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Review of
SAHARON SHELAH, *CARDINAL ARITHMETIC*

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Cardinal arithmetic, as Shelah observes, was the impetus for two major currents of modern set theory: generic extensions of the set theoretic universe, and inner model theory. Understanding the actual bounds or potential boundlessness of cardinal exponentiation — in particular for singular cardinals — and the underlying reasons for these phenomena are recurring themes of *Cardinal Arithmetic* by Saharon Shelah. However it is the main approach to cardinal arithmetic which is most notable in Shelah's book. *Cardinal Arithmetic* is a source book for Shelah's theory of possible cofinalities (pcf theory). It mines the mother lode of pcf theory down to what seems like the last mathematical speck, while pursuing applications which include spectacular results in cardinal arithmetic.

Pcf theory, which was first introduced by Shelah in a 1978 paper [5] to build Jónsson algebras on $\aleph_{\omega+1}$, is arguably a natural approach, or more accurately an effective alternative to cardinal exponentiation. Consider λ^κ, with $\kappa < \lambda$ and κ regular. Thinking of this cardinal exponentiation first as a set of functions from κ into λ leads to equating λ^κ with $|S_{\leq\kappa}(\lambda)|$, where $S_{\leq\kappa}(\lambda) = \{\, X \subset \lambda : |X| \leq \kappa \,\}$. The point here is that to understand the bounds or boundlessness of λ^κ, first effective ways of measuring the size of $S_{\leq\kappa}(\lambda)$ need to be studied. The study begins with the cofinalities of products $\prod \lambda_i$ modulo ideals J on κ, where $\lambda_i < \lambda$ are regular cardinals extending to λ. This results in a set of regular cardinals — the set of possible cofinalities of $\{\, \lambda_i : i < \kappa \,\}$ — which can be viewed as a set of candidates for the cofinality of $S_{\leq\kappa}(\lambda)$. Here lies the first step toward measuring the size of $S_{\leq\kappa}(\lambda)$.

Cardinal Arithmetic is, in part, a compilation of Shelah's work on pcf theory from work included in the 1978 paper [5], through Fall 1989 when he gave a series of lectures on pcf theory as part of the Logic Year at the Mathematical Sciences Research Institute in Berkeley,