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DHOMBRES TYPE FUNCTIONAL EQUATIONS WITH NON-TRIVIAL SOLUTIONS

This question concerns functional equation of the Dhombres type, namely

$$f(x \cdot f(x)) = \varphi(f(x)) \quad \text{where } x > 0.$$

In such equations the function φ is given and one looks for solution functions, f; that is, f is the "unknown". Interesting is the case when all functions are continuous. Several cases are well known; for example, if φ is an increasing homeomorphism of an interval $J \subseteq (0, \infty)$ then the range $R_f \subseteq J$ of any solution is an interval with the end-points fixed by φ, which contains no fixed point $\neq 1$. There is a characterization of these φ that allow only monotone solutions, and characterization of the monotone solutions; they form a "parametric family"where parameter is an initial monotone function defined on a compact subinterval of $\mathbb{R}_+$, see [1]. Also characterization of the continuous solutions in this case is known [2].

On the other hand, if φ is a decreasing homeomorphism then there can be no nonconstant solutions at all. The only known example of such solution is for the function $\varphi : y \mapsto \alpha/y$, with $\alpha \in (0, 1)$. In this case R_f consists of periodic points of period 2, except for the point $\sqrt{\alpha}$ which is fixed [3].

A general question is then this:

Question 1. How complicated can φ be and still support a non-trivial solution?

Or, more specifically:

Question 2. Can φ have periodic points other than those of period two and still support a non-trivial solution?

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