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ON THE THREE CRITICAL POINTS THEOREM

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1. Introduction

Let φ be a C^1 real function defined on $\mathbb{R}^m$. We assume that φ is coercive (i.e. $\varphi(x) \to \infty$ as $||x|| \to \infty$). It is well-known that under these assumptions φ reaches a minimum at some point x_0. Let now x_1 be a critical point of φ which is not a global minimum. M. A. Krasnosel'skiĭ [10] made the following observations: if x_1 is a nondegenerate singular point of the vector field $\nabla\varphi$ (i.e. the topological index $\operatorname{ind}(\nabla\varphi(x_1), 0)$ is different from zero), then φ admits a third critical point. In the sequel this statement became known as the "Three Critical Points Theorem" (TCPT).

The above result of Krasnosel'skiĭ was extended to the context of Banach spaces (see [1], [4], [8], [17]). Another generalization was obtained by Chang [5], [6] using the methods of Morse theory (the condition $\operatorname{ind}(\nabla\varphi(x_1), 0) \neq 0$ is replaced by the weaker assumption of nontriviality of Morse critical groups at x_1). Also, Brezis and Nirenberg [3] gave a very useful variant of TCPT for applications using the principle of local linking (see also [12]). In this paper we shall give a proof of TCPT based on a "strong" deformation lemma (see Lemma 2.1 below) thus avoiding standard minimax techniques. In contrast to the previous work in this field, we prove in fact the Lusternik–Schnirel'man type

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