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Capelli Elements in the Classical Universal Enveloping Algebras

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For any complex classical group $G = O_N, Sp_N$ consider the ring $\mathrm{Z}(\mathfrak{g})$ of G-invariants in the corresponding enveloping algebra $\mathrm{U}(\mathfrak{g})$. Let u be a complex parameter. For each $n = 0, 1, 2, \ldots$ and every partition ν of n into at most N parts we define a certain rational function $Z_\nu(u)$ which takes values in $\mathrm{Z}(\mathfrak{g})$. Our definition is motivated by the works of Cherednik and Sklyanin on the reflection equation, and also by the classical Capelli identity. The degrees in $\mathrm{U}(\mathfrak{g})$ of the values of $Z_\nu(u)$ do not exceed n. We describe the images of these values in the n-th symmetric power of $\mathfrak{g}$. Our description involves the plethysm coefficients as studied by Littlewood, see Theorem 3.4 and Corollary 3.6.

§1. Capelli elements in the algebra $\mathrm{U}(\mathfrak{gl}_N)$

We work with the general linear Lie algebra $\mathfrak{gl}_N$ over the complex field $\mathbb{C}$. In this section we recall the definition from [OO1, S] of the Capelli elements in the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$. Here we also recall an explicit construction from [N2,O] of these elements.

Let the indices i, j run through the set $\{1, \ldots, N\}$. Let the vectors e_i form the standard basis in $\mathbb{C}^N$. We fix in the Lie algebra $\mathfrak{gl}_N$ the basis of the standard matrix units E_{ij}. We will also regard E_{ij} as generators of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$. Now choose the Borel subalgebra in $\mathfrak{gl}_N$ spanned by the elements E_{ij} with $i \leqslant j$. Then choose the basis $E_{11}, \ldots, E_{NN}$ in the corresponding Cartan subalgebra.

Let ν be any partition of n into at most N parts. We will write $\nu = (\nu_1, \ldots, \nu_N)$. Let U_ν be the irreducible $\mathfrak{gl}_N$-module of highest weight ν. The module U_ν appears in the decomposition of the n-th tensor power of the defining $\mathfrak{gl}_N$-module $\mathbb{C}^N$. It is called the *polynomial* $\mathfrak{gl}_N$-module corresponding to the partition ν.

There is a distinguished basis in the centre $\mathrm{Z}(\mathfrak{gl}_N)$ of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$, parametrized by the same partitions ν. The

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element C_ν of this basis is determined up to multipler from $\mathbb{C}$ by the following proposition. This proposition is due to Sahi [S, Theorem 1]. Consider the canonical ascending filtration on the algebra $\mathrm{U}(\mathfrak{gl}_N)$. With respect to this filtration the subspace $\mathfrak{gl}_N \subset \mathrm{U}(\mathfrak{gl}_N)$ has degree one.

Proposition 1.1. *There is an element C_ν in $\mathrm{Z}(\mathfrak{gl}_N)$ of degree at most n such that for any partition $\lambda = (\lambda_1, \ldots, \lambda_N)$ of not more than n we have $C_\nu \cdot U_\lambda \neq \{0\}$ if and only if $\lambda = \nu$.*

We will call $C_\nu \in \mathrm{Z}(\mathfrak{gl}_N)$ the *Capelli element* in the algebra $\mathrm{U}(\mathfrak{gl}_N)$ corresponding to the partition ν. The elements C_ν corresponding to the partitions $\nu = (1, \ldots, 1, 0, \ldots, 0)$ were studied by Capelli in [C]. In the case $\nu = (n, 0, \ldots, 0)$ they were studied in [N1]. An explicit formula for the eigenvalue of the central element C_ν in the $\mathfrak{gl}_N$-module U_λ for any λ and ν was given by Okounkov and Olshanski in [OO1]. Let us reproduce this formula, it will fix the multiplier from $\mathbb{C}$ up to which the element $C_\nu \in \mathrm{Z}(\mathfrak{gl}_N)$ has been determined so far.

Let $a = (a_1, a_2, \ldots)$ be an arbitrary sequence of complex numbers. For each $k = 0, 1, 2, \ldots$ introduce the k-th *generalized factorial power* $(u\,|\,a)^k = (u - a_1) \cdots (u - a_k)$ of the variable u. Consider the function in N independent variables $y_1, \ldots, y_N$

$$s_\nu(y_1, \ldots, y_N\,|\,a) = \frac{\det\big[\,(y_j|a)^{\nu_i+N-i}\,\big]}{\det\big[\,(y_j|a)^{N-i}\,\big]} \tag{1.1}$$

where the determinants are taken with respect to $i, j = 1, \ldots, N$. This function is a symmetric polynomial in $y_1, \ldots, y_N$ which is called the *generalized factorial Schur polynomial,* see [M, Example I.3.20]. Note that here the denominator

$$\det\big[\,(y_j|a)^{N-i}\,\big] \;=\; \prod_{i<j} (y_i - y_j)$$

is the Vandermonde determinant. Thus the denominator in (1.1) does not depend on the sequence a.

If $a = (0, 0, \ldots)$ the polynomial $s_\nu(y_1, \ldots, y_N\,|\,a)$ is the ordinary Schur polynomial $s_\nu(y_1, \ldots, y_N)$. For the general sequence a by (1.1)

$$s_\nu(y_1, \ldots, y_N\,|\,a) = s_\nu(y_1, \ldots, y_N) \,+\, \text{lower degree terms.}$$

Therefore all the polynomials $s_\nu(y_1, \ldots, y_N\,|\,a)$ where the partitions ν have not more than N parts, form a linear basis in the ring of symmetric polynomials in the variables $y_1, \ldots, y_N$ with complex coefficients.

Proposition 1.2. *The Capelli element $C_\nu \in \mathrm{Z}(\mathfrak{gl}_N)$ can be chosen so that its eigenvalues in the irreducible $\mathfrak{gl}_N$-modules U_λ are respectively*

$$s_\nu(\lambda_1+N-1,\lambda_2+N-2,\ldots,\lambda_N \mid 0,1,2,\ldots). \tag{1.2}$$

By the Harish-Chandra theorem [D, Theorem 7.4.5], the eigenvalue of any element from $\mathrm{Z}(\mathfrak{gl}_N)$ in the irreducible module U_λ is a symmetric polynomial in $\lambda_1+N-1,\lambda_2+N-2,\ldots,\lambda_N$ and all the symmetric polynomials arise in this way. The proof of Proposition 1.2 consists of a direct verification that when $\lambda_1+\ldots+\lambda_N \leqslant n$, the expression (1.2) vanishes unless $\lambda=\nu$. The details can be found in [OO1, Section 3].

An explicit formula for the element $C_\nu \in \mathrm{U}(\mathfrak{gl}_N)$ in terms of the generators E_{ij} was given by [N2, Theorem 5.3] and [O, Theorem 1.3]. It generalizes the formula from [C] for C_ν with $\nu=(1,\ldots,1,0,\ldots,0)$ and employs the classical results of Young [Y1,Y2] about the irreducible representations of the symmetric group S_n. Let us recall the relevant results from [Y1,Y2] here.

Let W_ν be the irreducible S_n-module corresponding to the partition ν. We identify the partition ν with its Young diagram. Fix the chain

$$S_1 \subset S_2 \subset \cdots \subset S_n \tag{1.3}$$

of subgroups with the standard embeddings. There is a decomposition of the space W_ν into the direct sum of one-dimensional subspaces associated with this chain. These subspaces are parametrized by the *standard tableaux* of shape ν. Each of these tableaux is a bijective filling of the boxes of the Young diagram ν with the numbers $1,\ldots,n$ such that in every row and column the numbers increase from left to right and from top to bottom respectively. Denote by $\mathcal{T}_\nu$ the set of these tableaux.

For every tableau $T \in \mathcal{T}_\nu$ define a one-dimensional subspace W_T in W_ν as follows. For any $p \in \{1,\ldots,n\}$ take the tableau obtained from T by removing each of the numbers $p+1,\ldots,n$. Let the Young diagram ω be its shape. The subspace W_T is contained in an irreducible S_p-submodule of W_ν corresponding to ω. Any basis of W_ν formed by vectors $w_T \in W_T$ is called a *Young basis.* Fix an S_n-invariant inner product $\langle\ ,\ \rangle_\nu$ in W_ν. The subspaces W_T are then pairwise orthogonal. We shall be assuming that $\langle w_T, w_T\rangle_\nu = 1$ for each tableau $T \in \mathcal{T}_\nu$.

For any tableau $T \in \mathcal{T}_\nu$ consider the normalized diagonal matrix element of the S_n-module W_ν corresponding to the vector w_T

$$\Phi_T = \frac{\dim W_\nu}{n!} \sum_{\sigma\in S_n} \langle w_T, \sigma\cdot w_T\rangle_\nu\, \sigma \in \mathbb{C}\cdot S_n\,. \tag{1.4}$$

There is an explicit formula for this element of the group ring $\mathbb{C} \cdot S_n$. This formula is the most simple when $T \in \mathcal{T}_\nu$ is the *column tableau.* This tableau is obtained by filling the boxes of the diagram ν with $1, \ldots, n$ by columns from left to right, downwards in each column. We shall denote this tableau by T_c. Let S_ν and S'_ν be the subgroups in S_n preserving the collections of numbers appearing respectively in every row and column of the tableau T_c. Take the elements of the group ring $\mathbb{C} \cdot S_n$

$$\Theta_\nu = \sum_{\sigma \in S_\nu} \sigma \quad \text{and} \quad \Theta'_\nu = \sum_{\sigma \in S'_\nu} \sigma \cdot \operatorname{sgn} \sigma .$$

As usual, we denote by $\nu'_1, \nu'_2, \ldots$ the column lenghts of the diagram ν. Then by [Y1]

$$\Phi_{T_c} = \frac{\dim W_\nu}{n!} \cdot \frac{\Theta'_\nu \Theta_\nu \Theta'_\nu}{\nu'_1! \, \nu'_2! \ldots} .$$

There is an alternative description of the one-dimensional subspace W_T in W_ν due to Jucys [J]. Consider the sum of transpositions

$$z_p = (1,p) + (2,p) + \ldots + (p-1,p) \in \mathbb{C} \cdot S_n .$$

The elements $z_1, \ldots, z_n \in \mathbb{C} \cdot S_n$ are called the *Jucys-Murphy elements* corresponding to the standard chain (1.3). They pairwise commute. Fix a tableau $T \in \mathcal{T}_\nu$. For every $r = 1, \ldots, n$ put $c_p = k - l$ if the number p appears in the k-th column and l-th row of the tableau T. The number c_p is called the *content* of the box of the diagram ν occupied by p. Here on the left we show the column tableau of shape $\nu = (4,3,1)$:

1	4	6	8
2	5	7	
3			

0	1	2	3
-1	0	1	
-2			

On the right we have indicated the contents of the boxes of the Young diagram $\nu = (4,3,1)$. So here we get $(c_1, \ldots, c_8) = (0,-1,-2,1,0,2,1,3)$. Observe that the standard tableau $T \in \mathcal{T}_\nu$ can be always recovered from the sequence of contents $c_1, \ldots, c_n$. The next lemma is contained in [J].

Lemma 1.3. *We have $z_p \cdot w_T = c_p w_T$ in W_ν for any $p = 1, \ldots, n$.*

Let us now reproduce the explicit formula from [N2,O] for the element $C_\nu \in \mathrm{U}(\mathfrak{gl}_N)$. Consider the permutational action of the symmetric group S_n in the tensor product $(\mathbb{C}^N)^{\otimes n}$. Denote by Y_T the linear operator in $(\mathbb{C}^N)^{\otimes n}$ corresponding of the element (1.4). The image of this operator is equivalent to U_ν as a $\mathfrak{gl}_N$-module, see [W, Section IV.4]. Moreover, by definition we have the equality $Y_T^2 = Y_T$.

Further, denote by ι_p the embedding of the algebra $\mathrm{End}(\mathbb{C}^N)$ into the tensor product $\mathrm{End}(\mathbb{C}^N)^{\otimes n}$ as the p-th tensor factor:

$$\iota_p(X) = 1^{\otimes (p-1)} \otimes X \otimes 1^{\otimes (n-p)}; \qquad p = 1, \ldots, n. \tag{1.5}$$

We will use this notation throughout the present article. Now put

$$E(u) \;=\; -u + \sum_{ij} E_{ij} \otimes E_{ji} \;\in\; \mathrm{End}(\mathbb{C}^N) \otimes \mathrm{U}(\mathfrak{gl}_N)[u], \tag{1.6}$$

$$E_p(u) \;=\; (\iota_p \otimes \mathrm{id})\big(E(u)\big) \in \mathrm{End}(\mathbb{C}^N)^{\otimes n} \otimes \mathrm{U}(\mathfrak{g})[u]. \tag{1.7}$$

Let $\mathrm{tr} : \mathrm{End}(\mathbb{C}^N) \to \mathbb{C}$ be the usual matrix trace, so that $\mathrm{tr}(E_{ij})$ equals the Kronecker delta δ_{ij}. Now consider the product

$$E_1(u_1) \cdots E_n(u_n) \;\in\; \mathrm{End}(\mathbb{C}^N)^{\otimes n} \otimes \mathrm{U}(\mathfrak{gl}_N)[u_1, \ldots, u_n].$$

Theorem 1.4. *For any standard tableau $T \in \mathcal{T}_\nu$ we have*

$$C_\nu = (\mathrm{tr}^{\otimes n} \otimes \mathrm{id})\big(Y_T \otimes 1 \cdot E_1(c_1) \cdots E_n(c_n)\big). \tag{1.8}$$

The proofs of this theorem given in [N2,O] were rather involved. A more elegant proof was later found by Molev [M2, Theorem 8.2]. All these results were based on the notion of a fusion procedure introduced by Cherednik in [C2]. We keep using this notion in the present article.

Consider again the group ring $\mathbb{C} \cdot S_n$. For every two distinct indices $p, q = 1, \ldots, n$ introduce the rational function of two complex variables u, v valued in $\mathbb{C} \cdot S_n$

$$\varphi_{pq}(u,v) = 1 - \frac{(p,q)}{u - v}.$$

As direct calculations show, these rational functions satisfy the equations

$$\varphi_{pq}(u,v)\,\varphi_{pr}(u,w)\,\varphi_{qr}(v,w) = \varphi_{qr}(v,w)\,\varphi_{pr}(u,w)\,\varphi_{pq}(u,v) \tag{1.9}$$

for all pairwise distinct indices p, q, r. Consider the rational function of u, v, w appearing at either side of (1.9). The factor $\varphi_{pr}(u,w)$ in (1.9) has a pole at $u = w$. However, we have the following lemma.

Lemma 1.5. *The restriction of* (1.9) *to the set of all* (u,v,w) *such that $v = w \pm 1$, is regular at $u = w$.*

Proof. Under the condition $v = w \pm 1$ the rational function (1.9) can be written as

$$\left(1 - \frac{(p,q)+(p,r)}{u-w\mp 1}\right)\cdot\left(1\mp(q,r)\right)$$

which is a rational function of u,w manifestly regular at $u=w$ $\square$

Using Lemma 1.5 one can prove the next proposition, for details see [N2, Proposition 2.12]. Let the superscript $^\vee$ denote the group embedding $S_n \to S_{n+1}$ determined by the assignment $(p,q)\mapsto(p+1,q+1)$.

Proposition 1.6. *We have the identity in the algebra* $\mathbb{C}\cdot S_{n+1}(u)$

$$\left(1-\sum_{p=1}^{n}\frac{(1,p+1)}{u}\right)\cdot\Phi_T^\vee = \varphi_{12}(u,c_1)\cdots\varphi_{1,n+1}(u,c_n)\cdot\Phi_T^\vee\,.$$

The proof of the next proposition is similar and will be also omitted.

Proposition 1.7. *We have the identity in the algebra* $\mathbb{C}\cdot S_{n+1}(u)$

$$\left(1+\sum_{p=1}^{n}\frac{(p,n+1)}{u}\right)\cdot\Phi_T = \varphi_{1,n+1}(-c_1,u)\cdots\varphi_{n,n+1}(-c_n,u)\cdot\Phi_T\,.$$

We will also use an alternative definition of the element $\Phi_T\in\mathbb{C}\cdot S_n$ due to Cherednik [C2]. Suppose the numbers $1,\ldots,n$ appear respectively in the rows $l_1,\ldots,l_n$ of the standard tableau T. Order the set of all pairs p,q with $1\leqslant p<q\leqslant n$ lexicographically.

Theorem 1.8. *The rational function of u defined as the ordered product in $\mathbb{C}\cdot S_n(u)$ of the elements $\varphi_{pq}(c_p+l_p u, c_q+l_q u)$ over the pairs p,q is regular at $u=0$, and takes at $u=0$ the value $\Phi_T\cdot n!/\dim W_\nu$.*

One can prove this theorem by again using Lemma 1.5. This proof is contained in [N2, Section 2]. Another proof can be found in [JKMO].

We will close this section with a generalization of Theorem 1.4. Let us consider for a standard tableau $T\in\mathcal{T}_\nu$ the element of $\mathrm{U}(\mathfrak{gl}_N)[u]$

$$(\mathrm{tr}^{\otimes n}\otimes\mathrm{id})\big(Y_T\otimes 1\cdot E_1(u+c_1)\cdots E_n(u+c_n)\big)\,. \tag{1.10}$$

Corollary 1.9. *The element* (1.10) *belongs to* $\mathrm{Z}(\mathfrak{gl}_N)[u]$ *and does not depend on the choice of a tableau $T\in\mathcal{T}_\nu$. The eigenvalue of* (1.10) *in the irreducible $\mathfrak{gl}_N$-module U_λ is*

$$s_\nu(\lambda_1-u+N-1,\lambda_2-u+N-2,\ldots,\lambda_N-u\mid 0,1,2,\ldots)\,.$$

Proof. For any complex value of the parameter u consider the automorphism of the unital algebra $\mathrm{U}(\mathfrak{gl}_N)$ determined by the assignment $E_{ij} \mapsto E_{ij} - u \cdot \delta_{ij}$. The element (1.10) can be obtained by applying this automorphism to the left hand side of (1.8), see the definition (1.6). So the first statement of Corollary 1.9 follows from Theorem 1.4. By pulling back the $\mathfrak{gl}_N$-module U_λ through that automorphism we obtain the irreducible $\mathfrak{gl}_N$-module of highest weight $(\lambda_1 - u, \ldots, \lambda_N - u)$. The second statement of Corollary 1.9 now follows from Proposition 1.2 □

The principal aim of this article is to introduce the analogues of the elements (1.10) for the remaining classical Lie algebras $\mathfrak{so}_N$ and $\mathfrak{sp}_N$.

§2. Traceless tensors in the space $(\mathbb{C}^N)^{\otimes m}$

We will regard the orthogonal and symplectic Lie algebras $\mathfrak{so}_N$ and $\mathfrak{sp}_N$ as subalgebras in $\mathfrak{gl}_N$. From now on we will let the indices i, j run through the set $\{-M, \ldots, -1, 1, \ldots, M\}$ if $N = 2M$ and through the set $\{-M, \ldots, -1, 0, 1, \ldots, M\}$ if $N = 2M + 1$. Let e_i be the elements of the standard basis in $\mathbb{C}^N$. We will realize the complex orthogonal group O_N as the subgroup in GL_N preserving the symmetric bilinear form $\langle e_i, e_j \rangle = \delta_{i,-j}$ on $\mathbb{C}^N$. The complex symplectic group Sp_N will be realized as the subgroup in GL_N preserving the alternating form $\langle e_i, e_j \rangle = \delta_{i,-j} \cdot \operatorname{sgn} i$.

Let G be any of the subgroups O_N, Sp_N in GL_N. Denote by $\mathfrak{g}$ the corresponding Lie subalgebra in $\mathfrak{gl}_N$. Put $\varepsilon_{ij} = \operatorname{sgn} i \cdot \operatorname{sgn} j$ if $G = Sp_N$ and $\varepsilon_{ij} = 1$ if $G = O_N$. The Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}_N$ is then spanned by the elements

$$F_{ij} = E_{ij} - \varepsilon_{ij} \cdot E_{-j,-i}\,.$$

We will also regard F_{ij} as generators of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$. We choose the Borel subalgebra in $\mathfrak{g}$ spanned by the elements F_{ij} with $i \leqslant j$. Let us fix the basis $F_{-M,-M}, \ldots, F_{-1,-1}$ in the corresponding Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Any weight $\mu = (\mu_1, \ldots, \mu_M)$ of $\mathfrak{h}$ will be taken with respect to this basis. The half-sum of the positive roots of $\mathfrak{h}$ is

$$\rho = (\varepsilon + M - 1, \varepsilon + M - 2, \ldots, \varepsilon)$$

where $\varepsilon = 0\,, \frac{1}{2}\,, 1$ for $\mathfrak{g} = \mathfrak{so}_{2M}\,, \mathfrak{so}_{2M+1}\,, \mathfrak{sp}_{2M}$ respectively.

Now we assume that μ is a partition of m with at most M parts. Then μ can be regarded as a dominant weight of $\mathfrak{h}$. Let V_μ be the irreducible $\mathfrak{g}$-module of the highest weight μ. Note that if $\mathfrak{g} = \mathfrak{so}_{2M}$ then $\mu^* = (\mu_1, \ldots, \mu_{M-1}, -\mu_M)$ is again a dominant weight of $\mathfrak{h}$. We will also consider the corresponding irreducible $\mathfrak{so}_{2M}$-module V_{μ^*}. It is obtained from the module V_μ via the conjugation in $\mathfrak{so}_{2M} \subset \mathrm{End}(\mathbb{C}^N)$ by

$$E_{1,-1} + E_{-1,1} + E_{22} + E_{-2,-2} + \ldots + E_{MM} + E_{-M,-M} \in O_N\,.$$

All these irreducible $\mathfrak{g}$-modules appear in the decomposition of the m-th tensor power of the identity $\mathfrak{g}$-module $\mathbb{C}^N$. Take any two distinct indices p and q from the set $\{1, \ldots, m\}$. By applying the G-invariant bilinear form $\langle\ ,\ \rangle$ on $\mathbb{C}^N$ to an element $t \in (\mathbb{C}^N)^{\otimes m}$ in the p-th and q-th tensor factors we obtain a certain element $t' \in (\mathbb{C}^N)^{\otimes(m-2)}$. Then the element t is called *traceless* if $t' = 0$ for all possible indices $p \neq q$.

Now fix any embedding of the irreducible $\mathfrak{gl}_N$-module U_μ to $(\mathbb{C}^N)^{\otimes m}$. The subspace $U_\mu \cap V$ in U_μ is preserved by the action of the subalgebra $\mathfrak{g} \subset \mathfrak{gl}_N$. For $\mathfrak{g} = \mathfrak{so}_{2M+1}, \mathfrak{sp}_{2M}$ this subspace is isomorphic to V_μ as $\mathfrak{g}$-module. For $\mathfrak{g} = \mathfrak{so}_{2M}$ it is isomorphic to V_μ only if $\mu_M = 0$. Otherwise $U_\mu \cap V$ splits into the direct sum of the $\mathfrak{so}_{2M}$-modules V_μ and V_{μ^*}. All these statements are contained in [W, Section V.9].

We denote by $\mathrm{Z}(\mathfrak{g})$ the ring of invariants in the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ with respect to the adjoint action of the group G. The ring $\mathrm{Z}(\mathfrak{g})$ coincides with the centre of $\mathrm{U}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{so}_{2M+1}, \mathfrak{sp}_{2M}$ but is strictly contained in the centre of $\mathrm{U}(\mathfrak{g})$ when $\mathfrak{g} = \mathfrak{so}_{2M}$. Then any element of $\mathrm{Z}(\mathfrak{so}_{2M})$ acts in the irreducible $\mathfrak{so}_{2M}$-modules V_μ and V_{μ^*} by the same scalars.

There is a distinguished basis in the vector space $\mathrm{Z}(\mathfrak{g})$ analogous to the basis of the Capelli elements C_ν in $\mathrm{Z}(\mathfrak{gl}_N)$. This basis is labelled by the partitions μ and was introduced by Okounkov and Olshanski by generalizing Proposition 1.1. The element B_μ of this basis is determined up to multipler from $\mathbb{C}$ by the next proposition [OO2, Theorem 2.3]. Consider the canonical ascending filtration on the algebra $\mathrm{U}(\mathfrak{g})$. With respect to this filtration the subspace $\mathfrak{g} \subset \mathrm{U}(\mathfrak{g})$ has degree one.

Proposition 2.1. *There exists an element B_μ in $\mathrm{Z}(\mathfrak{g})$ of degree at most $2m$ such that for any partition $\lambda = (\lambda_1, \ldots, \lambda_M)$ of not more than m we have $B_\mu \cdot V_\lambda \neq \{0\}$ if and only if $\lambda = \mu$.*

Explicit formula for the eigenvalue of the element $B_\mu \in \mathrm{Z}(\mathfrak{g})$ in the irreducible $\mathfrak{g}$-module V_λ for any λ and μ has been also given in [OO2]. We will reproduce this formula, it fixes the multiplier from $\mathbb{C}$ up to which the element $B_\mu \in \mathrm{Z}(\mathfrak{g})$ is determined by Proposition 2.1. This formula again employs the definition (1.1).

Proposition 2.2. *The element $B_\mu \in \mathrm{Z}(\mathfrak{g})$ can be chosen so that* *its eigenvalues in the irreducible $\mathfrak{g}$-modules V_λ are respectively*

$$s_\mu\big((\lambda_1+\rho_1)^2,\dots,(\lambda_M+\rho_M)^2 \mid \varepsilon^2,(\varepsilon+1)^2,\dots\big).$$

The proof of this proposition does not differ significantly from that of Proposition 1.2. For details see [OO2, Theorem 2.5]. A certain explicit expression for the element $B_\mu \in \mathrm{U}(\mathfrak{g})$ in terms of the generators F_{ij} has been recently given by Olshanski in [O2]. This is an analogue of the expression [OO1, Theorem 14.1] for the element $C_\nu \in \mathrm{U}(\mathfrak{gl}_N)$ which is more complicated than (1.8). An analogue of the formula (1.8) for B_μ with the general partition μ is unknown. For $\mu=(1,\dots,1,0,\dots,0)$ and $\mu=(m,0,\dots,0)$ this analogue was given in [MN]. In the present article we will consider a natural generalization of the construction [MN]. But in general it yields elements of the ring $\mathrm{Z}(\mathfrak{g})$ different from B_μ.

Similarly to (1.5), for any element $X \in \mathrm{End}(\mathbb{C}^N)^{\otimes 2}$ and any two distinct indices $p,q \in \{1,\dots,n\}$ with fixed n we will denote

$$X_{pq}=(\iota_p\otimes\iota_q)(X) \in \mathrm{End}(\mathbb{C}^N)^{\otimes n}.$$

Along with Lemma 1.5, we will use one more simple observation. Denote

$$F(u) = -u-\eta+\sum_{ij} E_{ij}\otimes F_{ji} \in \mathrm{End}(\mathbb{C}^N)\otimes\mathrm{U}(\mathfrak{gl}_N)[u] \tag{2.1}$$

where we set $\eta=\frac{1}{2},-\frac{1}{2}$ for $\mathfrak{g}=\mathfrak{so}_N,\mathfrak{sp}_N$ respectively. Let

$$\tilde{E}(u) = -u+\sum_{ij} \varepsilon_{ij}\cdot E_{ij}\otimes E_{-i,-j} \in \mathrm{End}(\mathbb{C}^N)\otimes\mathrm{U}(\mathfrak{gl}_N)[u]$$

be the element obtained from $E(u)$ by applying the transposition with respect to the bilinear $\langle\ ,\ \rangle$ in the tensor factor $\mathrm{End}(\mathbb{C}^N)$. Now consider

$$\frac{\tilde{E}(\eta-u)\,E(\eta+u)}{u-\eta} \in \mathrm{End}(\mathbb{C}^N)\otimes\mathrm{U}(\mathfrak{gl}_N)(u). \tag{2.2}$$

We have the standard representation $\mathrm{U}(\mathfrak{gl}_N)\to\mathrm{End}(\mathbb{C}^N)^{\otimes m}$ which makes the element (2.2) acting in the space $(\mathbb{C}^N)^{\otimes(m+1)}$. The element $F(u)\in\mathrm{End}(\mathbb{C}^N)\otimes\mathrm{U}(\mathfrak{g})[u]$ also acts in the space $(\mathbb{C}^N)^{\otimes(m+1)}$ and the latter action preserves the subspace $\mathbb{C}^N\otimes V$. Here is a simple lemma.

Lemma 2.3. *Action of the element* (2.2) *in the space* $(\mathbb{C}^N)^{\otimes(m+1)}$ *preserves the subspace* $\mathbb{C}^N\otimes V$. *The action of the element* $F(u)$ *in this subspace coincides with the action of* (2.2).

Proof. Consider the elements of the algebra $\mathrm{End}\,(\mathbb{C}^N)^{\otimes 2}$

$$P=\sum_{ij} E_{ij}\otimes E_{ji} \qquad \text{and} \qquad Q=\sum_{ij} \varepsilon_{ij}\cdot E_{ij}\otimes E_{-i,-j}\,.$$

The element P corresponds to the exchange operator $e_i\otimes e_j \mapsto e_j\otimes e_i$ in $(\mathbb{C}^N)^{\otimes 2}$. The element Q is obtained from P by applying to either tensor factor of $\mathrm{End}\,(\mathbb{C}^N)^{\otimes 2}$ transposition with respect to $\langle\ ,\ \rangle$. Observe that

$$PQ=QP=\begin{cases} Q & \text{if } \mathfrak{g}=\mathfrak{so}_N\,, \\ -Q & \text{if } \mathfrak{g}=\mathfrak{sp}_N\,. \end{cases} \tag{2.3}$$

Further, by the definition of a traceless tensor $t\in \mathrm{End}\,(\mathbb{C}^N)^{\otimes m}$ we have the equality $Q_{pq}t=0$ for any two distinct indices $p,q\in\{1,\ldots,m\}$.

By definition the image of the element $F(u)\in \mathrm{End}\,(\mathbb{C}^N)\otimes \mathrm{U}(\mathfrak{g})\,[u]$ in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes(m+1)}\,[u]$ under the representation $\mathrm{U}(\mathfrak{g})\to \mathrm{End}\,(\mathbb{C}^N)^{\otimes m}$ is

$$P_{12}+\ldots+P_{1,m+1}-Q_{12}-\ldots-Q_{1,m+1}-u-\eta\,. \tag{2.4}$$

On the other hand, the image of the element (2.2) in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes(m+1)}\,(u)$ under the representation $\mathrm{U}(\mathfrak{gl}_N)\to \mathrm{End}\,(\mathbb{C}^N)^{\otimes m}$ is the product

$$\left(1+\frac{Q_{12}+\ldots+Q_{1,m+1}}{u-\eta}\right)\cdot\big(P_{12}+\ldots+P_{1,m+1}-u-\eta\big)\,.$$

By (2.3) and the definition of η this product equals (2.4) plus the sum

$$\sum_{p\neq q}\frac{Q_{1,q+1}\,P_{1,p+1}}{u-\eta}=\sum_{p\neq q}\frac{P_{1,p+1}\,Q_{p+1,q+1}}{u-\eta}\,.$$

But the action of the latter sum in $\mathbb{C}^N\otimes V$ is identically zero $\square$

Using this lemma we can easily prove the following proposition. It is a particular case of a more general result from [O1]. Denote

$$R(u,v)\,=1-\frac{P}{u-v} \qquad \text{and} \qquad \tilde{R}(u,v)\,=1+\frac{Q}{u+v}$$

in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes 2}(u,v)$. The first of these two functions is the *rational Yang R-matrix*. For any two distinct indices $p,q\in\{1,\ldots,n\}$ the element $R_{pq}(u,v)\in \mathrm{End}\,(\mathbb{C}^N)^{\otimes n}(u,v)$ corresponds to $\varphi_{pq}(u,v)\in\mathbb{C}\cdot S_n(u,v)$ under the permutational action of the symmetric group S_n in $(\mathbb{C})^{\otimes n}$.

Similarly to (1.7), for any fixed n and every index $p=1,\ldots,n$ let $F_p(u)\in \mathrm{End}(\mathbb{C}^N)^{\otimes n}\otimes \mathrm{U}(\mathfrak{g})[u]$ and $\tilde{E}_p(u)\in \mathrm{End}(\mathbb{C}^N)^{\otimes n}\otimes \mathrm{U}(\mathfrak{gl}_N)[u]$

be the images of $F(u)$ and $\tilde{E}(u)$ with respect to the embedding $\iota_p \otimes \mathrm{id}$. In the equations (2.5) to (2.9) below we will write $R(u,v)$ and $\tilde{R}(u,v)$ instead of $R(u,v)\otimes 1$ and $\tilde{R}(u,v)\otimes 1$ in $\mathrm{End}(\mathbb{C}^N)^{\otimes 2}\otimes \mathrm{U}(\mathfrak{g})$ for short.

Proposition 2.4. *We have the relation in* $\mathrm{End}(\mathbb{C}^N)^{\otimes 2}\otimes \mathrm{U}(\mathfrak{g})\,(u,v)$

$$R(u,v)\,F_1(u)\,\tilde{R}(u,v)\,F_2(v) = F_2(v)\,\tilde{R}(u,v)\,F_1(u)\,R(u,v)\,. \tag{2.5}$$

Proof. This proposition can be verified by direct calculation. Here we will give a conceptual proof which goes back to the origin [C2, S2] of the *reflection equation* (2.5). In the algebra $\mathrm{End}(\mathbb{C}^N)^{\otimes 2}\otimes \mathrm{U}(\mathfrak{gl}_N)\,(u,v)$

$$R(u,v)\,E_1(u)\,E_2(v) = E_2(v)\,E_1(u)\,R(u,v)\,, \tag{2.6}$$

$$R(u,v)\,\tilde{E}_1(-u)\,\tilde{E}_2(-v) = \tilde{E}_2(-v)\,\tilde{E}_1(-u)\,R(u,v)\,, \tag{2.7}$$

$$\tilde{E}_1(-u)\,\tilde{R}(u,v)\,E_2(v) = E_2(v)\,\tilde{R}(u,v)\,\tilde{E}_1(-u)\,, \tag{2.8}$$

$$E_1(u)\,\tilde{R}(u,v)\,\tilde{E}_2(-v) = \tilde{E}_2(-v)\,\tilde{R}(u,v)\,E_1(u)\,. \tag{2.9}$$

The relation (2.6) is well known and can be easily verified. The relation (2.7) is obtained from (2.6) by applying in the tensor factor $\mathrm{U}(\mathfrak{gl}_N)$ the automorphism $E_{ij}\mapsto -\varepsilon_{ij}\cdot E_{-j,-i}$. Applying to (2.6) transposition with respect to $\langle\ ,\ \rangle$ in the first tensor factor of $\mathrm{End}(\mathbb{C}^N)^{\otimes 2}$ we obtain (2.8). By applying to (2.7) transposition with respect to $\langle\ ,\ \rangle$ in the second tensor factor of $\mathrm{End}(\mathbb{C}^N)^{\otimes 2}$ we obtain (2.9).

Using (2.6) to (2.9) we get the equality in $\mathrm{End}(\mathbb{C}^N)^{\otimes 2}\otimes \mathrm{U}(\mathfrak{gl}_N)\,(u,v)$

$$R(u,v)\,\tilde{E}_1(\eta-u)(u)\,E_1(\eta+u)\,\tilde{R}(u,v)\,\tilde{E}_2(\eta-v)\,E_2(\eta+v) =$$
$$\tilde{E}_2(\eta-v)\,E_2(\eta+v)\,\tilde{R}(u,v)\,\tilde{E}_1(\eta-u)(u)\,E_1(\eta+u)\,R(u,v)\,.$$

The intersection of the kernels of all the representations $\mathrm{U}(\mathfrak{g})\mapsto \mathrm{End}\,V$ for $n=1,2,\ldots$ is zero [D, Theorem 2.5.7], therefore (2.5) follows from the above equality in $\mathrm{End}(\mathbb{C}^N)^{\otimes 2}\otimes \mathrm{U}(\mathfrak{gl}_N)\,(u,v)$ by Lemma 2.3 □

We will now introduce the main object of our study in this article. Let ν be any partition of n with at most N parts. Let T be any standard tableau of shape ν. It determines the sequence of contents $c_1,\ldots,c_n$. Consider the element of the algebra $\mathrm{End}(\mathbb{C}^N)^{\otimes n}\otimes \mathrm{U}(\mathfrak{g})(u)$

$$F_T(u) = (Y_T\otimes 1)\cdot \prod_{p=1}^{n}\Big(1+\frac{Q_{1p}\otimes 1+\ldots+Q_{p-1,p}\otimes 1}{2u+c_p}\Big)\,F_p(u+c_p)$$

where the (noncommuting) factors corresponding to $s=1,\ldots,n$ are arranged from the left to right. For example, for each of the partitions

$\nu=(2)$ and $\nu=(1,1)$ there is only one standard tableau of shape ν. For these partitions we get the elements of the algebra $\mathrm{End}(\mathbb{C}^N)^{\otimes 2}\otimes\mathrm{U}(\mathfrak{g})(u)$

$$F_T(u)=\bigl(1\pm P\otimes 1\bigr)\cdot F_1(u)\cdot\Bigl(1+\frac{Q\otimes 1}{2u\pm 1}\Bigr)\cdot F_2(u\pm 1)$$

respectively. Our main object of study is the rational function of u

$$Z_\nu(u)=(\mathrm{tr}^{\otimes n}\otimes\mathrm{id})\bigl(F_T(u)\bigr) \tag{2.10}$$

which by definition takes values in $\mathrm{U}(\mathfrak{g})$, cf. (1.10). As we will show later, this function does not depend on the choice of the tableau $T\in\mathcal{T}_\nu$.

Proposition 2.5. *The function $Z_\nu(u)$ takes values in the ring* $\mathrm{Z}(\mathfrak{g})$.

Proof. We regard the group G as a subgroup in $GL_N\subset\mathrm{End}(\mathbb{C}^N)$. Consider the adjoint action ad of the group G in the enveloping algebra $\mathrm{U}(\mathfrak{g})$. Observe that by the definition (2.1) for any element $g\in G$

$$(\mathrm{id}\otimes\mathrm{ad}\,g)\bigl(F(u)\bigr)=g\otimes 1\cdot F(u)\cdot g^{-1}\otimes 1\,.$$

Elements $Y_T\,,Q_{1p},\,\ldots,Q_{p-1,p}\in\mathrm{End}(\mathbb{C}^N)^{\otimes n}$ commute with $g^{\otimes n}$. So

$$(\mathrm{id}\otimes\mathrm{ad}\,g)\bigl(F_T(u)\bigr)=\bigl(g^{\otimes n}\otimes 1\bigr)\cdot F_T(u)\cdot\bigl((g^{-1})^{\otimes n}\otimes 1\bigr)\,.$$

Hence

$$(\mathrm{tr}^{\otimes n}\otimes\mathrm{ad}\,g)\bigl(F_T(u)\bigr)=(\mathrm{tr}^{\otimes n}\otimes\mathrm{id})\bigl(F_T(u)\bigr)\quad\square$$

We need one more formula for the element $F_T(u)$. It has motivated our definition of $Z_\nu(u)$. We will keep to the convention used in the definition of $F_T(u)$: in any product over a certain index the noncommuting factors are arranged from the left to the right, as this index increases.

Proposition 2.6. *Element $F_T(u)\in\mathrm{End}(\mathbb{C}^N)^{\otimes n}\otimes\mathrm{U}(\mathfrak{g})(u)$ equals*

$$(Y_T\otimes 1)\cdot\prod_{p=1}^{n}\Bigl(\prod_{q=1}^{p-1}\tilde{R}_{qp}(u+c_q,u+c_p)\otimes 1\Bigr)F_p(u+c_p)\,.$$

Proof. We use the induction on n. In the case $n=1$ the required equality is tautological. Assume we have the required equality for some partition ν of $n\geqslant 1$. Take any standard tableau U with $n+1$ boxes and not more than N rows, such that by removing the box with number $n+1$ we get T. Let c be the content of the removed box. Consider the projector $Y_U \in \operatorname{End}(\mathbb{C}^N)^{\otimes(n+1)}$. It is divisible on the right by $Y_T \otimes \mathrm{id}$. So by definition the element $F_T(u) \in \operatorname{End}(\mathbb{C}^N)^{\otimes(n+1)} \otimes \mathrm{U}(\mathfrak{g})(u)$ equals

$$(Y_U \otimes 1) \cdot F_T(u) \cdot \Big(1 + \frac{Q_{1,n+1} \otimes 1 + \ldots + Q_{n,n+1} \otimes 1}{2u + c}\Big) \cdot F_{n+1}(u + c)$$

But the element $F_T(u) \in \operatorname{End}(\mathbb{C}^N)^{\otimes n} \otimes \mathrm{U}(\mathfrak{g})(u)$ is divisible on the right by $Y_T \otimes 1$. This follows from Theorem 1.8 and Proposition 2.4, see also [MNO, Section 4.2]. The alternative expression for $F_U(u)$ is now provided by inductive assumption and by the identity in $\operatorname{End}(\mathbb{C}^N)^{\otimes(n+1)}(v)$

$$\begin{aligned} (2.11) \qquad & (Y_T \otimes \mathrm{id}) \cdot \big(1 + (Q_{1,n+1} + \ldots + Q_{n,n+1})/v\big) = \\ & (Y_T \otimes \mathrm{id}) \cdot \tilde{R}_{1,n+1}(c_1, v) \cdots \tilde{R}_{n,n+1}(c_n, v) \end{aligned}$$

with $v = 2u + c$. Let us verify this identity. By applying to both sides of the equation (2.11) the transposition with respect to $\langle\ ,\ \rangle$ in each of the first n tensor factors in $\operatorname{End}(\mathbb{C}^N)^{\otimes(n+1)}$ we get

$$\begin{aligned} (2.12) \qquad & \big(1 + (P_{1,n+1} + \ldots + P_{n,n+1})/v\big) \cdot (Y_T \otimes \mathrm{id}) = \\ & R_{1,n+1}(-c_1, v) \cdots R_{n,n+1}(-c_n, v) \cdot (Y_T \otimes \mathrm{id}) . \end{aligned}$$

We used the fact that the element $Y_T \otimes \mathrm{id} \in \operatorname{End}(\mathbb{C}^N)^{\otimes(n+1)}$ is invariant under this transposition. But (2.12) is provided by Proposition 1.7 □

Theorem 2.7. *$Z_\nu(u)$ does not depend on the choice of $T \in \mathcal{T}_\nu$.*

Proof. Any standard tableau of the shape ν can be obtained from the column tableau T_c by a chain of transformations $T \mapsto T'$ where the entries of the tableaux $T, T' \in \mathcal{T}_\nu$ differ by a single transposition $(r, r+1)$ such that $l_r > l_{r+1}$ for the tableau T. Let $c_1', \ldots, c_n'$ be the sequence of contents of the tableau T'. It is obtained from the sequence $c_1, \ldots, c_n$ by exchanging the terms c_r and c_{r+1}. Note that here we have $|c_r - c_{r+1}| > 1$, put $d = (c_r - c_{r+1})^{-1}$. Due to [Y2, Theorem IV] we have the relation

$$(2.13) \qquad \Phi_{T'} = \big((r, r+1) + d\big) \, \frac{\Phi_T}{1 - d^2} \, \big((r, r+1) + d\big)$$

in the group ring $\mathbb{C} \cdot S_n$, see the definition (1.4). Let X be the product $PR(c_{r+1}, c_r) = P + d$ in $\operatorname{End}(\mathbb{C}^N)^{\otimes 2}$, then the relation (2.13) implies

$$(2.14) \qquad Y_{T'} = X_{r,r+1} \, \frac{Y_T}{1 - d^2} \, X_{r,r+1}$$

in $\operatorname{End}(\mathbb{C}^N)^{\otimes n}$. On the other hand, by using Proposition 2.4 we obtain

$$X_{r,r+1} \cdot \prod_{p=1}^{n} \Big(\prod_{q=1}^{p-1} \tilde{R}_{1p}(u+c'_q, u+c'_p) \otimes 1 \Big) F_p(u+c'_p) =$$

$$\prod_{p=1}^{n} \Big(\prod_{q=1}^{p-1} \tilde{R}_{1p}(u+c_q, u+c_p) \otimes 1 \Big) F_p(u+c_p) \cdot X_{r,r+1} \tag{2.15}$$

in $\operatorname{End}(\mathbb{C}^N)^{\otimes n} \otimes \mathrm{U}(\mathfrak{g})(u)$. Combining the relations (2.14), (2.15) we get

$$F_{T'}(u) = \big(X_{r,r+1} \otimes 1 \big) \frac{F_T(u)}{1-d^2} \big(X_{r,r+1} \otimes 1 \big) .$$

We have the relation $X^2 = 2dX + 1 - d^2$ in the algebra $\operatorname{End}(\mathbb{C}^N)^{\otimes 2}$. The vectors w_T and $w_{T'}$ of the Young basis in the S_n-module W_ν are orthogonal, so (2.14) implies the equality $Y_T\, X_{r,r+1}\, Y_T = 0$. Therefore

$$Y_T\, X_{r,r+1}^2\, Y_T = (1-d^2) \cdot Y_T .$$

But the element $F_T(u) \in \operatorname{End}(\mathbb{C}^N)^{\otimes n} \otimes \mathrm{U}(\mathfrak{g})(u)$ is divisible by $Y_T \otimes 1$ on the right as well as on the left, see the proof of Proposition 2.6. Thus

$$(\operatorname{tr}^{\otimes n} \otimes \operatorname{id}) \big(F_{T'}(u) \big) = (\operatorname{tr}^{\otimes n} \otimes \operatorname{id}) \Big(\big(X_{r,r+1} \otimes 1 \big) \frac{F_T(u)}{1-d^2} \big(X_{r,r+1} \otimes 1 \big) \Big)$$

$$= (\operatorname{tr}^{\otimes n} \otimes \operatorname{id}) \Big(\big(X_{r,r+1}^2 \otimes 1 \big) \frac{F_T(u)}{1-d^2} \Big) = (\operatorname{tr}^{\otimes n} \otimes \operatorname{id}) \big(F_T(u) \big) \quad \square$$

§3. Leading terms of the element $Z_\nu(u)$

Throughout this section $\nu = (\nu_1, \ldots, \nu_N)$ will be any partition of n into not more than N parts. However, we will always have $n = 2m$. We fixed the basis $F_{-M,-M}, \ldots, F_{-1,-1}$ in the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

Consider again the standard ascending filtration of the algebra $\mathrm{U}(\mathfrak{g})$

$$\mathrm{U}_0(\mathfrak{g}) \subset \mathrm{U}_1(\mathfrak{g}) \subset \mathrm{U}_2(\mathfrak{g}) \subset \ldots \subset \mathrm{U}(\mathfrak{g}) .$$

Here $\mathrm{U}_0(\mathfrak{g}) = \mathbb{C}, \mathrm{U}_1(\mathfrak{g}) = \mathfrak{g}$. By definition the subspace $\mathrm{U}_n(\mathfrak{g}) \subset \mathrm{U}(\mathfrak{g})$ consists of all the elements with degree not more than n. We will identify the quotient space $\mathrm{U}_n(\mathfrak{g}) / \mathrm{U}_{n-1}(\mathfrak{g})$ with the subspace in the symmetric algebra $\mathrm{S}(\mathfrak{g})$ consisting of the homogeneous elements of degree n.

By (2.10) we get $Z_\nu(u) \in \mathrm{U}_n(\mathfrak{g}) \otimes \mathbb{C}(u)$. The image of $Z_\nu(u)$ in

$$\big(\mathrm{U}_n(\mathfrak{g}) / \mathrm{U}_{n-1}(\mathfrak{g}) \big) \otimes \mathbb{C}(u) \subset \mathrm{S}(\mathfrak{g}) \otimes \mathbb{C}(u)$$

is a homogeneous polynomial in $F_{ij} \in \mathfrak{g}$ of degree n with the coefficients from $\mathbb{C}(u)$. Due to Proposition 2.5 this polynomial is invariant under the adjoint action of the group G in $\mathrm{S}(\mathfrak{g})$. By the Chevalley theorem [D, Theorem 7.3.5] this polynomial is uniquely determined by its image

$$f_\nu(x_1, \ldots, x_M \,|\, u) \in \mathbb{C}[x_1, \ldots, x_M] \otimes \mathbb{C}(u) \tag{3.1}$$

with respect to the homomorphism $\eta : \mathrm{S}(\mathfrak{g}) \to \mathbb{C}[x_1, \ldots, x_M]$ defined by the assignment $F_{ij} \mapsto 0$ if $i \neq j$ or if $i = j = 0$, and by

$$F_{-M,-M} \mapsto x_1 \ , \ \ldots, \ F_{-1,-1} \mapsto x_M \,.$$

Moreover, the image (3.1) is a symmetric polynomial in $x_1^2, \ldots, x_M^2$. Our present aim is to determine the polynomial (3.1) for any partition ν of $n = 2m$. In particular, we will describe the partitions ν where the polynomial (3.1) is not identically zero.

Denote by Λ_M the ring of symmetric polynomials in $x_1, \ldots, x_M$ with complex coefficients. For any partition $\rho = (\rho_1, \rho_2, \ldots)$ put

$$p_\rho(x_1, \ldots, x_M) = \prod_{k=1}^{\ell(\rho)} \left(x_1^{\rho_k} + \ldots + x_M^{\rho_k} \right) \in \Lambda_M \,.$$

As usual, here $\ell(\rho)$ is the number of non-zero parts in the partition ρ.

We will use some elementary facts from the representation theory of the symmetric group S_{2n}. Consider the hyperoctahedral group H_n as the subgroup in S_{2n} that centralizes the product of transpositions $(1, n+1) \cdots (n, 2n) \in S_{2n}$. Thus $H_n = S_n \ltimes (\mathbb{Z}_2)^n$ where the subgroup $S_n \subset S_{2n}$ acts on $1, \ldots, 2n$ by simultaneous permutations of $1, \ldots, n$ and $n+1, \ldots, 2n$. Here the subgroup $(\mathbb{Z}_2)^n \subset S_{2n}$ is generated by the pairwise commuting transpositions $(1, n+1), \ldots, (n, 2n)$. Consider the one-dimensional representations χ_+ and χ_- of the group H_n which are trivial on its subgroup S_n while $\chi_\pm : (s, n+s) \mapsto \pm 1$ respectively. Take the corresponding minimal idempotents in the group ring $\mathbb{C} \cdot H_n$

$$h_\pm = \frac{1}{n!\,2^n} \sum_{\sigma \in H_n} \chi_\pm(\sigma)\, \sigma \,.$$

Note that the intersection of $h_-(\mathbb{C} \cdot S_{2n}) h_+$ with $h_+(\mathbb{C} \cdot S_{2n}) h_-$ is zero.

Proposition 3.1. *We can uniquely determine two linear maps*

$$\mathrm{ch}\colon h_-(\mathbb{C} \cdot S_{2n}) h_+ \longrightarrow \Lambda_M \quad \textit{and} \quad \mathrm{ch}\colon h_+(\mathbb{C} \cdot S_{2n}) h_- \longrightarrow \Lambda_M$$

by setting

$$\mathrm{ch}(h_- \sigma h_+) = \mathrm{ch}(h_+ \sigma h_-) = p_\rho(x_1^2, \ldots, x_M^2) \cdot 2^{\ell(\rho)}$$

for any permutation σ of $1, \ldots, n$ *with the cycle lengths* $2\rho_1, 2\rho_2, \ldots$.

Proof. Any double coset of the subgroup H_n in S_{2n} contains a permutation that acts on the numbers $n+1, \ldots, 2n$ trivially. Moreover, all permutations σ of $1, \ldots, n$ with the same cycle lengths belong to the same double coset. If any of these lengths is odd then $h_- \sigma h_+ = h_+ \sigma h_- = 0$. Now for each partition $\rho = (\rho_1, \rho_2, \ldots)$ of m choose a permutation σ with the cycle lengths $2\rho_1, 2\rho_2, \ldots$. All the corresponding elements $h_- \sigma h_+ \in \mathbb{C} \cdot S_{2n}$ are linearly independent. Therefore our definitions of two linear maps ch are self-consistent $\square$

We will call the two linear maps in Proposition 3.1 the *characteristic maps,* see [M, Section VII.2]. Now fix any standard tableau $T \in \mathcal{T}_\nu$ and take the corresponding minimal idempotent $\Phi_T \in \mathbb{C} \cdot S_n$. Regard Φ_T as an element of the group ring $\mathbb{C} \cdot S_{2n}$ where the subgroup $S_n \subset S_{2n}$ acts on the numbers $n+1, \ldots, 2n$ trivially. Consider the product

$$\Psi_T(u) = \prod_{p=1}^{n} \left(1 + \frac{(1, n+p) + \ldots + (p-1, n+p)}{2u + c_p} \right) \cdot \Phi_T \tag{3.2}$$

in $\mathbb{C}(u) \cdot S_{2n}$ where the (non-commuting) factors corresponding to the indices $p = 1, \ldots, n$ are as usual arranged from the left to the right. Computation of the homogeneous polynomial (3.1) in $x_1, \ldots, x_M$ hinges on the following observation.

Proposition 3.2. *For* $\mathfrak{g} = \mathfrak{so}_N$ *and* $\mathfrak{g} = \mathfrak{sp}_N$ *the polynomials* (3.1) *coincide with the images in* $\Lambda_M \otimes \mathbb{C}(u)$ *of* $h_- \Psi_T(u) h_+$ *and* $h_+ \Psi_T(u) h_-$ *respectively under the characteristic maps.*

Proof. Take the permutational action of the group S_{2n} in the space $(\mathbb{C}^N)^{\otimes 2n}$. Then the image in $\operatorname{End}(\mathbb{C}^N)^{\otimes 2n}(u)$ of $\Psi_T(u)$ is the product

$$\prod_{p=1}^{n} \left(1 + \frac{P_{1,n+p} + \ldots + P_{p-1,n+p}}{2u + c_p} \right) \cdot \left(Y_T \otimes \mathrm{id}^{\otimes n} \right).$$

Decompose this product with respect to the standard basis in the space $\operatorname{End}(\mathbb{C}^N)^{\otimes 2n}$. We get the sum

$$\sum_{i_1 \ldots i_{2n}} \sum_{j_1 \ldots j_{2n}} \psi^{i_1 \ldots i_{2n}}_{j_1 \ldots j_{2n}}(u) \cdot E_{i_1 j_1} \otimes \cdots \otimes E_{i_{2n} j_{2n}}$$

where the coefficients are certain rational functions of u valued in $\mathbb{C}$.

We will put $\varepsilon_i = \operatorname{sgn} i$ if $\mathfrak{g} = \mathfrak{sp}_N$ and set $\varepsilon_i = 1$ if $\mathfrak{g} = \mathfrak{so}_N$. Then $\varepsilon_{ij} = \varepsilon_i \varepsilon_j$ by definition. Denote

$$I_{ij}(u) = \varepsilon_i \cdot \big(F_{j,-i} - (u+\eta)\, \delta_{j,-i} \big) \in \mathrm{U}(\mathfrak{g})[u]\,, \qquad J_{ij} = \varepsilon_i \cdot \delta_{i,-j}\,.$$

By the definition of $F_T(u) \in \operatorname{End}(\mathbb{C}^N)^{\otimes n} \otimes \mathrm{U}(\mathfrak{g})(u)$ the element $Z_\nu(u)$ equals

$$\sum_{i_1 \ldots i_{2n}} \sum_{j_1 \ldots j_{2n}} \psi^{\,i_1 \ldots i_{2n}}_{j_1 \ldots j_{2n}}(u) \cdot I_{i_{n+1} i_1}(u + c_1)\, J_{j_{n+1} j_1} \cdots I_{i_{2n} i_n}(u + c_n)\, J_{j_{2n} j_n}$$

where we employed the definition (2.1) and the fact that the elements P, Q are obtained from each other by applying to the second tensor factor of $\operatorname{End}(\mathbb{C}^N)^{\otimes 2}$ the transposition with respect to $\langle\ ,\ \rangle$. This expression for $Z_\nu(u)$ shows that the polynomial $f_\nu(x_1, \ldots, x_M \,|\, u)$ equals the sum

$$\sum_{i_1 \ldots i_n} \sum_{j_1 \ldots j_n} \psi^{\,i_1 \ldots i_n, -i_1 \ldots -i_n}_{j_1 \ldots j_n, -j_1 \ldots -j_n}(u) \cdot \eta(F_{i_1 i_1})\, \varepsilon_{i_1} \varepsilon_{j_1} \cdots \eta(F_{i_n i_n})\, \varepsilon_{i_n} \varepsilon_{j_n}\,.$$

The product $\eta(F_{i_1 i_1})\, \varepsilon_{i_1} \varepsilon_{j_1} \cdots \eta(F_{i_n i_n})\, \varepsilon_{i_n} \varepsilon_{j_n}$ is invariant under the permutations of the indices $i_1, \ldots, i_n$ and of the indices $j_1, \ldots, j_n$. For $\mathfrak{g} = \mathfrak{so}_N$ it is also invariant under any substitution $j_p \mapsto -j_p$ with $p = 1, \ldots, n$ but changes the sign under the substitution $i_p \mapsto -i_p$. Inversely, for $\mathfrak{g} = \mathfrak{sp}_N$ this product is invariant under any substitution $i_p \mapsto -i_p$ but changes the sign under the substitution $j_p \mapsto -j_p$.

Now take any permutation of the first n tensor factors in $(\mathbb{C}^N)^{\otimes 2n}$ with the cycle lengths $2\rho_1, 2\rho_2, \ldots$ and decompose it in $\operatorname{End}(\mathbb{C}^N)^{\otimes 2n}$ as

$$\sum_{i_1 \ldots i_{2n}} \sum_{j_1 \ldots j_{2n}} \delta^{\,i_1 \ldots i_{2n}}_{j_1 \ldots j_{2n}} \cdot E_{i_1 j_1} \otimes \cdots \otimes E_{i_{2n} j_{2n}}$$

where each of the coefficients equals 0 or 1. It remains to show that

$$\sum_{i_1 \ldots i_n} \sum_{j_1 \ldots j_n} \delta^{\,i_1 \ldots i_n, -i_1 \ldots -i_n}_{j_1 \ldots j_n, -j_1 \ldots -j_n} \cdot \eta(F_{i_1 i_1})\, \varepsilon_{i_1 j_1} \cdots \eta(F_{i_n i_n})\, \varepsilon_{i_n j_n} \tag{3.3}$$

then equals $p_\rho(x_1^2, \ldots, x_M^2) \cdot 2^{\ell(\rho)}$ which is evident. Indeed, the latter expression and the sum (3.3) both are multiplicative with respect to the decomposition of our permutation of the first n tensor factors in $(\mathbb{C}^N)^{\otimes 2n}$ into the product of cycles. So we can assume that there is one single cycle of length $n = 2m$. In this case

$$\delta^{\,i_1 \ldots i_n, -i_1 \ldots -i_n}_{j_1 \ldots j_n, -j_1 \ldots -j_n} = \begin{cases} 1 & \text{if } i_1 = j_1 = \ldots = i_n = j_n\,, \\ 0 & \text{otherwise} \end{cases}$$

and (3.3) equals

$$\sum_i \left(\eta(F_{ii})\right)^n = 2\left(x_1^n + \ldots + x_M^n\right) \quad \square$$

Consider the two elements $h_-\,\Psi_T(u)\,h_+$ and $h_+\,\Psi_T(u)\,h_-$ of the ring $\mathbb{C}(u)\cdot S_{2n}$. According to Proposition 3.2, the first element corresponds to the case $\mathfrak{g}=\mathfrak{so}_N$ while the second corresponds to $\mathfrak{g}=\mathfrak{sp}_N$. We will evaluate the images of these two elements under the corresponding characteristic maps by studying their actions in irreducible S_{2n}-modules.

Let ω be any partition of $2n$. The irreducible S_{2n}-module W_ω contains a non-zero vector w_+ such that $\sigma\cdot w_+=\chi_+(\sigma)\,w_+$ for any $\sigma\in H_n$, if and only if every row of the Young diagram of ω has even length. Then the vector $w_+\in W_\nu$ is unique up to a scalar multiplier, and we will assume that $\langle w_+,w_+\rangle=1$. The module W_ω contains a non-zero vector w_- with $\sigma\cdot w_-=\chi_-(\sigma)\,w_-$ for any $\sigma\in H_n$, if and only if every column of ω has even length. The vector $w_-\in W_\nu$ is then unique up to a scalar multiplier, and we will assume that $\langle w_-,w_-\rangle=1$. All these facts are well known, see for instance [M, Section VII.2].

Now suppose that $\omega=(2\mu_1,2\mu_1,2\mu_2,2\mu_2,\ldots)$ for a certain partition $\mu=(\mu_1,\mu_2,\ldots)$ of m. We do not impose any restriction on the number of parts in μ yet. The partition ω satisfies both conditions above, so we have non-zero vectors $w_+,w_-\in W_\omega$. Let $b_1,\ldots,b_m$ be the contents of the diagram μ ordered arbitrarily.

Let χ_ν be the character of the irreducible S_n-module W_ν, take the element

$$\mathrm{X}_\nu=\frac{1}{n!}\sum_{\sigma\in S_n}\chi_\nu(\sigma)\,\sigma\;\in\mathbb{C}\cdot S_n\,.$$

We will also regard X_ν as an element of the group ring $\mathbb{C}\cdot S_{2n}$ by using the standard embedding $S_n\to S_{2n}$. Where the double signs $\pm$ and $\mp$ appear in the next proposition, one should simultaneously take only the upper signs or only the lower signs. Recall that in this section $n=2m$.

Proposition 3.3. *Action of the element $h_\mp\,\Psi_T(u)\,h_\pm\in\mathbb{C}(u)\cdot S_{2n}$ in the module W_ω coincides with the action of $h_\mp\,\mathrm{X}_\nu\,h_\pm\in\mathbb{C}\cdot S_{2n}$ times*

$$\frac{(u+b_1)(u+b_1\pm 1/2)\cdots(u+b_m)(u+b_m\pm 1/2)}{(u+c_1/2)(u+c_2/2)\cdots(u+c_{n-1}/2)(u+c_n/2)}\;\in\mathbb{C}(u)\,. \tag{3.4}$$

Proof. For each $p=1,\ldots,n$ consider the elements of the ring $\mathbb{C}\cdot S_{2n}$

$$z'_p=\sum_{q=1}^{p-1}(n+q,n+p)\quad\text{and}\quad z''_p=\sum_{q=1}^{p-1}(n+q,n+p)+(q,n+p)\,.$$

The elements $z_1', \ldots, z_n'$ are the images in $\mathbb{C} \cdot S_{2n}$ of the Jucys-Murphy elements $z_1, \ldots, z_n \in \mathbb{C} \cdot S_n$ under the embedding

$$S_n \to S_{2n} : (q, p) \mapsto (n+q, n+p) . \tag{3.5}$$

In particular, the elements $z_1', \ldots, z_n'$ pairwise commute. Note that the elements $z_1'', \ldots, z_n''$ also pairwise commute. The definition (3.2) can be now rewritten as

$$\Psi_T(u) = \prod_{p=1}^{n} \left(1 + \frac{z_p'' - z_p'}{2u + c_p} \right) \cdot \Phi_T . \tag{3.6}$$

But for any p the element z_p' commutes with each of $z_{p+1}'', \ldots, z_n''$. On the other hand, due to Lemma 1.3 we have the equalities in $\mathbb{C} \cdot S_{2n}$

$$z_p' \Phi_T \, h_\pm = \Phi_T \, z_p \, h_\pm = c_p \cdot \Phi_T \, h_\pm .$$

Therefore (3.6) implies the equality in the ring $\mathbb{C}(u) \cdot S_{2n}$

$$\Psi_T(u) \, h_\pm = \frac{(2u + z_1'') \cdots (2u + z_n'')}{(2u + c_1) \cdots (2u + c_n)} \, \Phi_T \, h_\pm . \tag{3.7}$$

The standard chain of subgroups (1.3) corresponds to the natural ordering of the numbers $1, \ldots, n$. Now consider the chain of subgroups

$$S_1 \subset S_2 \subset \ldots \subset S_{2n-1} \subset S_{2n}$$

corresponding the ordering $n+1, 1, n+2, 2, \ldots, 2n, n$. The elements $z_1'', \ldots, z_n'' \in \mathbb{C} \cdot S_{2n}$ are the Jucys-Murphy elements corresponding to the latter chain with the indices $1, 3, \ldots, 2n-1$. Take the Young basis in W_ν corresponding to this chain of subgroups in S_{2n}. The vectors w_U of this basis are parametrized by standard tableaux U of shape ω with the entries $1, \ldots, 2n$. But by [BG, Theorem 3.4] the vector $w_- \in W_\omega$ is a linear combination of the vectors w_U where $1, 3, \ldots, 2n-1$ occupy the first, third, ... rows of the tableau U. The collection of contents of the boxes in these rows is $2b_1, 2b_1+1, \ldots, 2b_m, 2b_m+1$. By Lemma 1.3 action of $h_- (2u + z_1'') \cdots (2u + z_n'')$ in W_ω coincides with the action of

$$(2u + 2b_1)(2u + 2b_1 + 1) \ldots (2u + 2b_m)(2u + 2b_m + 1) \cdot h_- .$$

Similarly, the vector $w_+ \in W_\omega$ is a linear combination of the vectors w_U where $1, 3, \ldots, 2n-1$ occupy the first, third, ... columns of the tableau U. The collection of contents of the boxes in these columns is

$2b_1, 2b_1 - 1, \ldots, 2b_m, 2b_m - 1$. Again due to Lemma 1.3 the action of $h_+(2u + z_1'') \cdots (2u + z_n'')$ in W_ω coincides with the action of

$$(2u + 2b_1)(2u + 2b_1 - 1) \ldots (2u + 2b_m)(2u + 2b_m - 1) \cdot h_+ .$$

Thus by (3.7) the action of the element $h_\mp \Psi_T(u) h_\pm$ in the module W_ν coincides with the action of $h_\mp \Phi_T h_\pm$ multiplied by the product (3.4).

To complete the proof of Proposition 3.3 it remains to observe that for any $\sigma \in S_n \subset S_{2n}$ we have $h_\mp \Phi_T h_\pm = h_\mp \sigma \Phi_T \sigma^{-1} h_\pm$. Therefore

$$h_\mp \Phi_T h_\pm = \frac{1}{n!} \sum_{\sigma \in S_n} h_\mp \sigma \Phi_T \sigma^{-1} h_\pm = h_\mp \mathrm{X}_\nu h_\pm \quad \square$$

Let us now formulate the main result of this section. Consider again the ring Λ_N of symmetric polynomials in the variables $y_1, \ldots, y_N$. We assume that $x_1, \ldots, x_M$ are independent of those N variables. Equip the vector space Λ_N with the standard inner product, so that the Schur polynomials $s_\nu(y_1, \ldots, y_N)$ where ν runs through the set of partitions with not more than N parts, constitute an orthonormal basis in Λ_N.

Symmetric polynomial $s_\mu(y_1^2, \ldots, y_N^2)$ is the *plethysm* of the Schur polynomial $s_\mu(y_1, \ldots, y_N)$ with the power sum $y_1^2 + \ldots + y_N^2$. Expand

$$s_\mu(y_1^2, \ldots, y_N^2) = \sum_\nu L_{\mu\nu}\, s_\nu(y_1, \ldots, y_N) \tag{3.8}$$

in Λ_N with respect to the basis of Schur polynomials. The polynomials $p_\rho(y_1, \ldots, y_N)$ form an othogonal basis in Λ_N. If $\rho_1 + \rho_2 + \ldots = m$ and the number of permutations in S_m with the cycle lengths $\rho_1, \rho_2, \ldots$ is $m!/z_\rho$ then the squared norm of $p_\rho(y_1, \ldots, y_N)$ is z_ρ. Further, then

$$s_\mu(y_1, \ldots, y_N) = \sum_\rho \chi_\mu^\rho\, p_\rho(y_1, \ldots, y_N)/z_\rho \tag{3.9}$$

where χ_μ^ρ denotes the value of the irreducible character χ_μ of S_m on a permutation with the cycle lengths $\rho_1, \rho_2, \ldots$. Therefore we have

$$L_{\mu\nu} = \sum_\rho \chi_\mu^\rho \chi_\nu^{2\rho} / z_\rho . \tag{3.10}$$

As usual, we denote $2\rho = (2\rho_1, 2\rho_2, \ldots)$. Note that $z_{2\rho} = 2^{\ell(\rho)} z_\rho$ then.

Combinatorial description of the coefficients $L_{\mu\nu}$ in the expansion (3.8) has been provided in [L, Section 5]. Another description of these coefficients is given by [CL, Theorem 5.3]. In particular, if $L_{\mu\nu} \neq 0$ then the Young diagram of ν can be split into horizontal and vertical blocks of two boxes each. These blocks are called *dominoes*, see [BG].

We put $\eta = \frac{1}{2}$ if $\mathfrak{g} = \mathfrak{so}_N$ and put $\eta = -\frac{1}{2}$ if $\mathfrak{g} = \mathfrak{sp}_N$. Recall that $c_1, \ldots, c_n$ are the contents of a standard tableau T of shape ν. The contents $b_1, \ldots, b_m$ of the boxes of μ have been ordered arbitrarily.

Theorem 3.4. *The polynomial $f_\nu(x_1, \ldots, x_M \,|\, u)$ equals the sum over all partitions μ of m into not more than M parts, of the products*

$$\frac{(u+b_1)(u+b_1+\eta)\cdots(u+b_m)(u+b_m+\eta)}{(u+c_1/2)(u+c_2/2)\cdots(u+c_{n-1}/2)(u+c_n/2)}\, L_{\mu\nu}\, s_\mu(x_1^2, \ldots, x_M^2)\,.$$

Proof. In this proof the upper signs in $\pm$ and $\mp$ correspond to the case $\mathfrak{g} = \mathfrak{so}_N$ while the lower signs correspond to $\mathfrak{g} = \mathfrak{sp}_N$. Initially let μ run through the set of all partitions of m, without any restriction on the number of parts. The elements

$$\Gamma_\mu = \frac{\dim W_\omega}{(2n)!} \sum_{\tau \in S_{2n}} \langle\, w_\mp\,, \tau \cdot w_\pm \,\rangle_\omega\, h_\mp \tau\, h_\pm \in \mathbb{C} \cdot S_{2n} \tag{3.11}$$

form a basis in the vector space $h_\mp\, (\mathbb{C} \cdot S_{2n})\, h_\pm$. Let us expand

$$h_\mp\, \Psi_T(u)\, h_\pm = \sum_\mu f_{\mu\nu}(u)\, \Gamma_\mu \tag{3.12}$$

with respect to this basis and compute the coefficients $f_{\mu\nu}(u) \in \mathbb{C}(u)$. The element $h_\mp \tau\, h_\pm$ acts in the S_{2n}-module W_ω as the linear operator $\langle\, \tau \cdot w_\pm\,, w_\mp \,\rangle_\omega E$ where $E : w \mapsto \langle\, w\,, w_\pm \,\rangle_\omega\, w_\mp$ for any vector $w \in W_\omega$.

The element Γ_μ acts as the operator E in the module W_ω and vanishes in any other irreducible S_{2n}-module. Denote by $d_{\mu\nu}(u)$ the rational function (3.4). By Proposition 3.3 and by the definition of X_ν

$$f_{\mu\nu}(u) = \frac{d_{\mu\nu}(u)}{n!} \sum_{\sigma \in S_n} \chi_\nu(\sigma)\, \langle\, \sigma \cdot w_\pm\,, w_\mp \,\rangle_\omega\,. \tag{3.13}$$

Here the factor $\langle\, \sigma \cdot w_\pm\,, w_\mp \,\rangle_\omega$ may be non-zero only if the permutation σ has the cycle lengths $2\rho_1, 2\rho_2, \ldots$ for some partition ρ of m. Then

$$\langle\, \sigma \cdot w_\pm\,, w_\mp \,\rangle_\omega = I_\mu \cdot 2^{\ell(\rho)}\, \chi_\mu^\rho \tag{3.14}$$

where I_μ depends only on the choice of the vectors $w_+\,, w_- \in W_\omega$ and

$$|\, I_\mu\,|^2 = \frac{(2n)!}{\dim W_\omega \cdot (2^n\, n!)^2}\,. \tag{3.15}$$

This result was independently obtained by Ivanov [I, Theorem 3.9] and Rains [R, Corollary 7.6]. Using (3.10) and (3.13) along with this result,

$$f_{\mu\nu}(u) \;=\; d_{\mu\nu}(u)\, I_\mu \cdot \sum_\rho \chi_\mu^\rho \chi_\nu^{2\rho} / z_\rho \;=\; d_{\mu\nu}(u)\, I_\mu\, L_{\mu\nu}\,. \tag{3.16}$$

To complete the proof of Theorem 3.4 it now remains to apply the characteristic map to each side of the equality (3.12). By Proposition 3.2 on the left-hand side we get the polynomial $f_\nu(x_1, \ldots, x_M \,|\, u)$. There are exactly $(2^n n!)^2/(4^{\ell(\rho)} z_\rho)$ elements in the double coset of the subgroup H_n in S_{2n} containing the permutation of $1, \ldots, n$ with the cycle lenghts $2\rho_1, 2\rho_2, \ldots$. By the definition (3.11) and again by (3.14),(3.15)

$$\mathrm{ch}(\Gamma_\mu) \;=\; I_\mu^{-1} \cdot \sum_\rho \chi_\mu^\rho\, p_\rho(x_1^2, \ldots, x_M^2)/z_\rho \;=\; I_\mu^{-1} \cdot s_\mu(x_1^2, \ldots, x_M^2)\,,$$

here we have also used Proposition 3.1 and the classical expansion (3.9). Thus the expression (3.16) for the coefficient in (3.12) shows that

$$f_\nu(x_1, \ldots, x_M \,|\, u) \;=\; \sum_\mu d_{\mu\nu}(u)\, I_\mu\, L_{\mu\nu} \cdot \mathrm{ch}(\Gamma_\mu) \;=$$
$$\sum_\mu d_{\mu\nu}(u)\, L_{\mu\nu}\, s_\mu(x_1^2, \ldots, x_M^2)\,.$$

The latter sum can be restricted to the partitions μ with not more than M parts, since for the other partitions we have $s_\mu(x_1^2, \ldots, x_M^2) = 0$ $\square$

Corollary 3.5. *If the polynomial* (3.1) *corresponding to* ν *is not identically zero, then the Young diagram of* ν *splits into dominoes.*

One can reformulate Theorem 3.4 as follows, cf. [OO2, Theorem 1.2]. Let us denote by $b_\mu(u)$ and $c_\nu(u)$ the numerator and the denominator of the fraction in (3.4). The upper signs in $b_\mu(u)$ correspond to $\mathfrak{g} = \mathfrak{so}_N$ while the lower signs correspond to $\mathfrak{g} = \mathfrak{sp}_N$.

Corollary 3.6. *For any fixed positive integers* m *and* N *we have*

$$\sum_\nu c_\nu(u)\, f_\nu(x_1, \ldots, x_M \,|\, u)\, s_\nu(y_1, \ldots, y_N) \;=$$
$$\sum_\mu b_\mu(u)\, s_\mu(x_1^2, \ldots, x_M^2)\, s_\mu(y_1^2, \ldots, y_N^2)\,. \tag{3.17}$$

where ν *and* μ *range respectively over all partitions of* $n = 2m$ *with at most* N *parts and all partitions of* m *with at most* $M = [N/2]$ *parts.*

Proof. By Theorem 3.4 for any partition ν of $n = 2m$ into not more than N parts the product $c_\nu(u)\, f_\nu(x_1, \ldots, x_M)\, s_\nu(y_1, \ldots, y_N)$ equals

$$\sum_{\mu} b_\mu(u)\, L_{\mu\nu}\, s_\mu(x_1^2, \ldots, x_M^2)\, s_\nu(y_1, \ldots, y_N)\,.$$

Taking here the sum over ν we obtain (3.17) by the definition (3.8) $\square$

We will complete this article with the following two examples. First, let us put $\mu = (m, 0, \ldots, 0)$ and $\nu = (2m, 0, \ldots, 0)$. Then the element $B_\mu \in \mathrm{Z}(\mathfrak{g})$ described in Section 2, coincides with the value of $Z_\nu(u)$ at $u = -m - \frac{1}{2}$ for $\mathfrak{g} = \mathfrak{so}_N$ and with the value of $(u + m - \frac{1}{2})/(u - \frac{1}{2}) \cdot Z_\nu(u)$ at the point $u = -m + \frac{1}{2}$ for $\mathfrak{g} = \mathfrak{sp}_N$; see [MN,Theorem 3.3].

Second, put $\mu = (1, \ldots, 1, 0, \ldots, 0)$ and $\nu = (1, \ldots, 1, 0, \ldots, 0)$ where the part 1 appears m and $2m$ times respectively. Then the element $(-1)^m B_\mu \in \mathrm{Z}(\mathfrak{g})$ coincides with the value of $Z_\nu(u)$ at $u = m + \frac{1}{2}$ for $\mathfrak{g} = \mathfrak{sp}_N$ and with the value of $(u - m + \frac{1}{2})/(u + \frac{1}{2}) \cdot Z_\nu(u)$ at the point $u = m - \frac{1}{2}$ for $\mathfrak{g} = \mathfrak{so}_N$; see [MN,Theorem 3.4].

It would be interesting to establish a link between our functions $Z_\nu(u)$ and the elements $B_\mu \in \mathrm{Z}(\mathfrak{g})$ with the general partitions μ.

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