INFINITE DETERMINANTS ASSOCIATED WITH HILL'S EQUATION

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1. Introduction and Summary. Hill's equation is the differential equation for a one-dimensional linear oscillator with a periodic potential. In most applications, the question of the existence of a periodic solution arises. The main purpose of this investigation is to examine the analytic character of the transcendental function, whose zeros determine the periodic solutions. For the special case of Mathieu's equation the results obtained here have previously been used for solving the inhomogeneous equation, and the cases where Hill's equation has two periodic solutions have been discussed in detail and applied to the construction of "transparent layers" [1].

We consider the differential equation of Hill's type:

$$y''+4(\omega^2+q(x))y=0\ , \tag{1.1}$$

where $q(x)$ is an even function of period π which can be expanded in a Fourier series

$$q(x)=2\sum_{n=1}^{\infty} t_n \cos 2nx\ . \tag{1.2}$$

We shall assume that the constants t_n satisfy

$$\sum_{n=1}^{\infty} |t_n| < \infty\ . \tag{1.3}$$

The most widely investigated problem connected with (1.1) is the question of the existence of solutions with period π or 2π. Let $y_1(x)$, $y_2(x)$ denote the solutions of (1.1) which satisfy the initial conditions

$$y_1(0)=1,\ y_1'(0)=0\ ;\quad y_2(0)=0,\ y_2'(1)=1. \tag{1.4}$$

Then the following elementary statements hold (see for instance Schaefke [5]: Equation (1.1) has

(α) an even solution of period π if and only if $y_1'(\pi/2)=0$
(α') an odd solution of period π if and only if $y_2(\pi/2)=0$
(β) an even solution of period 2π if and only if $y_1(\pi/2)=0$
(β') an odd solution of period 2π if and only if $y_2'(\pi/2)=0$.

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