A DUALITY THEOREM FOR AN ARBITRARY OPERATOR

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1. Introduction. This paper studies the spectral theory of an operator (or bounded linear transformation) on a Banach space B. Although it would be possible to modify the results to apply to operators on non-reflexive spaces, for simplicity only reflexive spaces are considered. The distinctive feature of the treatment is that we are primarily interested in the spectral properties of a completely arbitrary operator. That is, we seek a spectral theory which will be valid independently of any of the usual restrictions (such as normality or complete continuity). It is, of course, not to be expected, in view of many known counter examples, that such a theory will even approach in power the spectral theory of a Hermitian or normal operator on a Hilbert space. In fact, it is surprising that a spectral theory for an arbitrary operator exists at all. The results obtained here are incomplete, but it seems likely that any spectral theory which is valid for an arbitrary operator will be closely related to the theory developed here. It is interesting that certain known results in spectral theory, which imply the spectral theorem for a Hermitian or a unitary operator, can be obtained from the theory which we develop for an arbitrary operator. The principal ideas of this paper are developments of ideas which appeared in rudimentary form in unpublished portions of the author's University of Chicago doctoral dissertation.

The program of the paper is as follows. Instead of considering one type of spectral theory, we consider four. Since the content of these theories is that a certain duality exists between an operator and its adjoint, and since the term "spectral theory" is somewhat deceptive, we shall employ the term "duality theory" in its stead. The duality theories are not all of comparable strength. The weakest, type 4, will be shown to be valid for an arbitrary operator. This will be the principal result. This having been shown, it will then be possible to show that certain of the stronger and more conventional duality theories are satisfied provided that the operator is subject to appropriate restrictions. The duality theories will concern the existence and properties of spectral manifolds. Two types of spectral manifold are needed. The first type of spectral manifold, the strong type, is defined as one might expect, but it is the introduction of the second or weak type of spectral manifold that permits the construction of a theory which will apply to an arbitrary operator.

The two types of spectral manifold and the four types of duality

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