THE NILPOTENT PART OF A SPECTRAL OPERATOR

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1. Introduction. Throughout this paper, $\mathfrak{X}$ is a Banach space, T a bounded spectral operator on $\mathfrak{X}$ with scalar part S, nilpotent part N, and resolution of the identity $E(\sigma)$ for σ a Borel set in the complex plane. M is the bound for the norms of the $E(\sigma)$; $|E(\sigma)| \leq M$ for all Borel sets σ. The resolvent function for $T, (\lambda - T)^{-1}$, is denoted by $R(\lambda, T)$. The operator $R(\lambda, T)E(\sigma)$ has an unique analytic extension from the resolvent set of T to the complement of $\bar{\sigma}$, and on the subspace $E(\sigma)\mathfrak{X}$ it is equal to the operator $R(\lambda, T_\sigma)$ where T_σ is the restriction of T to $E(\sigma)\mathfrak{X}$. For material on spectral operators, we refer to the papers on N. Dunford [1], [2]. $\chi_\sigma(\xi)$ is the characteristic function of the Borel set σ: $\chi_\sigma(\xi) = 1$ if $\xi \in \sigma$, $\chi_\sigma(\xi) = 0$ if $\xi \notin \sigma$. For p a non-negative real number, μ_p is Hausdorff p-dimensional measure [3, pp. 102 *ff.*]; μ_2 is Lebesgue planar measure multiplied by $\pi/4$, and μ_1 restricted to an arc is majorized by arc length.

We assume throughout that there is an integer m for which the resolvent function for T satisfies the mth order rate of growth condition

$$|R(\lambda, T)E(\sigma)| \leq K \cdot d(\lambda, \sigma)^{-m}, \lambda \notin \bar{\sigma}, |\lambda| \leq |T| + 1 ,$$

where $d(\lambda, \sigma)$ is the distance from λ to σ and K is a constant independent of σ. If $\mathfrak{X}$ is Hilbert space, it is known that this growth condition implies $N^m = 0$ [1, p. 337]. In an arbitrary Banach space, this is no longer true; the best that can be done is $N^{m+2} = 0$. If $\mathfrak{X}$ is weakly complete, $N^{m+1} = 0$; or if σ is a set of μ_2 measure zero, $N^{m+1}E(\sigma) = 0$. If σ lies in an arc and either $\mathfrak{X}$ is weakly complete or σ has μ_1 measure zero, then $N^m E(\sigma) = 0$. Examples show that we cannot obtain lower indices of nilpotency in general.

2. The fundamental lemma and some easy consequences. If $f(\xi)$ is a bounded, scalar valued Borel function, the operator $\int f(\xi)E(d\xi)$ exists as a bounded operator with norm at most $4M \cdot \sup_\xi |f(\xi)|$ [1, p. 341], so that uniform convergence of a sequence of bounded Borel functions $f_n(\xi)$ implies convergence in the uniform operator topology of the operators $\int f_n(\xi)E(d\xi)$. Thus for a given bounded Borel function $f(\xi)$ and a given positive number η, there exist a finite number of disjoint Borel sets σ_i and points $\xi_i \in \sigma_i$ such that

Received February 5, 1959. This paper is a portion of a doctoral dissertation presented to Yale University, written under the direction of Professor E. Hille, while the author was an NSF fellow. Particular thanks are due to W. G. Bade who read the manuscript and discovered an error in the author's original proof of Theorem 3.1.