NEW BOUNDS FOR SOLUTIONS OF SECOND ORDER ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction In a previous paper [10] the authors presented methods for determining, with arbitrary and known accuracy, the Dirichlet integral and the value at a point of a solution of Laplace's equation. These methods have the advantage that upper and lower bounds are computed simultaneously. Moreover all error estimates are in terms of quadratic functionals of an arbitrary function, so that the Rayleigh-Ritz technique gives a systematic way of making the error arbitrarily small. These methods depend on an identity of F. Rellich [12]. As a consequence it is necessary to assume that the boundary is star-shaped with respect to some point, and it is not possible to treat differential equations with variable coefficients by these methods.

In this paper a generalization of Rellich's identity to general second order elliptic operators as well as to a large class of elliptic systems of second order operators is employed to extend the results of the previous paper to equations involving such operators and rather general domains.

The identity in question was obtained and used for hyperbolic operators by L. Hörmander [7] who, in a mimeographed note kindly communicated to the authors, has independently obtained therefrom some estimates for boundary values of the solution of a second order elliptic equation.

It is interesting to note the similarity in structure of the identity (2.4) and the formula for the first variation of Green's function for $\Delta u - pu$ given by Garabedian and Schiffer [6].

For the sake of simplicity only the case of a self-adjoint second order operator without zero order terms is treated in detail. However the method is easily extended to more general operators, and even systems, as is shown in §§6 and 7.

Section 2 is concerned with the above-mentioned identity. In §3 this identity is used to estimate several important quadratic functionals (including the generalized Dirichlet integral) in terms of Dirichlet data for a general non-homogeneous boundary value problem in N dimensions. We obtain an approximation to the generalized Dirichlet integral by means of a specific functional of an arbitrarily chosen function. The error estimate is a quadratic functional in the deviation of the Dirichlet data of the arbitrary function from the given data, and can be made

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