THE GEOMETRIC INTERPRETATION OF A SPECIAL CONNECTION

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1. Introduction. An esthetic problem remaining in the foundations of Riemannian geometry is this. To each Riemannian metric on a manifold one associates a connection on its bundle of bases. This connection is uniquely characterized by its having zero torsion and its parallel translation preserving the metric. By very definition at each tangent space of the bundle the connection assigns a complement to the vertical. Yet the various existence proofs of the Riemannian connection are computational and do not exhibit this complementary space (the horizontal space) directly. The problem is how can one do so directly from the metric, given (say) by the reduction of the bundle to the bundle of frames.

Under rather special circumstances, we exhibit a geometric and coordinate free construction of a connection in §2. This includes the class of complex analytic bundles and gives a geometric construction of a connection of type (1, 0) with curvature of type (1, 1), useful in algebraic geometry (§4). See [2] and [5]. For the case of complex vector bundles Nakano computes this connection in terms of coordinates [6]. For Kähler manifolds it turns out (§2) that this connection is the Riemannian connection restricted to the unitary subbundle. Hence we obtain a solution to the problem posed in the preceding paragraph for Kähler manifolds.

We assume the reader is familiar with the bundle approach to connections, say as in [1]. We use the notation of that paper.

2. The connection. Let $\mathfrak{B} = (B, M, G, G, \pi, \Phi)$ be a principal bundle. Throughout this discussion we shall suppose that G is the complexification of a compact[1] group K (such is the case for example if G is complex semisimple) and that B is an almost complex manifold. If $b \in B$, let J_b denote the linear transformation on B_b (the tangent space of B at b) giving the almost complex structure on B, so that $J_b^2 = -I$. We assume the following two conditions on this almost complex structure:

(i) the structure is invariant under right translation by G, that is $R_g J_b = J_{bg} R_g$

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[1] The results of this paper are valid whenever the Lie algebra $\mathfrak{g}$ of G is the complexification of the Lie algebra $\mathfrak{k}$ of K. The compactness of K is assumed only to guarantee that the reduction to K is possible.