SOME THEOREMS ON GENERALIZED DEDEKIND SUMS

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1. Introduction. Using a method developed by Rademacher [5], Apostol [1] has proved a transformation formula for the function

$$(1.1) \qquad G_p(x) = \sum_{m,\, n=1}^{\infty} n^{-p} x^{mn} \qquad (|x| < 1),$$

where p is a fixed odd integer > 1. The formula involves the coefficients

$$(1.2) \qquad c_r(h, k) = \sum_{\mu \,(\mathrm{mod}\; k)} P_{p+1-r}\left(\frac{\mu}{k}\right) P_r\left(\frac{h\mu}{k}\right) \qquad (0 \le r \le p+1),$$

where $(h, k) = 1$, the summation is over a complete residue system $(\mathrm{mod}\ k)$, and $P_r(x) = \overline{B}_r(x)$, the Bernoulli function.

We shall show in this note that the transformation formula for (1.1) implies a reciprocity relation involving $c_r(h, k)$, which for $r = p$ reduces to Apostol's reciprocity theorem [1, Th. 1; 2, Th. 2] for the generalized Dedekind sum $c_p(h, k)$. In addition, we prove some formulas for $c_r(h, k)$ which generalize certain results proved by Rademacher and Whiteman [6]. Finally we derive a representation of $c_r(h, k)$ in terms of so-called "Eulerian numbers".

2. Some preliminaries. It will be convenient to recall some properties of the Bernoulli function $P_r(x)$; by definition, $P_r(x) = B_r(x)$ for $0 \le x < 1$, and $P_r(x+1) = P_r(x)$. Also we have the formulas

$$(2.1) \qquad \sum_{r=0}^{k-1} P_r\left(t + \frac{r}{k}\right) = k^{1-m} P_r(kt), \quad P_r(-x) = (-1)^r P_r(x).$$

It follows from the second of (2.1) that $c_r(h, k) = 0$ for p even and $0 \le r \le p+1$. We have also

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