ON UNIFORM DISTRIBUTION MODULO A SUBDIVISION

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1. Let Δ be a subdivision of the interval $(0, \infty)$: $\Delta = (z_0, z_1, \cdots)$, where

$$0 = z_0 < z_1 < \cdots \quad \text{and} \quad \lim_{n \to \infty} z_n = \infty.$$

For $z_{n-1} \le x < z_n$, put

$$[x]_\Delta = z_{n-1}, \quad \delta(x) = z_n - z_{n-1}, \quad \langle x \rangle_\Delta = \frac{x - [x]_\Delta}{\delta(x)}, \quad \phi(x) = n + \langle x \rangle_\Delta,$$

so that $0 \le \langle x \rangle_\Delta < 1$. Let $\{x_k\}$ be an increasing sequence of positive numbers. If the sequence $\{\langle x_k \rangle_\Delta\}$ is uniformly distributed over $[0, 1]$, in the sense that the proportion of the numbers $\langle x_1 \rangle_\Delta, \cdots, \langle x_k \rangle_\Delta$ which lie in $[0, \alpha)$ approaches α as $k \to \infty$, for each $\alpha \in [0, 1)$, then we shall say that the sequence $\{x_k\}$ is *uniformly distributed modulo* Δ. If Δ is the subdivision Δ_0 for which $z_n = n$, this reduces to the ordinary concept of uniform distribution (mod 1), since then $[x]_\Delta = [x]$, $\delta(x) = 1$ for all x, and $\langle x \rangle_\Delta = x - [x]$ is the fractional part of x. Even in other cases, the generalization is more apparent than real, since the uniform distribution of one sequence (mod Δ) is equivalent to the uniform distribution of another sequence (mod 1). But most of the known theorems concerning uniform distribution (mod 1) are not applicable to the sequences $\{\langle x_k \rangle_\Delta\}$, if Δ is not Δ_0, for in such theorems x_k is ordinarily taken to be the value $f(k)$ of a function whose derivative exists and is monotonic for positive x. Here, on the other hand, $\langle x_k \rangle_\Delta \equiv \phi(x_k) \pmod 1$, and ϕ, although a continuous polygonal function, is not necessarily everywhere differentiable; and unless $\delta(x)$ is assumed monotonic, ϕ' is not monotonic even over the set on which it exists. This lack of monotonicity introduces serious difficulties; it is the object of the present work to show how they can be dealt with in certain cases.

For brevity, "uniformly distributed" will be abbreviated to "u.d.". The symbols "$\uparrow$", "$\nearrow$", "$\downarrow$" and "$\searrow$" indicate monotonic approach: increasing, non-decreasing, decreasing, and non-increasing, respectively.

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