ON EMBEDDING UNIFORM AND TOPOLOGICAL SPACES

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In this note we prove the following.

THEOREM. *Every space with separated uniform structure can be embedded as a closed subset of a separated convex linear space.*

Every metric space can be isometrically embedded as a closed subset of a normed linear space.

These statements follow at once from the theorem of § 3. Such an embedding is known for any *complete* metric space; and it is also known that any metric space is isometric which a relatively closed subset of a convex subset of a Banach space.

We also describe an embedding of an arbitrary T_1 space as a closed subset of a special homogeneous space.

1. Preliminaries.

(A) A *semi-metric* on a set X is a real non-negative function ρ on $X\times X$ such that $\rho(x, x)=0$, $\rho(x, y)=\rho(y, x)$, and $\rho(x, y)\leqq\rho(x, z)+\rho(z, y)$ for all $x, y, z \in X$. A semi-metric is a metric if and only if $\rho(x, y)=0$ implies $x=y$.

A collection of semi-metrics $(\rho_\alpha)_{\alpha\in A}$ on X indexed by a set A defines a uniform structure (and a topology) on X, generated by sets $U_{a\alpha}=\{(x, y): \rho_\alpha(x, y)<a\}$, where $a>0$ and $\alpha \in A$. Conversely, every uniform structure can be defined by a family of semi-metrics; see Bourbaki [1]. We will say that the uniform structure is *separated* if for every pair $x, y \in X$ there is a ρ_α such that $\rho_\alpha(x, y)\neq 0$.

(B) If X is a real linear space, a *semi-norm* on X is a real non-negative function s on X such that $s(\lambda x)=|\lambda| s(x)$ and $s(x+y)\leqq s(x)+s(y)$ for all $x, y \in X$ and for all real numbers λ. A semi-norm is a norm if and only if $s(x)=0$ implies $x=0$.

A collection of semi-norms $(s_\alpha)_{\alpha\in A}$ on X indexed by a set A defines a (locally) convex topology (and a uniform structure) compatible with the algebraic operations in X. Conversely, every convex topology can be described by a family of semi-norms; see Bourbaki [2]. We will say that the convex topology is *separated* if for every $x\neq 0$ in X there is an s_α such that $s_\alpha(x)\neq 0$.

(C) REMARK. Let X and X' be two sets with uniform structures

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