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σ-FINITE BOREL MEASURES ON THE REAL LINE*

Abstract

A characterization is given of those Borel measures on the real line that can be expressed as the total variation measure of an ACG_* function.

Let μ be a measure defined on the Borel subsets of an interval $[a, b]$. If μ is absolutely continuous with respect to Lebesgue measure (that is, if $\mu(N) = 0$ for every Borel set N of Lebesgue measure zero) and if $\mu([a, b]) < \infty$ then μ can be represented in the form

$$\mu(B) = \mu_f(B) = \int_B f'(x)\,dx \qquad (B \subset [a, b]), \tag{1}$$

where f is absolutely continuous on $[a, b]$ and μ_f is the corresponding Lebesgue-Stieltjes measure. Beginning students of analysis learn this material routinely.

It seems, though, that there has been little discussion of the σ-finite case. If μ is not finite, but is σ-finite, is there a representation similar to this available?

Part of such a representation is immediately available from the Radon-Nikodym theorem and a theorem of Lusin. Any absolutely continuous, σ-finite measure μ on $[a, b]$ can be represented as

$$\mu(B) = \int_B g(x)\,dx$$

for some measurable, finite a.e. function g. But Lusin's theorem (eg., see [1, p. 113]) asserts the existence of a continuous function f with $f' = g$ a.e. This gives

$$\mu(B) = \int_B f'(x)\,dx.$$

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