THEOREMS ON GENERALIZED DEDEKIND SUMS

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1. Introduction. Generalized Dedekind sums $s_p(h,k)$, defined by

$$(1) \qquad s_p(h,k) = \sum_{\mu=1}^{k-1} \frac{\mu}{k} \overline{B}_p\left(\frac{h\mu}{k}\right) = \sum_{\mu=1}^{k-1} \frac{\mu}{k} B_p\left(\frac{h\mu}{k} - \left[\frac{h\mu}{k}\right]\right),$$

were introduced by the author [1]. The integers h and k are assumed relatively prime, $\overline{B}_p(x)$ is the p-th Bernoulli function, $B_p(x)$ the p-th Bernoulli polynomial (for definitions see [1; (2.11), (2.12)]), and $[x]$ is the greatest integer $\leq x$. For even values of the integer p the sums (1) are trivial (see [1; (4.13)]) and we assume in what follows that p is odd. These sums enjoy a reciprocity law, namely;

$$(2) \qquad \begin{aligned} &(p+1)\,(hk^p\, s_p(h,k) + kh^p\, s_p(k,h)) \\ &\qquad = p\, B_{p+1} + \sum_{s=0}^{p+1} \binom{p+1}{s} (-1)^s B_s B_{p+1-s}\, h^s k^{p+1-s}. \end{aligned}$$

The B's being Bernoulli numbers*. An arithmetic proof of (2) is given in [1] by a method closely related to a general summation technique recently developed by Mordell [5]. When $p = 1$, the sums

$$(3) \qquad s_1(h,k) = \sum_{\mu=1}^{k-1} \frac{\mu}{k}\left(\frac{h\mu}{k} - \left[\frac{h\mu}{k}\right] - \frac{1}{2}\right)$$

are known as Dedekind sums and are usually denoted by $s(h,k)$. Aside from being of interest from an arithmetical standpoint, these sums also occur in the asymptotic theory of partitions and have been studied in a large number of papers, for example [1], [3], [5], [6], [7], [8], [9], [10], and [11].

In this paper we establish a connection between the sums (1) and certain finite sums involving Hurwitz zeta functions which makes it possible to give a short analytic proof of (2).

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* When $p > 1$, the factor $(-1)^s$ may be suppressed in the summand in (2) because the terms corresponding to odd values of s vanish.

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