ON A THEOREM OF L. A. SANTALÓ

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1. Introduction. L. A. Santaló [2] proved, as a by-product of other investigations, the following theorem:

Let a set of parallel line segments be given in the plane. If every three of the segments can be intersected (that is, met) by a straight line, then there exists a straight line intersecting all the segments.

This result was rediscovered by M. Dresher and T. E. Harris (cf. [1]). An interesting generalisation was obtained by H. Rademacher and I. J. Schoenberg [1]. Their generalisation—extending a special case due to T. E. Harris—is obtained from Santaló's theorem on replacing in it "*three*" by "$m + 2$" and "*straight line*" by "*polynomial line* $y = \alpha_0 x^m + \cdots + \alpha_m$".

The proof of Rademacher and Schoenberg (as well as a proof of Santaló's original theorem by J. Rey Pastor, cf. [2]) is based on Helly's theorem on convex sets.

The principal aim of the present paper is to give a generalisation of Santaló's theorem in a different direction. We shall restrict ourselves to intersections by straight lines, but, on the other hand, shall allow much greater freedom in the choice of sets to be intersected. As far as we are aware, our theorems cannot be deduced from Helly's theorem on convex sets.

We shall use the following customary definition:

Two sets S and S^* in the plane are said to be separated by a straight line L if $S \subset H \cup L$ and $S^* \subset H^* \cup L$, where H and H^*, $H \neq H^*$ are the two open half-planes determined by L. The separation is strict, if $S \subset H$ and $S^* \subset H^*$.

For the sake of brevity we shall use also the following:

Definition. A family of point sets in the plane is said to have property $\mathcal{S}$ if, either (i) there are three sets belonging to the family which cannot be intersected simultaneously by a straight line, or (ii) there exists a straight

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