COMPLETENESS OF SETS OF TRANSLATED COSINES

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1. Introduction. Conditions for the completeness on $(0, \pi)$ of sets $\{\cos \lambda_n x\}$ are well known. Here we shall consider sets $\{\cos(\lambda_n x + q_n)\}$. Such sets seem first to have been considered by Ditkin [3], who proved that $\{\cos(nx + q_n)\}_0^\infty$ is L-complete in $(0, \pi)$ if $0 \le q_n < \pi/2$.

Ditkin's very simple proof uses Fourier series and does not seem capable of extension to the more general sets considered here. Our principal object is to show how the problem may be attacked by complex-variable methods; we shall not attempt an exhaustive discussion.

As a specimen we quote the following case. If $\lambda_n \ge 0$ and $|\lambda_n - n| \le \delta < 1/2$, then the sets $\{\cos(\lambda_n x + q_n)\}_0^\infty$ and $\{\sin(\lambda_n x + q_n)\}_1^\infty$ are L-complete in $(0, \pi)$ if $\pi\delta/2 \le q_n < \pi(1 - \delta)/2$. (The statement "$\{f_n(x)\}$ is L^p-complete" means that the only functions of L^p which are orthogonal to all $f_n(x)$ are almost everywhere zero.) A further result, not covered by the present paper, has been given by Bitsadze [1], who showed that every function satisfying a Hölder condition admits a uniformly convergent expansion in terms of the set $\{\cos(nx + \pi/4)\}$; he indicates an application of this result to the Tricomi partial differential equation.

We remark that although Ditkin's set $\{\cos(nx + q_n)\}_0^\infty$ remains complete when all $q_n = \pi/2$, it may fail to be complete if some but not all $q_n = \pi/2$. In fact, the set $\{1, \sin x, \cos 2x, \cos 3x, \cdots\}$ is orthogonal to $\cos x$. However, we shall show that not only is the set $\{\sin(nx + q_n)\}_0^\infty$ complete if $0 \le q_n < \pi/2$, but even the set $\{\sin(nx + q_n)\}_1^\infty$ is complete.

By applying the completeness theorem of Paley and Wiener [5, p. 100] to the equivalent set $\{\cos nx + a_n \sin nx\}$, $0 \le |a_n| < 1$, we can show at once that $\{\cos(nx + q_n)\}_0^\infty$ is L^2-complete if either $0 \le |q_n| \le \delta < \pi/4$ for all n or else $\pi/4 < \delta \le |q_n| \le \pi/2$ for all n. The problem of necessary and sufficient conditions for the completeness of $\{\cos(nx + q_n)\}$ remains open.

2. A general theorem. We shall obtain our results on $\{\cos(\lambda_n x + q_n)\}$ as

Received December 4, 1950.

Pacific J. Math. 1 (1951), 321-328.