EXCEPTIONAL REAL LEHMER SEQUENCES

L. K. DURST

1. Introduction. If L and M are rational integers and L is positive, the sequence

$$(P): \quad P_0, \; P_1, \; P_2, \; \cdots, \; P_n, \; \cdots$$

is called the *Lehmer sequence* generated by

$$f(z) = z^2 - L^{1/2}z + M \; ,$$

if

$$\begin{aligned} P_n &= (\alpha^n - \beta^n)/(\alpha - \beta), \text{ for } n \text{ odd} \; , \\ &= (\alpha^n - \beta^n)/(\alpha^2 - \beta^2), \text{ for } n \text{ even} \; , \end{aligned}$$

where α, β are the roots of $f(z) = 0$. Since $P_0 = 0$, $P_1 = 1$ and the remaining terms of (P) satisfy the recursion relations

$$\begin{aligned} P_{2n} &= P_{2n-1} - MP_{2n-2} \\ P_{2n+1} &= LP_{2n} - MP_{2n-1} \; , \end{aligned}$$

it is clear that every Lehmer sequence is a sequence of rational integers. In Lehmer [1], P_n is denoted by $\overline{U}_n$.

The sequence (P) is called *real* if $K = L - 4M$, the discriminant of $f(z)$, is positive. An index n greater than 2 is called *exceptional* if each prime dividing P_n also divides a term P_m, where $0 < m < n$. The sequence (P) is called *exceptional* if it contains a term whose index is exceptional.

This paper continues the classification of exceptional real Lehmer sequences begun by Morgan Ward [2]. The main result is the following theorem.

THEOREM 1.0. *For real Lehmer sequences, the only possible exceptional indices are six and twelve. Twelve is exceptional only in the sequences determined by*

$$L = 1, \; M = -1 \text{ and } L = 5, \; M = 1.$$

Six is exceptional if and only if

$$L = -3K + 2^{s+2}, \; M = -K + 2^s,$$

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