DERIVATIONS OF JORDAN ALGEBRAS

BRUNO HARRIS

Introduction. Let J be a finite-dimensional semisimple Jordan algebra over a field of characteristic zero, and D a derivation of J into a module M. Jacobson has shown, in [6], that D is inner in the sense that there exist elements z_i in J, m_i in M such that for all x in J,

$$D(x) = \sum_i (z_i, x, m_i) \tag{1}$$

(where (z_i, x, m_i) denotes the associator $(z_i \cdot x) \cdot m_i - z_i \cdot (x \cdot m_i)$ and $x \cdot y$ denotes the product in J or the product of an element of J and one of M). This theorem is the analogue for Jordan algebras of the first Whitehead lemma for semisimple Lie algebras of characteristic zero. In this paper we will consider two problems: first to generalize the above theorem to arbitrary characteristic p (excluding $p = 2$ but allowing $p = 3$); second, to express the group of derivations modulo inner derivations of any Jordan algebra (not necessarily finite-dimensional or semisimple) as a cohomology group. The second problem is part of a much more general one: that of developing a cohomology theory for Jordan algebras analogous to the existing theories for associative and Lie algebras (see [9]).

Our results are as follows: with respect to the first problem, we show that if J is finite-dimensional and separable, then every derivation of J into a module is inner (i.e. satisfies (1)) if and only if J satisfies the additional condition that it has no simple ideal which is special and whose degree is divisible by the characteristic of the base field. (This latter condition is directly related to the fact that the Lie algebra of all $n \times n$ matrices over a field of characteristic p cannot be expressed as the direct sum of the derived algebra and the center if p divides n.) For the proof we use the representation theory of Jordan algebras given in [7] and rely to a certain extent on the classification of simple algebras; however, it may be possible to give a proof not relying on the classification by using a Casimir operator (as is done for Lie and alternative algebras in [3]).

As for the second problem, our results cover only special Jordan algebras and certain types of modules. We use the concept of a bimodule with involution (introduced in [7]) for an associative algebra with involution, and introduce cohomology groups which are like the usual cohomology groups of associative algebras but also take into account

Received September 22, 1958. This work was supported in part by a grant from the National Science Foundation.