ON THE REALIZABILITY OF HOMOTOPY GROUPS AND THEIR OPERATIONS

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1. Introduction. Let B be a given arcwise connected topological space and b_0 a basic point of B. Then we obtain a sequence of homotopy groups

$$\pi_1(B),\ \pi_2(B),\ \cdots,\ \pi_n(B),\ \cdots.$$

The fundamental group $\pi_1(B)$ is in general non-abelian and written multiplicatively. All higher homotopy groups $\pi_n(B)$, $n \geq 2$, are abelian and written additively. The group $\pi_1(B)$ operates on the left of every higher homotopy group $\pi_n(B)$, $n \geq 2$; that is to say, for every $w \in \pi_1(B)$ and every $a \in \pi_n(B)$, a unique element $wa \in \pi_n(B)$ is determined, and

$$w(a_1 + a_2) = wa_1 + wa_2, \quad w_1(w_2 a) = (w_1 w_2)a, \quad 1a = a.$$

For arbitrary elements $a \in \pi_m(B)$ and $b \in \pi_n(B)$, $m \geq 2$, $n \geq 2$, a Whitehead product $a \circ b$ is defined [10, p. 411], which is an element of $\pi_{m+n-1}(B)$. The Whitehead product is known to be bilinear; namely,

$$(a_1 + a_2) \circ b = a_1 \circ b + a_2 \circ b, \quad a \circ (b_1 + b_2) = a \circ b_1 + a \circ b_2.$$

Roughly speaking, the realizability problem is whether these homotopy groups and mutual operations described above are otherwise completely arbitrary. It can be formulated precisely as follows. Let

$$\pi_1,\ \pi_2,\ \cdots,\ \pi_n,\ \cdots$$

be a given sequence of abstract groups. All groups except the first one are abelian and additive, while π_1 is written multiplicatively. There are given two kinds of operations between these groups. First, the group π_1 operates on the left of every group π_n with $n \geq 2$. Secondly, for arbitrary elements $\alpha \in \pi_m$, $\beta \in \pi_n$, $m \geq 2$, $n \geq 2$, a bilinear product $\alpha \circ \beta$ is defined and is an element of the group π_{m+n-1}.

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