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NEW DEVELOPMENTS ON THE GINZBURG-LANDAU MODEL

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Dedicated to Jean Leray with admiration

The starting point of the new developments around the Ginzburg-Landau equation is a "frustrating" lemma. Suppose $\Omega \subset \mathbb{R}^2$ is a smooth bounded simply connected domain. Fix a smooth boundary condition $g : \partial\Omega \to S^1$ (S^1 = the unit circle in $\mathbb{R}^2 = \mathbb{C}$) and consider the class of functions

$$H_g^1 = H_g^1(\Omega, S^1) = \{u : \Omega \to S^1 : \nabla u \in L^2 \text{ and } u = g \text{ on } \partial\Omega\}.$$

Lemma 1. *The class H_g^1 is not empty if and only if*

$$\deg(g, \partial\Omega) = 0.$$

Here deg refers to the usual (Brouwer) degree, also called the winding number, of g, considered as a map from $\partial\Omega\,(\simeq S^1)$ into S^1. It is a pleasure to acknowledge the pioneering role played by J. Leray in the development of degree theory and its use in analysis (see J. Leray and J. Schauder [14]).

The proof of Lemma 1 is not straightforward, especially the implication $\Rightarrow$. One method consists in taking some $u \in H_g^1$ and using it to homotopy g to a constant, for example via its restriction to circles when Ω is a disc. Of course u need not be continuous and thus one cannot use the standard degree theory. Instead one relies on the $H^{1/2}$ degree theory—a notion introduced by L. Boutet