

A SHORT PROOF OF PILLAI'S THEOREM ON NORMAL NUMBERS

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1. Introduction. The object of this paper is to give a short proof of the Pillai theorem [2] on normal numbers using the Niven-Zuckerman result [1] as a tool.

DEFINITION 1. A number σ is *simply normal* to the base r if, in the expansion to the base r of the fractional part of σ , we have $\lim_{n \rightarrow \infty} n_c/n = 1/r$ for all c , where n_c is the number of occurrences of the digit c in the first n digits of σ .

DEFINITION 2. A number σ is *normal* to the base r if $\sigma, r\sigma, r^2\sigma, \dots$ are each simply normal to all the bases r, r^2, r^3, \dots .

THEOREM (Pillai). *A necessary and sufficient condition that a number σ be normal to the base r is that it be simply normal to the bases r, r^2, r^3, \dots .*

2. Proof. The necessity of the condition follows from the definition of normality.

To prove sufficiency, assume that σ is simply normal to the bases r, r^2, \dots . Let $A = (a_1 a_2 \dots a_v)$ be any fixed sequence of digits (to base r), where $v = hr - s$, $h > 0$, $0 \leq s < r$; and consider the occurrence of A in σ . Count the number of occurrences of A in the collection of sequences of length hr . There are s digits free after v of the hr digits are fixed. Thus there are $(s + 1)r^s$ different occurrences of A in these sequences.

For any positive integer n , define $f_n(A)$ to be the frequency of the occurrences of A in σ except in places where A will straddle the middle of sequences of length $2h2^{n-1}r$ starting in places congruent to 1 (mod $2h2^{n-1}r$), or where A will straddle the middle of sequences of length $4h2^{n-1}r$ starting in places congruent to 1 (mod $4h2^{n-1}r$), or \dots , or where A will straddle the middle of sequences of length $2^s h2^{n-1}r$ starting in places congruent to 1 (mod $2^s h2^{n-1}r$), and so on.

Certainly $\lim_{n \rightarrow \infty} f_n(A)$, if it exists, will be equal to $f(A)$, the frequency of A in σ .

We have

$$f_1(A) = \frac{(s + 1)r^s}{hr r^{hr}} = \frac{1}{r^v} - \frac{v - 1}{hr^v + 1},$$

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