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AN ELEMENTARY PROOF OF THE BANACH–ZARECKI THEOREM

Abstract

In this paper we shall give a new, elementary proof of the Banach–Zarecki theorem, based on the following classical result [3] (p. 183): If $\{A_i\}_i$ is a sequence of decreasing sets in a measurable space $(X, \mathcal{M}, \mu)$ and $\mu(A_1) < +\infty$ then $\mu(\cap_i A_i) = \lim_{i\to\infty} \mu(A_i)$.

There is a very rich literature concerning the Banach–Zarecki Theorem, such as the books of Saks [5] (p. 227), Natanson [4] (p. 250), Foran [3] (p. 357), Ene [1] (pp. 81, 104) and a paper of Varberg [6] (p. 835). This theorem asserts that *if a continuous and VB function satisfies Lusin's condition (N) on an interval then it is also AC on that interval.*

The proofs in [5], [3], [6] and [1] (p. 81) are based on the following result (see Theorem 6.5 of [5], p. 227; Theorem 1 of [6], p. 834; Theorem 8.1 of [3]): *If a function F is derivable at every point of a measurable set D, then* $m^*(F(D)) \le (\mathcal{L}) \int_D |F^{'}(x)| dx$.

In [4], the Banach–Zarecki Theorem is proved in a totally different way, namely using Lebesgue's Convergence Theorem as well as the fact that *the Banach indicatrix for a continuous and VB function on $[a, b]$ is summable* (see Theorem 3 of [4], p. 225).

In [1] (p. 104), the Banach–Zarecki Theorem is a consequence of some general notion (AC_∞, VB_∞ etc.). Here the Banach indicatrix has also an important role, but the proof is different from that in [4].

In this paper we shall give a new, elementary proof of the Banach–Zarecki theorem, based on the following classical result [3] (p. 183): *If $\{A_i\}_i$ is a sequence of decreasing sets in a measurable space $(X, \mathcal{M}, \mu)$ and $\mu(A_1) < +\infty$ then $\mu(\cap_i A_i) = \lim_{i\to\infty} \mu(A_i)$.*

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