COINCIDENCE PROPERTIES OF BIRTH AND DEATH PROCESSES

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A birth and death process (for brevity referred to henceforth as process B) is a stationary Markov process whose state space is the non-negative integers and whose transition probability matrix

$$P_{ij}(t) = \Pr\{x(t) = j \mid x(0) = i\} \tag{1}$$

satisfies the conditions (as $t \to 0$)

$$P_{ij}(t) = \begin{cases} \lambda_i t + o(t) & \text{if } j = i+1 \\ \mu_i t + o(t) & \text{if } j = i-1 \\ 1 - (\lambda_i + \mu_i)t + o(t) & \text{if } j = i \end{cases} \tag{2}$$

where $\lambda_i > 0$ for $i \geq 0$, $\mu_i > 0$ for $i \geq 1$ and $\mu_0 \geq 0$. We further assume that $P_{ij}(t)$ satisfies the foward and backward equation in the usual form. In this paper we restrict attention to the case $\mu_0 = 0$ so that when the particle enters the state zero it remains there a random length of time according to an exponential distribution with parameter λ_0 and then moves into state one etc.

In order to avoid inessential difficulties we assume henceforth that the infinitesimal birth and death rates λ_i and μ_i uniquely determine the process. This is equivalent to the condition $\sum_{n=0}^{\infty}(\pi_n + 1/\lambda_n\pi_n) = \infty$ where

$$\pi_0 = 1 \text{ and } \pi_n = \frac{\lambda_0\lambda_1\lambda_2 \cdots \lambda_{n-1}}{\mu_1\mu_2\mu_3 \cdots \mu_n} \qquad [2].$$

In the companion paper we show that for all $t > 0$

$$\det(P_{i_\mu, j_\nu}(t)) = P\begin{pmatrix} & i_1, i_2, \cdots, i_n \\ t\,; & \\ & j_1, j_2, \cdots, j_n \end{pmatrix} \qquad \begin{matrix} i_1 < i_2 < i_3 < \cdots < i_n \\ \\ j_1 < j_2 < j_3 < \cdots < j_n \end{matrix} \tag{3}$$

has the following interpretation: Start n labled particles at time zero in states $i_1, i_2, \cdots, i_n$ respectively, each governed by the transition law (1) and acting independently. The determinant (3) is equal to the probability that at time t particle 1 is located in state j_1, particle 2 is located in state j_2 etc., without any two of these particles having occupied simultaneously a common state at some earlier time $\tau < t$. We refer to this event as a transition in time t of n particles from initial states

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