SOME SPECIAL EQUATIONS IN A FINITE FIELD

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1. The equation (1.1). Let $f_i(u)$, $i = 1, \cdots, r$ denote polynomials with coefficients in the finite field $GF(q)$, $q = p^n$. We consider the equation

$$(1.1) \qquad f_1(\xi_1) + \cdots + f_r(\xi_r) = \alpha \qquad (\xi_i, \alpha \in GF(q));$$

let N denote the number of solutions of (1.1).

For $\beta \in GF(q)$, put

$$e(\beta) = e^{2\pi i t(\beta)/p}, \quad t(\beta) = \beta + \beta^p + \cdots + \beta^{p^{n-1}}.$$

Then we may write

$$(1.2) \qquad qN = \sum_{\beta} e(-\alpha\beta) \sum_{\xi_1, \cdots, \xi_r} e(\beta f_1(\xi_1) + \cdots + \beta f_r(\xi_r)),$$

where the summation extends over all numbers β, ξ_i of $GF(q)$. Now put

$$(1.3) \qquad S(f) = \sum_{\xi} e(f(\xi)),$$

where f is any polynomial with coefficients in $GF(q)$. Then (1.2) becomes

$$qN = q^r + \sum_{\beta \neq 0} e(-\alpha\beta) \prod_{i=1}^{r} S(\beta f_i).$$

2. Estimate for N. If $\deg f \leq 2$, $S(f)$ can be evaluated explicitly. However, we are primarily interested in the case $\deg f > 2$. An estimate for $S(f)$ is given by the following:

THEOREM 1. *If* $k = \deg f < p$, *then*

$$(2.1) \qquad S(f) = O(q^{1-1/k}) \qquad (q \longrightarrow \infty).$$

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