GENERALIZED RANDOM VARIABLES

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We will consider random variables on a denumerably infinite sample space. However, the range $\boldsymbol{R}$ of the random variables will not necessarily be a set of real numbers. In Part I the range will be a subset of a given metric space, and in Part II it will be an arbitrary set. Since each distribution on the sample space determines a distribution on $\boldsymbol{R}$ (for a given random variable), the sample space may be ignored entirely, and we may restrict our attention to distributions on $\boldsymbol{R}$. Thus, instead of discussing means and variances of random variables on the sample space, we will discuss means and variances of distributions on the set $\boldsymbol{R}$.

In classical probability theory $\boldsymbol{R}$ would be a set of real numbers, and the mean and variance of a distribution on $\boldsymbol{R}$ would also be real numbers. Of these restrictions only one will be kept, namely that the variance will always be a non-negative real number. As indicated above, $\boldsymbol{R}$ may be a more general space, and the means will also be selected from more general spaces. The defining property of a mean will be the property of minimizing the variance of the given distribution. It will be shown that these means still have many of the classical properties, though in general means are not unique, and in certain circumstances there may be no mean.

While the mean is classically taken to be a real number, it need not be an element of $\boldsymbol{R}$. For example, the mean of a set of integers may be a fraction. This approach is extended in Part I, where the means may be arbitrary points of a certain metric space $\boldsymbol{T}$, and $\boldsymbol{R}$ is any subset of $\boldsymbol{T}$. Even the form chosen for the variance is the same as in classical probability theory.

In Part II the concept of a random variable and of means is further generalized. Here $\boldsymbol{R}$ is an arbitrary set, and the topological space $\boldsymbol{T}$ from which means are chosen need not be metric and need bear no relation to $\boldsymbol{R}$. The variance is still a numerical function on $\boldsymbol{T}$, but of a much more general form than in Part I. In both frameworks an analogue of the strong law of large numbers is proved, to show that classical results can be generalized to these new kinds of random variables.

In Part III we consider certain generalizations. The positive result in this part is that the restriction to independent random variables in Parts I and II is unnecessary; the results hold for any metrically

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