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# A POLYNOMIAL FIXED-POINT PROBLEM

This problem arose in an earlier, unsuccessful, attempt to answer a question about the Dubins-Freedman construction of random distributions that has in the meantime been answered affirmatively in the paper [1].

For $n \in \mathbf{N}$, let $\mathcal{P}_n$ denote the set of polynomials of the form

$$\sum_{i=0}^{2k} x^{n-s(i)}(1-x)^{s(i)}$$

where $0 \leq k \leq 2^{n-1}-1$ and $s(i)$ is the number of 1's in the binary expansion of $i$. Thus,

$$\begin{aligned}
&\mathcal{P}_1 = \{x\},\\
&\mathcal{P}_2 = \{x^2, x^2+2x(1-x)\},\\
&\mathcal{P}_3 = \{x^3, x^3+2x^2(1-x), x^3+3x^2(1-x)+x(1-x)^2,\\
&\qquad\qquad x^3+3x^2(1-x)+3x(1-x)^2\},
\end{aligned}$$

etc.

Let $\mathcal{P} = \cup_{n=1}^{\infty}\mathcal{P}_n$. Note that all members of $\mathcal{P}$ are partition polynomials which map 0 to 0 and 1 to 1, and are increasing in between. (A *partition polynomial* is a polynomial of the form $\sum_{i=0}^{n} a_i x^i(1-x)^{n-i}$, where each $a_i$ is integer with $0 \leq a_i \leq \binom{n}{i}$.) However, there are many increasing partition polynomials with this property which are not members of $\mathcal{P}$. (For example, $x^3+x^2(1-x)+x(1-x)^2$.)

Let $\mathcal{L}$ denote the set of those members of $\mathcal{P}$ which are $< x$ on $(0,1)$, and $\mathcal{R}$ the set of those members of $\mathcal{P}$ which are $> x$ on $(0,1)$. Then $\mathcal{P} = \mathcal{L} \cup \{x\} \cup \mathcal{R}$. Furthermore, if $p \in \mathcal{R}$ then $p(x) = x + (1-x)r(x)$ for some $r \in \mathcal{P}$; and if $q \in \mathcal{L}$ then $q(x) = xs(x)$ for some $s \in \mathcal{P}$.

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