EIGENVALUES OF THE UNITARY PART OF A MATRIX

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1. Introduction. It is well known that every matrix A (square and with complex entries) has a polar decomposition $A = P_1U_1 = U_2P_2$, where U_i are unitary and P_i are unique positive semi-definite Hermitian matrices. If A is non-singular then $U_1 = U_2 = U$, where U is also unique. In this case we call U the unitary part of A. The eigenvalues of P_1 are the same as those of P_2.

In [2] the following problem was solved. Given the eigenvalues of P_1, what is the exact range of variation of the eigenvalues of A ? The answer shows that a knowledge of the eigenvalues of P_1 puts restrictions only on the moduli of the eigenvalues of A. In this paper we are going to consider the corresponding question for the unitay part U of A. In turns out that a knowledge of the eigenvalues of U restricts only the arguments of the eigenvalues of A.

Before stating the result, we need some definitions. An ordered pair of n-tuples (λ_i), (α_i) of complex numbers is said to be *realizable* if there exists a non-singular matrix A of order n with eigenvalues λ_i such that the unitary part of A has eigenvalues α_i. If (γ_j) is an n-tuple of complex numbers of modulus 1, and if two of the γ_j are of the form e^{ib}, e^{ic} with $0 < b - c < \pi$ and $0 \leqq d \leqq (b - c)/2$, then the operation of replacing e^{ib}, e^{ic} by $e^{i(b-d)}$, $e^{i(c+d)}$ is called a *pinch* of (γ_j). In other words, a pinch of (γ_j) consists in choosing two of the γ_j which do not lie on the same line through 0 and turning them toward each other through equal angles.

If (a_i), (b_i) are n-tuples of real numbers, and if (a'_i), (b'_i) are their rearrangements in non-decreasing order, then we write $(a_i) \prec (b_i)$ when $\sum_r^n a'_i \leqq \sum_r^n b'_i$, $r = 2, \cdots, n$ and $\sum_1^n a'_i = \sum_1^n b'_i$. It is easily seen that the conditions are equivalent to the conditions $\sum_1^r a'_i \geqq \sum_1^r b'_i$, $r = 1, \cdots, n-1$, and $\sum_1^n a'_i = \sum_1^n b'_i$.

Our main theorem is the following.

THEOREM 1. *Let (λ_i), (α_i) be n-tuples of complex numbers such that $\lambda_i \neq 0$ and $|\alpha_i| = 1$. Then the following statements are equivalent:*

(1) *the pair (λ_i), (α_i) is realizable;*

(2) *(α_i) can be reduced to $(\lambda_i/|\lambda_i|)$ by a finite sequence of pinches;*

(3) *$\prod_1^n \alpha_i = \prod_1^n (\lambda_i/|\lambda_i|)$, and exactly one of the following hold:*

(a) *there is a line through 0 containing all the α_i and $(\lambda_i/|\lambda_i|)$ is a rearrangement of (α_i);*

(b) *there is no line through 0 containing all α_i but there is*

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