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SPHERES AND SYMMETRY: BORSUK'S ANTIPODAL THEOREM

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Dedicated to the memory of Karol Borsuk (1905–1982)

Last century, the mathematicians left the area of normal intuition when they started to consider not only Euclidean spaces of dimensions 1, 2, and 3, but of any finite dimension. By this, they won a huge amount of new objects, among which the spheres S^n belong to the simplest and most important ones.

It was very encouraging that even these seemingly simple objects turned out to have highly interesting topological properties. Some of the basic facts were found during the first decades of this century:

a) From the Brouwer fixed point theorem ([18]; Poincaré [61] and Bohl [12] already knew equivalent results) it followed that the spheres S^n are not contractible (cf. Borsuk [13, p. 162]).

b) Hopf [37] gave a complete description of the homotopy classes of maps $f : S^n \to S^n$ by relating these to the Brouwer degree.

c) Moreover, Hopf [38] gave examples for the surprising fact that there are essential maps $f : S^n \to S^m$ with $n > m$, i.e., maps which cannot be continuously deformed to a constant map. This started the highly complicated theory of homotopy groups of spheres.

Borsuk's antipodal theorem introduced a new concept to these considerations: symmetry. Spaces are now considered as topological spaces with some symmetry, e.g. the antipodal symmetry on spheres; maps between such spaces should respect the symmetry.