ASYMPTOTIC LOWER BOUNDS FOR THE FUNDAMENTAL FREQUENCY OF CONVEX MEMBRANES

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1. Introduction. Let the bounded, simply connected, open region R of the (x, y)-plane have the boundary curve C. If a uniform ideal elastic membrane of unit density is uniformly stretched upon C with unit tension across each unit length, then λ, the square of the fundamental frequency, satisfies the conditions (subscripts denote differentiation)

$$\text{(1a)} \qquad \begin{cases} \Delta u \equiv u_{xx} + u_{yy} = -\lambda u & \text{in} \quad R\,, \\ \lambda = \text{minimum}\,, \end{cases}$$

with the boundary condition

$$\text{(1b)} \qquad u(x, y) = 0 \quad \text{on} \quad C\,.$$

Variational methods of the Rayleigh-Ritz type are frequently used to approximate λ. They always yield upper bounds for λ, and the upper bounds can be made arbitrarily close.

Another common practical method of approximating λ is to calculate the least eigenvalue λ_h of a suitably chosen finite-difference operator Δ_h over a network with small mesh width h. For one choice of Δ_h it was shown by Courant, Friedrichs, and Lewy [3, p. 57] without details that $\lambda_h \to \lambda$ as $h \to 0$. For convex regions R of a special polygonal form the author has shown [4] that a special case of (11) below is valid for a common choice of Δ_h, and hence that λ_h is asymptotically a lower bound for λ as $h \to 0$. For an unusual finite-difference approximation to problem (1) when R is the union of squares of the network, Polya [12] has found that $\lambda_h > \lambda$ for all h, and also for the higher eigenvalues. The author knows of no other study of the sign or order of decrease of $\lambda - \lambda_h$ to 0.

In the present paper the investigation of [4] is extended to a much wider class of regions: those with piecewise analytic boundary curves and convex corners. The new theorems are stated and proved in §§ 3 and 4. Theorem 2 contains the theorem of [4] as a special case. Lemmas used in the proof of Theorem 1 are given in § 5. Identity (31) of Lemma 7 is interesting in itself.

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