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FIXED POINT THEOREMS AND CHARACTERIZATIONS OF METRIC COMPLETENESS

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1. Introduction

Let X be a metric space with metric d. A mapping T from X into itself is called *contractive* if there exists a real number $r \in [0,1)$ such that $d(Tx,Ty) \le rd(x,y)$ for every $x,y \in X$. It is well know that if X is a complete metric space, then every contractive mapping from X into itself has a unique fixed point in X. However, we exhibit a metric space X such that X is not complete and every contractive mapping from X into itself has a fixed point in X; see Section 4. On the other hand, in [1], Caristi proved the following theorem: Let X be a complete metric space and let $\phi : X \to (-\infty,\infty)$ be a lower semicontinuous function, bounded from below. Let $T : X \to X$ be a mapping satisfying

$$d(x,Tx) \le \phi(x) - \phi(Tx)$$

for every $x \in X$. Then T has a fixed point in X. Later, characterizations of metric completeness have been discussed by Weston [8], Takahashi [7], Park and Kang [6] and others. For example, Park and Kang [6] proved the following: Let X be a metric space. Then X is complete if and only if for every selfmap T of X with a uniformly continuous function $\phi : X \to [0,\infty)$ such that

$$d(x,Tx) \le \phi(x) - \phi(Tx)$$

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