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ON THE DECOMPOSITION THEOREMS OF LEBESGUE AND JORDAN

Abstract

In the first part of this paper we show a powerful special case of Lebesgue's decomposition theorem, namely: if F is a VB function, satisfying Lusin's condition (N) on $[a,b]$, then $F(x) - F(a) = s_F(x) + (\mathcal{L})\int_a^x F'(t)\,dt$, where s_F is the saltus function of F. In the second part we show that if F satisfies Lusin's condition (N) on $[a,b]$ then the functions (from the decomposition theorem of Jordan) $V_F(x) := V(F;[a,x])$ and $G(x) := F(x) - V_F(x)$ also satisfy (N).

The following decomposition theorem of Lebesgue is well known:

Theorem A (Lebesgue's decomposition theorem)**.** ([7], p. 119).
If F is an additive function of bounded variation of an interval, the derivative F' is summable, and the function F is the sum of a singular additive function of an interval and of the indefinite integral of the derivative F'.

Moreover, if the function F is non-negative, we have for every interval I_o

$$F(I_o) \geq \int_{I_o} F'(t)\,dt\,,$$

equality holding only in the case in which the function F is absolutely continuous on I_o.

In the first part of this paper, for the special case of a function defined on an interval $[a,b]$, with bounded variation and satisfying Lusin's condition (N), Theorem A becomes

$$F(x) - F(a) = s_F(x) + (\mathcal{L})\int_a^x F'(t)\,dt\,,$$

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