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ON THE CONTINUITY OF THE NONLINEAR TSCHEBYSCHEFF OPERATOR

R. B. BARRAR AND H. L. LOEB

An existence theorem and a Lipschitz continuity theorem for uniform nonlinear Tschebyscheff approximation are given. These theorems include as special cases known results on generalized rational functions but also yield new results for exponential families.

The classical theorems on the uniqueness and characterization of the best uniform approximations by polynomials have been extended to nonlinear approximating families in the papers of Motzkin [11], Tornheim [16] and Rice [13]. These papers introduce the important ideas of unisolvent and varisolvent families.

Meinardus and Schwedt [10] have stressed the importance of a gradient function in the theory of nonlinear approximation.

In the present paper, we combine both these concepts and are thus enabled to extend the strong unicity theorem and theorems on the continuity of the Tschebyscheff operator to nonlinear approximating families. The strong unicity theorem in the linear case is due to Newman and Shapiro [12]. The continuity theorem for ordinary rational approximation is proved in Maehly and Witzgall [9]. The theorems for generalized rational approximation appear in Cheney and Loeb [3], and Cheney [2].

Our assumptions appear to cover many of the nonlinear approximating families in current use. Some examples are given in the last section of the paper.

2. **Notation.** Let P be an open subset of real Euclidean M dimensional space E_M. We consider a family V of real valued functions $F(A, x)$ where $A = (a_1, \cdots, a_M)$ belongs to P and x belongs to $[0, 1]$. The functions $F(A, x)$ and $\partial F(A, x)/\partial a_i$, $i = 1 \cdots M$ are assumed to be continuous in A and x.

We further assume that the family V satisfies the following conditions:

(A) To each $A \in P$, the functions $\partial F(A, x)/\partial a_i$, $i = 1 \cdots M$ generate a Haar Subspace $W(A)$ of dimension $d(A)$ where $d(A) \geqq 1$. For convenience of notation and without loss of generality, we will assume in the statements and proofs of all lemmas and theorems in this paper that $\partial F(A, x)/\partial a_i$, $i = 1 \cdots d(A)$, generate the Haar Subspace.

(B) For each $A \in P$, $F(A, x) \not\equiv F(A_1, x)$ implies $F(A, x) - F(A_1, x)$ has at most $d(A) - 1$ zeroes. $d(A)$ is sometimes called the degree