GENERATING SETS OF ELEMENTS IN COMPACT GROUPS

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1. Preliminaries. It is well known that compact topological groups have many properties similar to those of finite groups, which are of course special cases of compact topological groups under the discrete topology. The program of this paper is to characterize sets of elements in a compact topological group which generate a given subgroup and, conversely, to determine properties of the subgroup generated by a given set of elements by an investigation of the properties of this set. Tools for our investigation are the convolution algebra of continuous complex-valued functions on the group and the system of irreducible representations of the group. We shall also formulate the results using those concepts. Our results are straightforward generalizations of known theorems on generating sets of elements in finite groups[1].

From now on G will denote a compact topological group which, as a topological space, is T_1. It follows that G is Hausdorff and, therefore, also normal. Let e denote the identity of G. A subset H of G will be called a subgroup of G if it is an abstract subgroup of G and closed, unless the contrary is specifically stated. Let μ denote the normalized Haar measure on G: $\mu(G) = 1$.

A subgroup H with positive measure $\mu(H) > 0$ is necessarily both open and closed, as are all (left) cosets of H. Thus a compact group G with such a subgroup is disconnected and the quotient-spaces G/H (with respect to left cosets of H) is finite and discrete in the quotient topology. Then $1/\mu(H)$ is the index of H in G. The quotient space of G with respect to left cosets of a subgroup of measure 0 contains infinitely many elements and is again compact, Hausdorff and normal.

Let C denote the field of complex numbers and $C(G)$ the set of all complex-valued continuous functions on G. Defining scalar multiplication and addition in $C(G)$ pointwise as usual, $C(G)$ becomes a Banach-space under the uniform norm: $\|f\| = \sup_{x \in G}\{|f(x)|\}$ $(f \in C(G))$. Defining multiplication in $C(G)$ by convolution,

$$(f * g)(x) = \int_G f(xy^{-1})g(y)dy ,$$

$C(G)$ becomes a Banach algebra. Left and right translations of $f \in C(G)$ by $s \in G$ are defined by ${}_sf(x) = f(sx)$ and $f_s(x) = f(xs)$ respectively. Both ${}_sf$ and f_s are functions in $C(G)$ and every $f \in C(G)$ is both left

[1] See [2].

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