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ORDER-INDUCED TOPOLOGICAL PROPERTIES

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Each topology $\mathscr{T}$ on a set X may be associated with a preorder relation $R_{\mathscr{T}}$ on X defined by $\langle a, b\rangle \in R_{\mathscr{T}}$ iff every open set containing b contains a. Although the correspondence is many-to-one, there is always a least topology, $\mu(R)$, and a greatest topology, $\nu(R)$, having a given preorder R. This leads to a natural correspondence between order properties and some topological properties and to the concept of an order-induced topological property. We show that a number of familiar topological properties (mostly lower separation axioms) are order-induced and also consider some new properties suggested by order properties. Let T_p be an order-induced topological property with associated order property K_p. We characterize minimal and maximal T_p as follows: A topological space $(X, \mathscr{T})$ is maximal T_p iff $\mathscr{T} = \nu(R_{\mathscr{T}})$ and $R_{\mathscr{T}}$ is minimal K_p. With the imposition of a further condition on the class K_p (satisfied by most properties under discussion), $(X, \mathscr{T})$ is minimal T_p iff $\mathscr{T} = \mu(R_{\mathscr{T}})$ and $R_{\mathscr{T}}$ is maximal K_p. We apply these general theorems to a number of order-induced properties and conclude with an example to show that, for two particular properties, $\mathscr{T}$ may be minimal T_p even though $R_{\mathscr{T}}$ is not maximal K_p.

1. Introduction. Correspondences between topologies and preorders on X similar to that assigning $R_{\mathscr{T}}$ to $\mathscr{T}$ have been described by several mathematicians. Ore in 1943 **[14]** associated with each closure operator on a fixed set X a preorder relation which, for the topological closure operators, is exactly the same as $R_{\mathscr{T}}$. Others have restricted their attention to the "principal" or "discrete" spaces in which arbitrary intersections of open sets are open. Linfield in his thesis **[11]** of 1925 studied principal topologies whose preorders were equivalence relations [see 7], and in 1935 both Alexandroff **[1]** and Tucker **[20]** described a one-to-one correspondence between T_0 principal topologies and partial orders. Destouches in 1937 drew on the work of Linfield and Alexandroff to study principal spaces in general **[6]**, and Steiner in 1966 showed that the lattice of principal topologies is anti-isomorphic to the lattice of preorder relations on X **[16]**. Alexandroff, Tucker, and Steiner all assigned the relation $R_{\mathscr{T}}^{-1}$ to $\mathscr{T}$, and Lorrain (1969) used both $R_{\mathscr{T}}^{-1}$ and $R_{\mathscr{T}}$ to define functors from the category of principal spaces to the category of preordered sets **[13]**.

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