Real Analysis Exchange
Vol. 23(1), 1998-99, pp. 303–311

Charles Swartz, Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003, USA, e-mail:cswartz@nmsu.edu

UNIFORM INTEGRABILITY AND MEAN CONVERGENCE FOR THE VECTOR-VALUED MCSHANE INTEGRAL

Abstract

We show that a pointwise convergent, uniformly integrable sequence of Banach space valued, McShane integrable functions converges in mean. We also show that uniform integrability holds in a vector-valued generalization of the Beppo Levi convergence theorem.

It has been observed in [3, 4, 5], [7] that uniform integrability for the Henstock-Kurzweil integral is a sufficient condition to "take the limit under the integral sign." In this note we point out that uniform integrability for the McShane integral is actually a sufficient condition for mean or L^1 convergence. Our methods extend easily to functions with values in a Banach space so we consider this case where the results give significant improvements to the scalar case. We also show that the conclusion of the vector-valued generalization of the Monotone Convergence (Beppo Levi) Theorem given in [10] can be improved to uniform integrability.

We fix the notation and terminology which will be used in the sequel. It should be noted that we will work in $\mathbb{R}$ whereas the results in [3, 4, 5] are for compact intervals in $\mathbb{R}$. Let X be a (real) Banach space and let $\mathbb{R}^*$ be the extended real line with the points $\pm\infty$ added. If f is any function $f : \mathbb{R} \to X$, we always assume that f is extended to $\mathbb{R}^*$ by setting $f(\pm\infty) = 0$.

A gauge is a function γ on $\mathbb{R}^*$ whose value at a point t is a neighborhood $\gamma(t)$ of t, where $\gamma(t)$ is bounded whenever $t \in \mathbb{R}$. [A neighborhood of ∞ is an interval of the form $(a, \infty]$; similarly for $-\infty$.] A partition of $\mathbb{R}$ is a finite collection of left-closed intervals $\{I_i : i = 1, \ldots, n\}$ such that $I_i \cap I_j = \phi$ for $i \neq j$ and $\mathbb{R} = \bigcup_{i=1}^{n} I_i$ (here we agree that $(-\infty, a)$ is left-closed). A tagged partition of $\mathbb{R}$ is a finite collection of pairs $\{(I_i, t_i) : i = 1, \ldots, n\}$ such that

Key Words: McShane integral, uniform integrability, mean convergence
Mathematical Reviews subject classification: 28B05
Received by the editors March 25, 1997