcal{A}_x^c \cap \mathcal{A}_y} \frac{A_{H'}}{y - b_{H'}} \right) v_0. \end{aligned}$$

Similarly, for $H \in \mathcal{A}_x \cap \mathcal{A}_y^c$ we have

$$\begin{aligned} w_H &= \sum_{H' \in \mathcal{A}_x \cap \mathcal{A}_y} \frac{A_{H'}}{y - c_{HH'}} (v_0 - v_0) + \sum_{H' \in \mathcal{A}_x^c \cap \mathcal{A}_y} \frac{A_{H'}}{y - b_{H'}} v_0 \\ &= \left(\sum_{H' \in \mathcal{A}_x^c \cap \mathcal{A}_y} \frac{A_{H'}}{y - b_{H'}} \right) v_0. \end{aligned}$$

Thus, for every $H \in \mathcal{A}_x$, w_H becomes the same vector, which we denote by w_0.

Take any $H' \in \mathcal{A}_x^c \cap \mathcal{A}_y$. Let p be an intersection point of H' and a hyperplane in $\mathcal{A}_x$. We denote by $\mathcal{B}_p$ the set of hyperplanes in $\mathcal{A}_x$ which pass through p. Suppose $H \in \mathcal{B}_p$. If $H \in \mathcal{A}_x \cap \mathcal{A}_y$, we have

$$\{H'' \in \mathcal{C}_{H,y}\,;\, c_{HH''} = b_{H'}\} = \mathcal{B}_p \setminus \{H\}.$$

If $H \in \mathcal{A}_x \cap \mathcal{A}_y^c$, we have

$$\{H'' \in \mathcal{A}_x \cap \mathcal{A}_y\,;\, c_{HH''} = b_{H'}\} = \mathcal{B}_p \setminus \{H\}.$$

Thus the integrability conditions (9) and (11) can be written in the same form as

$$\left[A_H, \sum_{H'' \in \mathcal{B}_p \setminus \{H\}} A_{H''} + A_{H'}\right] = O$$

for every $H \in \mathcal{B}_p$, and hence we get

$$\left[A_H, \sum_{H'' \in \mathcal{B}_p} A_{H''} + A_{H'}\right] = O.$$

Summing up these relations for all $H \in \mathcal{B}_p$, we get

$$\left[\sum_{H \in \mathcal{B}_p} A_H, \sum_{H'' \in \mathcal{B}_p} A_{H''} + A_{H'}\right] = O,$$

from which we obtain

$$\left[\sum_{H \in \mathcal{B}_p} A_H, A_{H'}\right] = O.$$

Since $\bigcup_p \mathcal{B}_p = \mathcal{A}_x$, we have

$$O = \sum_p \left[\sum_{H \in \mathcal{B}_p} A_H, A_{H'}\right] = \left[\sum_{H \in \mathcal{A}_x} A_H, A_{H'}\right]$$

for every $H' \in \mathcal{A}_x^c \cap \mathcal{A}_y$. Using this relation, we get

$$\begin{aligned}\left(\sum_{H\in\mathcal{A}_x} A_H + \lambda\right) w_0 &= \left(\sum_{H\in\mathcal{A}_x} A_H + \lambda\right)\left(\sum_{H'\in\mathcal{A}_x^c\cap\mathcal{A}_y} \frac{A_{H'}}{y-b_{H'}}\right) v_0 \\ &= \left(\sum_{H'\in\mathcal{A}_x^c\cap\mathcal{A}_y} \frac{A_{H'}}{y-b_{H'}}\right)\left(\sum_{H\in\mathcal{A}_x} A_H + \lambda\right) v_0 \\ &= 0,\end{aligned}$$

which shows $w \in \mathcal{L}$.

Finally we consider the case $\lambda = 0$. Take any $v = (v_H)_{H\in\mathcal{A}_x} \in \mathcal{L}$. Then we have

$$\sum_{H\in\mathcal{A}_x} A_H v_H = 0.$$

We set

$$G(y)v = w = (w_H)_{H\in\mathcal{A}_x}.$$

Then we have, for $H \in \mathcal{A}_x \cap \mathcal{A}_y$,

$$\begin{aligned}w_H &= \frac{1}{y-b_H}\sum_{H'\in\mathcal{A}_x} A_{H'}v_{H'} + \sum_{H'\in\mathcal{C}_{H,y}} \frac{A_{H'}}{y-c_{HH'}}(v_H - v_{H'}) \\ &\quad + \sum_{H'\in\mathcal{A}_x^c\cap\mathcal{A}_y} \frac{A_{H'}}{y-b_{H'}} v_{H'} \\ &= \sum_{H'\in\mathcal{C}_{H,y}} \frac{A_{H'}}{y-c_{HH'}}(v_H - v_{H'}) + \sum_{H'\in\mathcal{A}_x^c\cap\mathcal{A}_y} \frac{A_{H'}}{y-b_{H'}} v_{H'},\end{aligned}$$

and for $H \in \mathcal{A}_x \cap \mathcal{A}_y^c$,

$$w_H = \sum_{H'\in\mathcal{A}\cap\mathcal{A}_y} \frac{A_{H'}}{y-c_{HH'}}(v_H - v_{H'}) + \sum_{H'\in\mathcal{A}_x^c\cap\mathcal{A}_y} \frac{A_{H'}}{y-b_{H'}} v_{H'}.$$

Take any $H' \in \mathcal{A}_x^c \cap \mathcal{A}_y$. We calculate the residue of $\sum_{H\in\mathcal{A}_x} A_H w_H$ at the pole $y = b_{H'}$. By the help of the integrability conditions (9) and

(11), the residue becomes

$$
\begin{aligned}
&\sum_{H\in\mathcal{A}_x\cap\mathcal{A}_y} A_H \left\{ \sum_{\substack{H''\in\mathcal{C}_{H,y}\\ c_{HH''}=b_{H'}}} A_{H''}(v_H - v_{H''}) + A_{H'}v_H \right\} \\
&\quad + \sum_{H\in\mathcal{A}_x\cap\mathcal{A}_y^c} A_H \left\{ \sum_{\substack{H''\in\mathcal{A}_x\cap\mathcal{A}_y\\ c_{HH''}=b_{H'}}} A_{H''}(v_H - v_{H''}) + A_{H'}v_H \right\} \\
&= \sum_{H\in\mathcal{A}_x\cap\mathcal{A}_y} \left(\sum_{\substack{H''\in\mathcal{C}_{H,y}\\ c_{HH''}=b_{H'}}} A_{H''} + A_{H'} \right) A_H v_H \\
&\quad - \sum_{H\in\mathcal{A}_x\cap\mathcal{A}_y} A_H \sum_{\substack{H''\in\mathcal{C}_{H,y}\\ c_{HH''}=b_{H'}}} A_{H''} v_{H''} \\
&\quad + \sum_{H\in\mathcal{A}_x\cap\mathcal{A}_y^c} \left(\sum_{\substack{H''\in\mathcal{A}_x\cap\mathcal{A}_y\\ c_{HH''}=b_{H'}}} A_{H''} + A_{H'} \right) A_H v_H \\
&\quad - \sum_{H\in\mathcal{A}_x\cap\mathcal{A}_y^c} A_H \sum_{\substack{H''\in\mathcal{A}_x\cap\mathcal{A}_y\\ c_{HH''}=b_{H'}}} A_{H''} v_{H''} \\
&= \sum_{H\in\mathcal{A}_x\cap\mathcal{A}_y} \left(\sum_{\substack{H''\in\mathcal{C}_{H,y}\\ c_{HH''}=b_{H'}}} A_{H''} \right) A_H v_H \\
&\quad - \sum_{H\in\mathcal{A}_x\cap\mathcal{A}_y} A_H \sum_{\substack{H''\in\mathcal{C}_{H,y}\\ c_{HH''}=b_{H'}}} A_{H''} v_{H''} \\
&\quad + \sum_{H\in\mathcal{A}_x\cap\mathcal{A}_y^c} \left(\sum_{\substack{H''\in\mathcal{A}_x\cap\mathcal{A}_y\\ c_{HH''}=b_{H'}}} A_{H''} \right) A_H v_H \\
&\quad - \sum_{H\in\mathcal{A}_x\cap\mathcal{A}_y^c} A_H \sum_{\substack{H''\in\mathcal{A}_x\cap\mathcal{A}_y\\ c_{HH''}=b_{H'}}} A_{H''} v_{H''}.
\end{aligned}
$$

In the last hand side, the coefficient of $A_H v_H$ for $H \in \mathcal{A}_x \cap \mathcal{A}_y$ is

$$\sum_{\substack{H'' \in \mathcal{C}_{H,y} \\ c_{HH''} = b_{H'}}} A_{H''} - \sum_{\substack{H'' \in \mathcal{C}_{H,y} \cap \mathcal{A}_y \setminus \{H\} \\ c_{HH''} = b_{H'}}} A_{H''} - \sum_{\substack{H'' \in \mathcal{A}_x \cap \mathcal{A}_y^c \\ c_{HH''} = b_{H'}}} A_{H''} = O,$$

and that for $H \in \mathcal{A}_x \cap \mathcal{A}_y^c$ is

$$\sum_{\substack{H'' \in \mathcal{A}_x \cap \mathcal{A}_y \\ c_{HH''} = b_{H'}}} A_{H''} - \sum_{\substack{H'' \in \mathcal{A}_x \cap \mathcal{A}_y \\ c_{HH''} = b_{H'}}} A_{H''} = O.$$

Thus the residue at $y = b_{H'}$ is O. In a similar way, we see that the residue of $\sum_{H \in \mathcal{A}_x} A_H w_H$ at $y = c_{HH'}$ which does not coincide with any b_K ($K \in \mathcal{A}_x^c \cap \mathcal{A}_y$) is also O. Hence we have

$$\sum_{H \in \mathcal{A}_x} A_H w_H = O,$$

which shows $w = (w_H)_{H \in \mathcal{A}_x} \in \mathcal{L}$. Q.E.D.

Thanks to Proposition 2.2, we can derive from the convolution system (18) a Pfaffian system

$$d\tilde{V} = \tilde{\Omega}\tilde{V} \tag{19}$$

on the quotient space $(\mathbb{C}^n)^{\mathcal{A}_x}/(\mathcal{K} + \mathcal{L})$. The matrix 1-form $\tilde{\Omega}$ is induced from $\hat{\Omega}$ as the action on $(\mathbb{C}^n)^{\mathcal{A}_x}/(\mathcal{K} + \mathcal{L})$.

Definition 2.2. We call the operation which sends the Pfaffian system (4) to the system (19) the *middle convolution* in x_i-direction with parameter λ, and denote it by $mc_\lambda^{x_i}$.

Theorem 2.2. *Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{C}^l$, and, for each $H \in \mathcal{A}$, A_H be an $n \times n$-constant matrix. Define the 1-form Ω by (5). Assume that the Pfaffian system (4) is completely integrable.*

Then for any i ($1 \le i \le l$) and $\lambda \in \mathbb{C}$, the middle convolution system (19) of (4) in x_i-direction with parameter λ is completely integrable, and admits an integral representation of solutions of the form

$$\begin{aligned} &\tilde{V}(x_1, x_2, \ldots, x_l) \\ &= Q\left(\int_\Delta \frac{U(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_l)}{t - a_H}(t - x_i)^\lambda\, dt\right)_{H \in \mathcal{A}_{x_i}}, \end{aligned} \tag{20}$$

where $U(x_1, x_2, \ldots, x_l)$ *is a solution of (4),* Q *a constant linear transformation, and* Δ *a twisted 1-cycle. If the* x_i*-equation of (4) is irreducible, the middle convolution system (18) is also irreducible.*

Remark 2.2. The linear transformation Q is given as follows. We set $N = n \cdot \#\mathcal{A}_{x_i}$ and $M = \dim(\mathcal{K}+\mathcal{L})$. Let P be an $N \times N$ non-singular matrix whose first M columns make a basis of $\mathcal{K}+\mathcal{L}$. Then we can take

$$Q = (O_{N-M,M}, I_{N-M})P^{-1}.$$

Proof. By the construction, solutions $V = (V_H)_{H \in \mathcal{A}_{x_i}}$ of the convolution system (18) are given by the integral (13). This implies that the convolution system (18) is completely integrable. The complete integrability of the middle convolution system (19) follows from Proposition 2.2.

Let P be as in the above Remark. Then, thanks to Proposition 2.2, the Pfaffian system of differential equations for $P^{-1}V$ becomes of block upper triangular form, and the solutions $\tilde{V}$ of the middle convolution system are related with V by

$$P^{-1}V = \begin{pmatrix} * \\ \tilde{V} \end{pmatrix}.$$

Thus we have

$$\tilde{V} = (O_{N-M,M}, I_{N-M})P^{-1}V = QV,$$

from which the expression (20) follows.

Since the x_i-equation of the middle convolution system (19) is just the middle convolution of the x_i-equation of (4) as ordinary differential equations, the irreducibility follows from the result by Katz [7, Theorem 2.9.8]. Q.E.D.

To illustrate the results in this section, we give an example. Let $A_1, A_2, \ldots, A_5$ be constant $m \times m$-matrices. We consider the Pfaffian system

$$dU = \left[A_1 \frac{dx}{x} + A_2 \frac{dy}{y} + A_3 \frac{dx}{x-1} + A_4 \frac{dy}{y-1} + A_5 \frac{d(x-y)}{x-y}\right] U, \tag{21}$$

which is called of KZ- type. We assume that the system (21) is completely integrable. The corresponding arrangement is $\mathcal{A} = \{H_1, H_2, \ldots, H_5\}$ with

$$H_1 = \{x = 0\}, \ H_2 = \{y = 0\}, \ H_3 = \{x = 1\},$$
$$H_4 = \{y = 1\}, \ H_5 = \{x = y\}.$$

Then we have

$$\mathcal{A}_x = \{H_1, H_3, H_5\}, \quad \mathcal{A}_y = \{H_2, H_4, H_5\}.$$

When we consider the (middle) convolution in x-direction, by the definition we have

$$\begin{aligned} &a_{H_1} = 0,\ a_{H_3} = 1,\ b_{H_2} = 0,\ b_{H_4} = 1, \\ &c_{H_1 H_5} = 0,\ c_{H_3 H_5} = 1. \end{aligned}$$

The x-equation of the (middle) convolution of (21) in x-direction is nothing but the (middle) convolution of the x-equation

$$\frac{\partial U}{\partial x} = \left(\frac{A_1}{x} + \frac{A_3}{x-1} + \frac{A_5}{x-y} \right) U$$

of (21) as a system of ordinary differential equations, which is explained in the Introduction. To write down the y-equation (16) and (17) of the convolution in this case, we note that the unknown vector is given by $V = {}^t(V_{H_1}, V_{H_3}, V_{H_5})$, and that

$$\mathcal{A}_x \cap \mathcal{A}_y = \{H_5\}, \quad \mathcal{C}_{H_5,y} = \emptyset.$$

Then the y-equation of the convolution of (21) in x-direction with parameter λ is given by

$$\begin{aligned} \frac{\partial V_{H_1}}{\partial y} &= \frac{A_5}{y}(V_{H_1} - V_{H_5}) + \frac{A_2}{y} V_{H_1} + \frac{A_4}{y-1} V_{H_1}, \\ \frac{\partial V_{H_3}}{\partial y} &= \frac{A_5}{y-1}(V_{H_3} - V_{H_5}) + \frac{A_2}{y} V_{H_3} + \frac{A_4}{y-1} V_{H_3}, \\ \frac{\partial V_{H_5}}{\partial y} &= \frac{1}{y-x}(A_1 V_{H_1} + A_3 V_{H_3} + (A_5 + \lambda) V_{H_5}) + \frac{A_2}{y} V_{H_5} + \frac{A_4}{y-1} V_{H_5}. \end{aligned}$$

An explicit example will be given in the last section.

§3. Additivity of the middle convolution

We regard $\mathbb{C}^n$ as $\{A_H\,;\, H \in \mathcal{A}\}$-module, and denote it by $\mathcal{V}$. When we regard $\mathbb{C}^n$ as $\{A_H\,;\, H \in \mathcal{A}_x\}$-module, we use $\mathcal{V}_x$ instead of $\mathcal{V}$. Then the convolution and the middle convolution can be understood as operations for $\mathcal{V}$ or $\mathcal{V}_x$.

We assume

$$\bigcap_{H' \in \mathcal{A}_x \setminus \{H\}} \mathrm{Ker} A_{H'} \cap \mathrm{Ker}(A_H - c) = 0 \quad (\forall H \in \mathcal{A}_x, \forall c \in \mathbb{C}) \tag{22}$$

and

$$\sum_{H' \in \mathcal{A}_x \setminus \{H\}} \mathrm{Im} A_{H'} + \mathrm{Im}(A_H - c) = \mathbb{C}^n \quad (\forall H \in \mathcal{A}_x, \forall c \in \mathbb{C}). \tag{23}$$

Then, as is mentioned in Introduction, we have the additivity

$$mc_0(\mathcal{V}_x) \simeq \mathcal{V}_x, \quad mc_{\lambda_2} \circ mc_{\lambda_1}(\mathcal{V}_x) \simeq mc_{\lambda_1+\lambda_2}(\mathcal{V}_x). \tag{24}$$

We shall show that the additivity also holds for the middle convolution in x_i-direction.

Theorem 3.1. *Let i be an index $(1 \le i \le l)$, and assume the conditions (22) and (23) with $x = x_i$.*

Then we have

$$mc_0^{x_i}(\mathcal{V}) \simeq \mathcal{V}, \quad mc_{\lambda_2}^{x_i} \circ mc_{\lambda_1}^{x_i}(\mathcal{V}) \simeq mc_{\lambda_1+\lambda_2}^{x_i}(\mathcal{V}). \tag{25}$$

Proof. Set $x_i = x$. First we note that $mc_\lambda^x(\mathcal{V}_x) = mc_\lambda(\mathcal{V}_x)$. Then, thanks to (24), there exist the isomorphisms (25) as vector spaces. It remains to show that these isomorphisms are compatible with the action of A_H $(H \in \mathcal{A})$.

As is shown in [1], the isomorphism $mc_0(\mathcal{V}_x) \simeq \mathcal{V}_x$ is induced from the homomorphism

$$\begin{array}{cccc} \phi_1 : & c_0(\mathcal{V}_x) & \to & \mathcal{V}_x \\ & (v_H)_{H \in \mathcal{A}_x} & \mapsto & \displaystyle\sum_{H \in \mathcal{A}_x} A_H v_H. \end{array}$$

Take any $j \neq i$, and set $x_j = y$. Let us write the y-equation of $\mathcal{V}$ as

$$\frac{\partial U}{\partial y} = g(y)U,$$

and that of $c_0^x(\mathcal{V})$ as

$$\frac{\partial V}{\partial y} = G(y)V.$$

In order to show the compatibility of $mc_0^x(\mathcal{V}) \simeq \mathcal{V}$, it is enough to check

$$\phi_1 \circ G(y) = g(y) \circ \phi_1 \tag{26}$$

for any value of y.

Take any $v = (v_H)_{H\in\mathcal{A}_x} \in (\mathbb{C}^n)^{\mathcal{A}_x}$, and set $G(y)v = (w_H)_{H\in\mathcal{A}_x}$. Every w_H can be calculated by using (16) with $\lambda = 0$ and (17), and then by the help of Theorem 2.1 we have

$$
\begin{aligned}
&\phi_1 \circ G(y)v \\
&= \sum_{H\in\mathcal{A}_x} A_H w_H \\
&= \sum_{H\in\mathcal{A}_x\cap\mathcal{A}_y} A_H \Bigg(\frac{1}{y-b_H} \sum_{H'\in\mathcal{A}_x} A_{H'} v_{H'} \\
&\qquad\qquad + \sum_{H'\in\mathcal{C}_{H,y}} \frac{A_{H'}}{y-c_{HH'}} (v_H - v_{H'}) + \sum_{H'\in\mathcal{A}_x^c\cap\mathcal{A}_y} \frac{A_{H'}}{y-b_{H'}} v_H \Bigg) \\
&\quad + \sum_{H\in\mathcal{A}_x\cap\mathcal{A}_y^c} A_H \Bigg(\sum_{H'\in\mathcal{A}_x\cap\mathcal{A}_y} \frac{A_{H'}}{y-c_{HH'}} (v_H - v_{H'}) \\
&\qquad\qquad + \sum_{H'\in\mathcal{A}_x^c\cap\mathcal{A}_y} \frac{A_{H'}}{y-b_{H'}} v_H \Bigg) \\
&= \sum_{H\in\mathcal{A}_x\cap\mathcal{A}_y} \frac{A_H}{y-b_H} \phi_1(v) \\
&\quad + \sum_{H\in\mathcal{A}_x\cap\mathcal{A}_y} \Bigg(\sum_{H'\in\mathcal{C}_{H,y}} \frac{A_{H'}}{y-c_{HH'}} A_H v_H + \sum_{H'\in\mathcal{A}_x^c\cap\mathcal{A}_y} \frac{A_{H'}}{y-b_{H'}} A_H v_H \\
&\qquad\qquad - A_H \sum_{H'\in\mathcal{C}_{H,y}} \frac{A_{H'}}{y-c_{HH'}} v_{H'} \Bigg) \\
&\quad + \sum_{H\in\mathcal{A}_x\cap\mathcal{A}_y^c} \Bigg(\sum_{H'\in\mathcal{A}_x\cap\mathcal{A}_y} \frac{A_{H'}}{y-c_{HH'}} A_H v_H + \sum_{H'\in\mathcal{A}_x^c\cap\mathcal{A}_y} \frac{A_{H'}}{y-b_{H'}} A_H v_H \\
&\qquad\qquad - A_H \sum_{H'\in\mathcal{A}_x\cap\mathcal{A}_y} \frac{A_{H'}}{y-c_{HH'}} v_{H'} \Bigg) \\
&= \sum_{H\in\mathcal{A}_x\cap\mathcal{A}_y} \frac{A_H}{y-b_H} \phi_1(v) \\
&\quad + \sum_{H'\in\mathcal{A}_x^c\cap\mathcal{A}_y} \frac{A_{H'}}{y-b_{H'}} \Bigg(\sum_{H\in\mathcal{A}_x\cap\mathcal{A}_y} A_H v_H + \sum_{H\in\mathcal{A}_x\cap\mathcal{A}_y^c} A_H v_H \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{H\in\mathcal{A}_x\cap\mathcal{A}_y} \frac{A_H}{y-b_H}\phi_1(v) + \sum_{H\in\mathcal{A}_x^c\cap\mathcal{A}_y} \frac{A_H}{y-b_H}\phi_1(v) \\
&= \sum_{H\in\mathcal{A}_y} \frac{A_H}{y-b_H}\phi_1(v) \\
&= g(y)\phi_1(v).
\end{aligned}
$$

This shows (26).

The isomorphism $mc_{\lambda_2} \circ mc_{\lambda_1}(\mathcal{V}_x) \simeq mc_{\lambda_1+\lambda_2}(\mathcal{V}_x)$ is induced from the homomorphism

$$
\begin{array}{rccc}
\phi_2 : & c_{\lambda_2} \circ c_{\lambda_1}(\mathcal{V}_x) & \to & c_{\lambda_1+\lambda_2}(\mathcal{V}_x) \\
& (\hat{v}_H)_{H\in\mathcal{A}_x} & \mapsto & \displaystyle\sum_{H\in\mathcal{A}_x} G_H\hat{v}_H,
\end{array}
$$

where G_H $(H \in \mathcal{A}_x)$ are the residue matrices of the x-equation

$$
\frac{\partial V}{\partial x} = \left(\sum_{H\in\mathcal{A}_x} \frac{G_H}{x-a_H} \right) V
$$

of $c^x_{\lambda_1}(\mathcal{V})$. We set the y-equations of $c^x_{\lambda_1}(\mathcal{V})$, $c^x_{\lambda_1+\lambda_2}(\mathcal{V})$ and $c^x_{\lambda_2} \circ c^x_{\lambda_1}(\mathcal{V})$ as

$$
\frac{\partial V}{\partial y} = \left(\sum_{H\in\tilde{\mathcal{A}}_y} \frac{G_H}{y-b_H} \right) V,
$$

$$
\frac{\partial V}{\partial y} = \left(\sum_{H\in\tilde{\mathcal{A}}_y} \frac{G'_H}{y-b_H} \right) V,
$$

$$
\frac{\partial \hat{V}}{\partial y} = \left(\sum_{H\in\tilde{\mathcal{A}}_y} \frac{\hat{G}_H}{y-b_H} \right) \hat{V},
$$

respectively, where $\tilde{\mathcal{A}}$ is the arrangement of hyperplanes corresponding to the Pfaffian system (18) of $c^x_{\lambda_1}(\mathcal{V})$. Then, to prove the compatibility of $mc^{x_i}_{\lambda_2} \circ mc^{x_i}_{\lambda_1}(\mathcal{V}) \simeq mc^{x_i}_{\lambda_1+\lambda_2}(\mathcal{V})$, it is enough to check

$$
\phi_2 \circ \left(\sum_{H\in\tilde{\mathcal{A}}_y} \frac{\hat{G}_H}{y-b_H} \right) = \left(\sum_{H\in\tilde{\mathcal{A}}_y} \frac{G'_H}{y-b_H} \right) \circ \phi_2. \tag{27}
$$

Note that the y-equation of $c^x_{\lambda_1}(\mathcal{V})$ (resp. $c^x_{\lambda_1+\lambda_2}(\mathcal{V})$) can be obtained by replacing λ by λ_1 (resp. $\lambda_1+\lambda_2$) and $\mathcal{A}_y$ by $\tilde{\mathcal{A}}_y$ in (16) and (17).

Then in particular we have

$$G_H = G'_H$$

for any $H \in \mathcal{A}_x^c \cap \tilde{\mathcal{A}}_y$. Similarly, the y-equation of $c^x_{\lambda_2} \circ c^x_{\lambda_1}(\mathcal{V})$ can be obtained by replacing λ by λ_2, $\mathcal{A}_y$ by $\tilde{\mathcal{A}}_y$ and A_H by G_H in (16) and (17).

Take any $\hat{v} = (\hat{v}_H)_{H\in\mathcal{A}_x}$, and set

$$\left(\sum_{H\in\tilde{\mathcal{A}}_y} \frac{\hat{G}_H}{y-b_H}\right)\hat{v} = (\hat{w}_H)_{H\in\mathcal{A}_x}.$$

Then, for $H \in \mathcal{A}_x \cap \tilde{\mathcal{A}}_y$, we have

$$\begin{aligned}\hat{w}_H = {} & \frac{1}{y-b_H}\sum_{H'\in\mathcal{A}_x}(G_{H'}+\delta_{HH'}\lambda_2)\hat{v}_{H'} \\ & + \sum_{H'\in\mathcal{C}_{H,y}} \frac{G_{H'}}{y-c_{HH'}}(\hat{v}_H - \hat{v}_{H'}) + \sum_{H'\in\mathcal{A}_x^c\cap\tilde{\mathcal{A}}_y} \frac{G_{H'}}{y-b_{H'}}\hat{v}_H,\end{aligned}$$

and for $H \in \mathcal{A}_x \cap \tilde{\mathcal{A}}_y^c$ we have

$$\hat{w}_H = \sum_{H'\in\mathcal{A}_x\cap\tilde{\mathcal{A}}_y} \frac{G_{H'}}{y-c_{HH'}}(\hat{v}_H - \hat{v}_{H'}) + \sum_{H'\in\mathcal{A}_x^c\cap\tilde{\mathcal{A}}_y} \frac{G_{H'}}{y-b_{H'}}\hat{v}_H.$$

By using them, we get

$$\begin{aligned}\phi_2 \circ \left(\sum_{H\in\tilde{\mathcal{A}}_y} \frac{\hat{G}_H}{y-b_H}\right)(\hat{v}) &= \sum_{H\in\mathcal{A}_x} G_H\hat{w}_H \\ &= \sum_{H\in\mathcal{A}_x} \frac{G_H}{y-b_H}\phi_2(\hat{v}) + \lambda_2 \sum_{H\in\mathcal{A}_x\cap\tilde{\mathcal{A}}_y} \frac{G_H}{y-b_H}\hat{v}_H.\end{aligned}$$

By (15), we see that

$$G_H = \sum_{H'\in\mathcal{A}_x} E_{HH'} \otimes (A_{H'} + \delta_{HH'}\lambda_1)$$

for $H \in \mathcal{A}_x$. Then, if we set $\hat{v}_H = (v_{HK})_{K\in\mathcal{A}_x}$, we have

$$G_H\hat{v}_H = E_H \otimes \left(\sum_{K\in\mathcal{A}_x} A_K v_{HK} + \lambda_1 v_{HH}\right),$$

where E_H denotes the unit vector in $\mathbb{C}^{\mathcal{A}_x}$ with the only non-zero entry 1 in the H-th position. Moreover we have

$$G_H\phi_2(\hat{v}_H) = E_H \otimes (A_H + \lambda_1)\left(\sum_{K\in\mathcal{A}_x} A_K v_{HK} + \lambda_1 v_{HH}\right).$$

Thus, for $H \in \mathcal{A}_x \cap \tilde{\mathcal{A}}_y$, we have

$$\begin{aligned} G_H\phi_2(\hat{v}) + \lambda_2 G_H\hat{v}_H &= E_H \otimes (A_H + \lambda_1 + \lambda_2)\left(\sum_{K\in\mathcal{A}_x} A_K v_{HK} + \lambda_1 v_{HH}\right) \\ &= G'_H\phi_2(\hat{v}). \end{aligned}$$

As noticed before, for $H \in \mathcal{A}_x^c \cap \tilde{\mathcal{A}}_y$ we have $G_H = G'_H$. Hence we get

$$\phi_2 \circ \left(\sum_{H\in\tilde{\mathcal{A}}_y} \frac{\hat{G}_H}{y - b_H}\right)(\hat{v}) = \left(\sum_{H\in\tilde{\mathcal{A}}_y} \frac{G'_H}{y - b_H}\right) \circ \phi_2(\hat{v}),$$

which proves (27). Q.E.D.

It follows from Theorem 3.1 that the middle convolution in x_i-direction is invertible as a homomorphism of $\{A_H\,;\,H \in \mathcal{A}\}$-modules. Then combining with Theorem 1.1, we get the following assertion.

Theorem 3.2. *Assume that, after a linear transformation of coordinates of $\mathbb{C}^l$ if necessary, for some index i $(1 \le i \le l)$, the x_i-equation of the Pfaffian system (4) is irreducible and rigid as an ordinary differential equation. Then the system (4) can be obtained from a Pfaffian system of rank 1 by a finite iteration of middle convolutions in x_i-direction and additions.*

§4. Examples

Lauricella's hypergeometric series F_D is expressed by the integral

$$F_D(x_1, x_2, \ldots, x_n) = C\int_0^1 t^{a_0}(t-1)^{a_1}\prod_{j=1}^{n}(t - x_j)^{b_j}\,dt$$

with some constant C. A Pfaffian system in n variables for F_D can be obtained by the middle convolution in the following way.

We consider the function

$$u(x_1, x_2, \ldots, x_n) = {x_1}^{a_0}(x_1 - 1)^{a_1} \prod_{j=2}^{n} (x_1 - x_j)^{b_j},$$

where $a_0, a_1, b_2, \ldots, b_n \in \mathbb{C}$. This function satisfies the Pfaffian system

$$du = \left(a_0 d\log x_1 + a_1 d\log(x_1 - 1) + \sum_{j=2}^{n} b_j d\log(x_1 - x_j) \right) u \tag{28}$$

of rank 1. The singular locus of this system is given by the arrangement $\mathcal{A} = \{H_0, H_1, \ldots, H_n\}$, where

$$H_0 = \{x_1 = 0\},\ H_1 = \{x_1 = 1\},\ H_j = \{x_1 = x_j\} \quad (2 \le j \le n).$$

The middle convolution of the system (28) in x_1-direction with parameter b_1 makes a Pfaffian system

$$dV = \Omega V, \tag{29}$$

which is a Pfaffian system for F_D.

The singular locus of (29) is obtained as follows. Take any j $(2 \le j \le n)$. For each H_k $(0 \le k \le n,\ k \ne j)$, we have

$$\mathcal{C}_{H_k, x_j} = \{H_j\}$$

and

$$c_{H_k H_j} = \begin{cases} 0 & (k = 0), \\ 1 & (k = 1), \\ x_k & (2 \le k \le n,\ k \ne j). \end{cases}$$

Then the singular points of the x_j-equation of (29) is given by

$$x_j = x_1,\ x_j = 0,\ x_j = 1,\ x_j = x_k \quad (2 \le k \le n,\ k \ne j).$$

Hence the singular locus of the system (29) is given by the arrangement $\tilde{\mathcal{A}} = \mathcal{A} \cup \{H_{jk}\,;\, 2 \le j \le n,\ 0 \le k \le n,\ k \ne j\}$, where

$$H_{j0} = \{x_j = 0\},\ H_{j1} = \{x_j = 1\},\ H_{jk} = \{x_j = x_k\} \quad (2 \le k \le n,\ k \ne j).$$

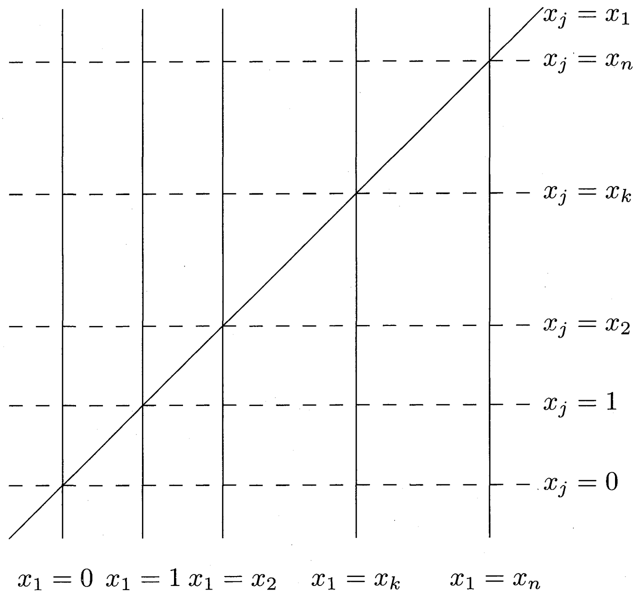

Fig. 2. Singular locus in $x_1 x_j$-plane

We have another example related to Appell's hypergeometric series F_4. We quote the results from our previous work [4].

We consider the KZ-type system (21) of rank 4, where $A_1, A_2, \ldots, A_5$ are given by

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1-\gamma & 0 & 0 \\ 0 & \epsilon & 0 & 1 \\ 0 & 0 & 0 & 1-\gamma \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \epsilon & 1 \\ 0 & 0 & 1-\gamma & 0 \\ 0 & 0 & 0 & 1-\gamma \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\alpha\beta & -\gamma' & \epsilon & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(\alpha+\epsilon)(\beta+\epsilon) & -\gamma' \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha\beta & \epsilon & -\gamma' & 0 \\ 0 & -(\alpha+\epsilon)(\beta+\epsilon) & 0 & -\gamma' \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon & -\epsilon & 0 \\ 0 & -\epsilon & \epsilon & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$\alpha, \beta, \gamma, \gamma' \in \mathbb{C}$, and

$$\epsilon := \gamma + \gamma' - \alpha - \beta - 1.$$

This system is obtained by Kato [6] as a transform of a Pfaffian system for Appell's F_4. The restriction of the system to the singular locus $x = y$ becomes a system of ordinary differential equations of rank 3, which is explicitly given by

$$\frac{du}{dx} = \left(\frac{C_1}{x} + \frac{C_2}{x-1}\right) u \tag{30}$$

with

$$C_1 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & \gamma' - \alpha - \beta & 1 \\ 0 & 0 & 2(1-\gamma) \end{pmatrix},$$
$$C_2 = \begin{pmatrix} 0 & 0 & 0 \\ -\alpha\beta & \gamma - \alpha - \beta - 1 & 0 \\ 0 & -2(\epsilon + \alpha)(\epsilon + \beta) & -2\gamma' \end{pmatrix}.$$

We see that the index of rigidity of the restriction is 0.

The middle convolution of (21) in x-direction with parameter $\lambda \neq 0, \alpha, \beta$ gives the system

$$dW = \left[B_1 \frac{dx}{x} + B_2 \frac{dy}{y} + B_3 \frac{dx}{x-1} + B_4 \frac{dy}{y-1} + B_5 \frac{d(x-y)}{x-y}\right] W \tag{31}$$

with the same singular locus, where

$$B_1 = \begin{pmatrix} \lambda+1-\gamma & 0 & -1 & \epsilon & \epsilon \\ 0 & \lambda+1-\gamma & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \epsilon & (\alpha+\epsilon)(\beta+\epsilon) & 0 & 0 & -\epsilon \\ 0 & 1-\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha\beta+\gamma'\epsilon & 0 \\ 0 & 0 & 0 & 1-\gamma & 0 \\ \gamma-1+\epsilon & (\alpha+\epsilon)(\beta+\epsilon) & 0 & 0 & 1-\gamma-\epsilon \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha\beta+(1-\gamma)\gamma' & (1-\gamma)\gamma'\epsilon & \lambda-\gamma' & 0 & \gamma'\epsilon \\ \epsilon & (\alpha+\epsilon)(\beta+\epsilon)+(1-\gamma)\gamma' & 0 & \lambda-\gamma' & -\epsilon \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & -\gamma' & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma'\epsilon & -\gamma'\epsilon \\ 0 & 0 & -1 & -\gamma'+\epsilon & \epsilon \\ 0 & 0 & 1 & -\epsilon & -\gamma'-\epsilon \end{pmatrix},$$

$$B_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1-\gamma-\epsilon & -(\alpha+\epsilon)(\beta+\epsilon) & -1 & \epsilon & \lambda+2\epsilon \end{pmatrix}.$$

As a restriction of (31) to the singular locus $x = y$, we get a system of ordinary differential equations of the rank 4. The explicit form is given by

$$\frac{dv}{dx} = \left(\frac{D_1}{x} + \frac{D_2}{x-1}\right) v, \tag{32}$$

where

$$D_1 = \begin{pmatrix} \lambda+1-c+\epsilon & \frac{\epsilon}{1-\gamma-\epsilon} & -\frac{1}{1-\gamma-\epsilon} & 0 \\ 0 & 1-\gamma & 0 & 0 \\ 0 & \alpha\beta+\gamma'\epsilon & 0 & 0 \\ 0 & -(\alpha+1-\gamma) & 0 & \lambda+2(1-\gamma) \end{pmatrix},$$

$$D_2 = \begin{pmatrix} -\gamma'-\epsilon & -\frac{\epsilon}{1-\gamma-\epsilon} & \frac{1}{1-\gamma-\epsilon} & 0 \\ -(\lambda+2\epsilon)\epsilon & \lambda+2(\epsilon-\gamma')-\frac{(1-\gamma)\epsilon}{1-\gamma-\epsilon} & -\frac{1-\gamma-2\epsilon}{1-\gamma-\epsilon} & -\frac{(\beta+1-\gamma)(1-\gamma)}{1-\gamma-\epsilon} \\ -(\lambda+2\epsilon)(\alpha\beta+(1-\gamma)\gamma') & -\frac{\epsilon(\alpha\beta+\gamma'(2-2\gamma-\epsilon)}{1-\gamma-\epsilon} & \lambda+\frac{\alpha\beta+\gamma'\epsilon}{1-\gamma-\epsilon} & \frac{(\beta+1-\gamma)(\alpha\beta+\gamma'\epsilon)}{1-\gamma-\epsilon} \\ (\lambda+2\epsilon)(\alpha+1-\gamma) & \frac{(\alpha+1-\gamma)\epsilon}{1-\gamma-\epsilon} & -\frac{\alpha+1-\gamma}{1-\gamma-\epsilon} & \frac{(\alpha+1-\gamma)(\beta+1-\gamma)}{1-\gamma-\epsilon} \end{pmatrix}.$$

We observe that the index of rigidity of the last system is -2 unless $\lambda = 2\gamma - 2$ or $\lambda = 2\gamma'$, in which cases we have the index of rigidity 0. This is remarkable, because the index of rigidity is invariant under the middle convolution for ordinary differential equations.

The system (30) appears in the conformal field theory [3], and the system (32) appears in the theory of Heckman–Opdam hypergeometric systems [8]. These systems may also be related to special solutions of difference Painlevé equations. Thus, we expect that the combination of prolongation-restriction process and the middle convolution will be a good tool which relate several interesting equations in various areas of mathematics and physics.

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