

**NONLINEAR ERGODIC THEOREMS
FOR ALMOST NONEXPANSIVE CURVES OVER
COMMUTATIVE SEMIGROUPS**

OSAMU KADA — WATARU TAKAHASHI

Dedicated to Ky Fan on the occasion of his 80th birthday

1. Introduction

Let S be a commutative semigroup with identity, and let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.

We also denote by \mathbb{Z} , \mathbb{Z}^+ , \mathbb{R} and \mathbb{R}^+ the sets of all integers, nonnegative integers, real numbers and nonnegative real numbers, respectively. Let C be a subset of H . Then a mapping $T : C \rightarrow C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The first nonlinear ergodic theorem for nonexpansive mappings (in a Hilbert space) was established by Baillon [1]: Let C be a nonempty closed convex subset of H and let T be a nonexpansive mapping of C into itself. If T has a fixed point, then the Cesàro means $(1/n) \sum_{k=0}^{n-1} T^k x$ converge weakly as $n \rightarrow \infty$ to a fixed point y of T . In this case, put $y = Px$ for each $x \in C$. Then P is a nonexpansive retraction of C onto the set $\text{Fix}(T)$ of fixed points of T such that $PT^n = T^n P = P$ for all $n \in \mathbb{Z}^+$, and $Px \in \text{clco}\{T^n x : n \in \mathbb{Z}^+\}$ for each $x \in C$, where $\text{clco } A$ is the closure of the convex hull of A . In [33, 34], Takahashi proved the existence of such an ergodic retraction for an amenable semigroup of nonexpansive mappings in a Hilbert space. And also Rodé [30] found a sequence of means on the semigroup, generalizing the Cesàro means on the positive integers, such that the corresponding sequence of mappings converges to an ergodic

1991 *Mathematics Subject Classification*. Primary 47A35; Secondary 47H20.