Constructive Logic and the Paradoxes

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§0. INTRODUCTION

Heyting's formalization of intuitionistic mathematics started many discussions about the meaning of the logical connectives in terms of proof and construction. We focus on the ideas and results related to the interpretation of implication and on formal systems that have different rules for implication. Some of these systems are not intended to contribute to the discussions mentioned above, but are related to the Basic Calculus introduced in §§3 and 4.

The set-theoretic paradoxes of the turn of the century shocked many mathematicians into realizing that their simple intuitions about sets and logic were inconsistent. Constructive mathematics along the lines of Brouwer, Markov, and Bishop is not intended to resolve this issue, and doesn't. The most common solutions favored by mathematicians involve reducing one's attention to a hierarchical class of sets, thereby excluding the paradoxical ones. A few mathematicians and logicians kept searching for the Holy Grail of set theory with full comprehension by changing the rules of equality or logic. Of special interest to the Basic Calculus and the set theory F of §5 is Fitch's system with the additional implication hierarchy introduced by Myhill. It seems that this approach replaces a hierarchy of sets by a (simpler) hierarchy of implications.

In an attempt to find a non-circular proof interpretation for the logical connectives, we change from Heyting's axiomatization to a subsystem of intuitionistic logic with a limited *modus ponens*: Basic Calculus. In set theory with full comprehension over this subsystem, Russell's Paradox turns into a proof of Löb's Rule, a rule that is relatively inconsistent with *modus ponens*.

§1. THE PROOF INTERPRETATION

L. E. J. Brouwer's introduction of intuitionistic mathematics was not a reaction to the paradoxes, although its influence may have been felt; it offered an alternative to the formalist and logicist approaches. Consequently, a naïve extension of intuitionistic mathematics to set theory with full comprehension does not solve the paradoxes. Brouwer's Ph.D. thesis of 1907, and later work, expounded the intuitionist's point of view (see [Brouwer 1975]). One aspect of this point of view was a proscription of the use of logical principles as a guide to mathematics, the most well-known among these being the Principle of the Excluded Middle. The